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WILLMORE TORI IN A WIDE FAMILY OF CONFORMAL STRUCTURES ON ODD DIMENSIONAL SPHERES

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ABSTRACT. We obtain a variable reduction principle for the Willmore variational problem in an ample class of conformal structures on S^{2n+1}. This variational problem is transformed into another one, associated with an elastic-energy functional with potential, on spaces of curves in CP^n. Then, we give a simple method to construct Willmore tori in certain conformal structures on S^{2n+1}. Moreover, we exhibit some families of Willmore tori for the standard conformal class on S^3 and S^7.

1. Introduction. Let $\mathbf{S}^{2n+1}$ be the unit sphere in $\mathbf{C}^{n+1}$ endowed with the standard metric $\bar{g}$. The unit circle $\mathbf{S}^1$ acts naturally on $\mathbf{S}^{2n+1}$ to produce $\mathbf{C}P^n$ as orbit space. The canonical projection $\pi : (\mathbf{S}^{2n+1}, \bar{g}) \rightarrow (\mathbf{C}P^n, g)$ is a Riemannian submersion, where g denotes the Fubini-study metric of constant holomorphic sectional curvature 4. A vertical, unit global vector field V is defined on $\mathbf{S}^{2n+1}$ by $V(z) = iz$, for all $z \in \mathbf{S}^{2n+1}$. The horizontal distribution $\mathcal{H}$ is defined to be the $\bar{g}$-orthogonal complementary to the orbits. As usual, overbars will denote horizontal lifts of the corresponding objects in a Riemannian submersion (see [**6**], [**13**] for details about notation and terminology). In particular, the Levi-Civita connections $\bar{\nabla}$ and ∇ of $\bar{g}$ and g, respectively, are related via the following well-known formulae:

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} - \bar{g}(i\bar{X}, \bar{Y})V, \tag{1.1}$$

$$\bar{\nabla}_{\bar{X}}V = \bar{\nabla}_V \bar{X} = i\bar{X}, \tag{1.2}$$

$$\bar{\nabla}_V V = 0. \tag{1.3}$$

Remark 1. (i) It should be noticed that the last formula shows the geodesic nature of the orbits in $(\mathbf{S}^{2n+1}, \bar{g})$. (ii) Since π may also be

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regarded as the projection of a principal fiber bundle with structure group $\mathbf{S}^1$ and $\mathcal{H}$ is $\mathbf{S}^1$-invariant, it defines a principal connection whose connection 1-form will be denoted by ω. (iii) We can use the nice argument of Pinkall (see [**15**]), to show that an immersed surface M in $\mathbf{S}^{2n+1}$ is $\mathbf{S}^1$-invariant if and only if $M = M_\gamma = \pi^{-1}(\gamma)$ for some immersed curve γ in $\mathbf{C}P^n$. In particular, if γ is closed, then M_γ is a torus, which is embedded if γ is free of self-intersections in $\mathbf{C}P^n$.

Let h be a Riemannian metric on $\mathbf{C}P^n$ and u a positive smooth function on $\mathbf{C}P^n$. We define

$$\bar{h}_u = \pi^*(h) + \varepsilon(u \circ \pi)^2 \omega^*(dt^2), \tag{1.4}$$

where dt^2 is the usual metric on $\mathbf{S}^1$ and $\varepsilon = \pm 1$. It is clear that $\bar{h}_u$ is a metric on $\mathbf{S}^{2n+1}$, which is Riemannian or Lorentzian according to whether ε is $+1$ or -1, respectively. These metrics are called the *generalized Kaluza-Klein* metrics on $\mathbf{S}^{2n+1}$ ([**9**]).

It is not difficult to see that the $\mathbf{S}^1$-action on $\mathbf{S}^{2n+1}$ is made up through isometries of $(\mathbf{S}^{2n+1}, \bar{h}_u)$. Furthermore, $\pi : (\mathbf{S}^{2n+1}, \bar{h}_u) \to (\mathbf{C}P^n, h)$ is a pseudo-Riemannian submersion, which has geodesic fibers if and only if u is constant. In this case the scalar curvature of $(\mathbf{S}^{2n+1}, \bar{h}_u)$ is constant. Moreover, if γ is a curve with curvature function k in $(\mathbf{C}P^n, h)$, then the mean curvature function α of M_γ in $(\mathbf{S}^{2n+1}, \bar{h}_u)$ satisfies [**1**]:

$$\alpha^2 = \frac{1}{4}(k^2 \circ \pi). \tag{1.5}$$

Let $\mathcal{N}$ be the space of immersions of a genus one compact surface N in $\mathbf{S}^{2n+1}$. For any semi-Riemannian metric $\widetilde{h}$ on $\mathbf{S}^{2n+1}$, we have the Willmore functional $\mathcal{W} : \mathcal{N} \to \mathbf{R}$ defined by

$$\mathcal{W}(\varphi) = \int_N (\alpha^2 + S)\, dv, \tag{1.6}$$

where α is the mean curvature function of φ, S is the sectional curvature function of $(\mathbf{S}^{2n+1}, \widetilde{h})$ along φ and dv is the volume element of $\varphi^*(\widetilde{h})$ on N. The critical points of this functional are the so-called *Willmore tori*. This functional is an invariant under conformal changes of the ambient metric $\widetilde{h}$ ([**7**]). Therefore, if $\mathcal{C}(\bar{h}_u)$ denotes the conformal class associated to $\bar{h}_u$, it is natural to pose the following problem:

Studying the existence and characterization of
$\mathbf{S}^1$*-invariant Willmore tori in* $(\mathbf{S}^{2n+1}, \mathcal{C}(\bar{h}_u))$.

Some particular answers to this problem have been obtained in [**1**], [**5**], [**15**]. On the other hand, we consider the *total squared curvature* functional acting on closed curves (or curves satisfying given first order boundary data) in a Riemannian manifold (M, g). The extremal points of this functional are called *free elastic curves* in (M, g) (see [**10**], [**11**], [**12**]). In this note we show that the existence of Willmore tori in $(\mathbf{S}^{2n+1}, \mathcal{C}(\bar{h}_u))$ which are invariant under the natural $\mathbf{S}^1$-action on $\mathbf{S}^{2n+1}$ is equivalent to the existence of critical points of the functional

$$\mathcal{F}(\gamma) = \int_\gamma (k^2 + \phi(\gamma'))\, ds, \tag{1.7}$$

acting on closed curves γ in $(\mathbf{C}P^n, (1/u^2)h)$, where k is the curvature function of γ and $\phi(\gamma') = 4\varepsilon\,(g(\gamma', \gamma'))^2$ works as a potential. A ϕ*-elastic curve* is a critical point of (1.7). Then, we will use the Euler-Lagrange equation associated with the functional (1.7), to construct Willmore tori in a wide family of conformal structures on $\mathbf{S}^{2n+1}$ (see Corollary 3.1). In particular, we obtain families of Willmore tori in $(\mathbf{S}^3, \mathcal{C}(\bar{g}))$ and $(\mathbf{S}^7, \mathcal{C}(\bar{g}))$ (see Corollaries 3.2 and 3.3).

2. The main theorem.

Theorem 2.1. *$M_\gamma = \pi^{-1}(\gamma)$ is a Willmore torus in $(\mathbf{S}^{2n+1}, \mathcal{C}(\bar{h}_u))$ if and only if γ is a closed curve in $\mathcal{C}P^n$, which is a critical point of the following elastic-energy functional on $(\mathcal{C}P^n, (1/u^2)h)$:*

$$\mathcal{F}(\gamma) = \int_\gamma (k^2 + \phi(\gamma'))\, ds, \tag{2.1}$$

where k is the curvature function of γ and $\phi(\gamma') = 4\varepsilon(g(\gamma', \gamma'))^2$.

Proof. Since the Willmore variational problem is invariant under conformal changes of the ambient space metric, we choose the following metric in $\mathcal{C}(\bar{h}_u)$:

$$\widetilde{h}_u = \frac{1}{(u\circ\pi)^2}\bar{h}_u = \pi^*\left(\frac{1}{u^2}h\right) + \varepsilon\omega^*(dt^2). \tag{2.2}$$

This choice has the following advantage: $\pi : (\mathbf{S}^{2n+1}, \widetilde{h}_u) \to (\mathcal{C}P^n, (1/u^2)h)$ has geodesic fibers.

It is clear that the Willmore functional is $\mathbf{S}^1$-invariant, that is, $\mathcal{W}(e^{i\theta}\varphi) = \mathcal{W}(\varphi)$. We define the submanifold $\mathcal{N}_{\mathbf{S}^1}$ of $\mathbf{S}^1$-invariant immersions which can be identified (see (iii) of Remark 1) with $M_\gamma = \{\pi^{-1}(\gamma) \mid \gamma$ is a closed curve immersed in $\mathbf{C}P^n\}$. Let Σ be the set of critical points of $\mathcal{W}$ (Willmore tori), and denote by $\Sigma_{\mathbf{S}^1}$ the set of critical points of $\mathcal{W}$ when restricted to $\mathcal{N}_{\mathbf{S}^1}$. Then we use the principle of symmetric criticality ([**14**]) to get

$$\Sigma \cap \mathcal{N}_{\mathbf{S}^1} = \Sigma_{\mathbf{S}^1}. \tag{2.3}$$

Therefore, to obtain Willmore tori in $(\mathbf{S}^{2n+1}, \mathcal{C}(\bar{h}_u))$ which do not break the $\mathbf{S}^1$-symmetry of the problem, we only need to compute $\mathcal{W}$ on $\mathcal{N}_{\mathbf{S}^1}$ and then to proceed in due course.

To compute $\mathcal{W}(\pi^{-1}(\gamma))$, we parametrize γ by its arc length in $(\mathcal{C}P^n, (1/u^2)h)$ and observe that $T_p(M_\gamma)$ is a mixed section of $T_p(\mathbf{S}^{2n+1})$ for any $p \in M_\gamma$. Since $\widetilde{h}_u(V,V) = \varepsilon$, the term S in the integrand of $\mathcal{W}$ is given by ([**6**]):

$$S = \varepsilon\widetilde{h}_u(\widetilde{D}^u_{\bar{X}}V, \widetilde{D}^u_{\bar{X}}V), \tag{2.4}$$

where $\bar{X}$ is the horizontal lift of $X = \gamma'$ and $\widetilde{D}^u$ is the Levi-Civita connection of $\widetilde{h}_u$. Take a local horizontal frame $\{\bar{X}, i\bar{X}, Y_2, iY_2, \dots, Y_n, iY_n\}$ along M_γ. Then we use (1.2) to get

$$\begin{aligned}\widetilde{D}^u_{\bar{X}}V = &-\frac{1}{2}\widetilde{h}_u([\bar{X}, i\bar{X}], V)i\bar{X} - \frac{1}{2}\sum \widetilde{h}_u([\bar{X}, Y_j], V)Y_j \\ &- \frac{1}{2}\sum \widetilde{h}_u([\bar{X}, iY_j], V)iY_j.\end{aligned}$$

To calculate the Lie brackets appearing in the last formula, we use (1.1) and then

$$\widetilde{D}^u_{\bar{X}}V = \varepsilon g(\gamma', \gamma')i\bar{X}, \tag{2.5}$$

g being the Fubini-study metric in $\mathbf{C}P^n$.

Using (1.5), (2.3) and (2.4), we have

$$\begin{aligned}\mathcal{W}(\pi^{-1}(\gamma)) &= \int_0^L \int_0^{2\pi} \left(\frac{1}{4}\kappa^2 + \varepsilon(g(\gamma', \gamma'))^2\right) ds\, dt \\ &= \frac{\pi}{2}\int_0^L (\kappa^2 + \phi(\gamma'))\, ds,\end{aligned}$$

where L is the length of γ in $(\mathbf{C}P^n, (1/u^2)h)$ and $\phi(\gamma') = 4\varepsilon(g(\gamma', \gamma'))^2$.

This completes the proof of the theorem. □

In particular, if $n = 1$, we identify $\mathbf{C}P^1$ with $\mathbf{S}^2$ in the standard fashion to obtain the usual Hopf map $\pi : \mathbf{S}^3 \to \mathbf{S}^2$. On the other hand, as a consequence of the uniformization theorem for Riemann surfaces, we can choose in the conformal class of $\bar{h}_u$ a metric

$$\bar{g}_u = \pi^*(g) + \varepsilon(u \circ \pi)^2 \omega^*(dt^2), \tag{2.6}$$

where g is the canonical metric of constant Gaussian curvature 4 in $\mathbf{S}^2$. So we have

Corollary 2.2. *Let γ be a closed immersed curve in $\mathbf{S}^2$. Then $M_\gamma = \pi^{-1}(\gamma)$ is a Willmore torus in $(\mathbf{S}^3, \mathcal{C}(\bar{g}_u))$ if and only if γ is a ϕ-elastica with potential $\phi(\gamma') = 4\varepsilon(g(\gamma', \gamma'))^2$ in $(\mathbf{S}^2, (1/u^2)g)$.*

3. Further discussions and applications. Let γ be a ϕ-elastica in $(\mathcal{C}P^n, (1/u^2)h)$. The potential ϕ is a smooth function, defined on the unit tangent vector bundle of $(\mathcal{C}P^n, (1/u^2)h)$. It is clear that ϕ is a basic function on that bundle, i.e., a function on $\mathcal{C}P^n$ if and only if h is chosen in the conformal class of the Fubini-study metric g on $\mathcal{C}P^n$. In this case, without loss of generality, we can take $h = g$. For basic potentials, the Euler-Lagrange equations of ϕ-elasticae can be computed using Lemma 1.1 of [**10**] in a standard argument which involves some integration by parts. Then we have

$$2\widehat{\nabla}^3_T T + 3\widehat{\nabla}_T(\kappa^2 T) + 2\widehat{R}(\widehat{\nabla}_T T, T)T + \widehat{\nabla}\phi - \phi\widehat{\nabla}_T T - T(\phi)T = 0, \tag{3.1}$$

where the elements appearing in this formula are taken in $(\mathcal{C}P^n, (1/u^2)h)$, in particular $\widehat{R}$ is the Riemann curvature of this metric and $\phi = 4u^4$.

Let γ be a closed curve in $\mathbf{C}P^n$ and denote by η its unit normal vector field in $(\mathbf{C}P^n, g)$. Put $\mathcal{F}_+^\gamma$ to name the space of positive smooth functions, f, on $\mathbf{C}P^n$ such that $\eta(f) = 0$ (along γ). We have

Corollary 3.1. *Let γ be a geodesic in $(\mathbf{C}P^n, g)$ and $u \in \mathcal{F}_+^\gamma$. Then M_γ is a Willmore torus in $(\mathbf{S}^{2n+1}, \mathcal{C}(\bar{g}_u))$ which is conformally minimal in $(\mathbf{S}^{2n+1}, \bar{g}_u)$.*

Proof. A direct computation shows that (3.1) can be written as

$$2\widehat{\nabla}_T^3 + 3\widehat{\nabla}_T(\kappa^2 T) + 2\widehat{R}(\widehat{\nabla}_T T, T)T - \phi^3 \nabla_{T^*} T^* = 0, \tag{3.2}$$

where T^* is the unit tangent of γ computed in $(\mathbf{C}P^n, g)$. Now it is obvious that if γ is a geodesic in $(\mathbf{C}P^n, g)$ and $u \in \mathcal{F}_+^\gamma$, then it is also a geodesic in $(\mathbf{C}P^n, (1/u^2)g)$ and so a ϕ-elastica in $(\mathbf{C}P^n, (1/u^2)g)$ with $\phi = 4u^4$. Now the statement follows from the main theorem. □

Corollary 3.2. *Let γ be any great circle in $(\mathbf{S}^2, g)$ and $u \in \mathcal{F}_+^\gamma$. Then $M_\gamma = \pi^{-1}(\gamma)$ is a Willmore torus in $(\mathbf{S}^3, \mathcal{C}(\bar{g}_u))$.*

It is obvious that minimal surfaces of the standard sphere $(\mathbf{S}^m, \bar{g})$ are Willmore. If we pay attention to the spectral behavior of the position vector of those surfaces in $\mathbf{R}^{m+1}$ [**16**], then it seems natural to look for Willmore surfaces in $(\mathbf{S}^m, \bar{g})$ which can be constructed in $\mathcal{R}^{m+1}$ using eigenfunctions of the Laplacian coming from exactly two different eigenvalues (2-type surfaces [**8**]). These surfaces have been completely classified in [**3**]. They are certain flat tori which fully yield in $(\mathbf{S}^5, \bar{g})$ or in $(\mathbf{S}^7, \bar{g})$. Since the family of Willmore tori in $(\mathbf{S}^5, \bar{g})$ has been studied in [**5**], in this note we are going to deal with those surfaces in $(\mathbf{S}^7, \bar{g})$.

It was shown in [**3**] that the map $Y : \mathbf{R}^2 \to \mathbf{C}^4$, given by

$$Y(s,t) = e^{it}(c_1 \cos As,\ c_1 \sin As,\ c_2 \cos Bs,\ c_2 \sin Bs), \tag{3.3}$$

with $c_1^2 + c_2^2 = 1$, defines an isometric immersion of a flat torus, say T, in $(\mathbf{S}^7, \bar{g})$ which is of 2-type when $A \neq B$ and Willmore for certain choices of (A, B) which involve the isometry type of that flat torus (see [**3**]). It is evident that these Willmore tori are $\mathbf{S}^1$-invariant. Namely, if we put $\gamma(s) = \pi(c_1 \cos As, c_1 \sin As, c_2 \cos Bs, c_2 \sin Bs)$, then $T = \pi^{-1}(\gamma)$.

Furthermore, one can prove that $\gamma(s)$ is a helix which yields into a three-dimensional, Lagrangian and totally geodesic $\mathbf{R}P^3$ of $\mathbf{C}P^3$. It should be noticed that, according to our main theorem, the curves $\gamma(s)$ are closed helices, which are ϕ-elasticae (in this case elasticae, [**10**]) with $\phi = 4$ (because of its constancy, ϕ works as a Lagrange multiplier, [**10**]). Now we get Willmore tori in $(\mathbf{S}^7, \mathcal{C}(\bar{g}))$ by lifting closed helices, which are elasticae (with $\phi = 4$) in $(\mathcal{C}P^3, g)$. We can use a similar argument to that used in [**2**] and [**4**] to obtain a one-parameter family of elastic helices, $\phi = 4$, in $\mathbf{R}P^3$. In particular, this family contains a rational one-parameter subfamily of closed elastic helices. Now we regard $\mathbf{R}P^3$ as a Lagrangian and totally geodesic submanifold in $\mathbf{C}P^3$ to obtain, via our main theorem, the following family of Willmore tori in $(\mathbf{S}^7, \mathcal{C}(\bar{g}))$ which includes those of 2-type given in (3.3).

Corollary 3.3. *A rational one-parameter family of Willmore tori exists in* $(\mathbf{S}^7, \mathcal{C}(\bar{g}))$ *which have nonzero constant mean curvature in* $(\mathbf{S}^7, \bar{g})$. *Moreover, they yield fully in the seven sphere.*

Remark 2. It should be noticed that the family of Willmore tori given in this corollary is different from that obtained in [**3**]. However, both families have a nonempty intersection made up by immersions defined in (3.3).

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