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VECTOR BUNDLES ON A FORMAL NEIGHBORHOOD OF A CURVE IN A SURFACE

E. BALLICO AND E. GASPARIM

ABSTRACT. Here we study vector bundles in a formal or tubular neighborhood of a smooth projective curve C in a complex surface W. In several cases (e.g., if C has genus 0 and its normal bundle has degree -1 or -2) we attach to every such bundle a series of discrete invariants and "simpler" bundles, and we study the set of all bundles with fixed invariants.

0. Introduction. Let W be either a smooth connected quasi-projective surface defined over an algebraically closed field $\mathbf{K}$ or a smooth connected two-dimensional complex manifold. Let $C \subset W$ be a smooth connected curve of genus $q \geq 0$ and U either the formal completion of W along C or, in the complex analytic case, a small tubular neighborhood of C in W for the Euclidean topology. We want to study algebraic (or complex analytic) vector bundles on U. Even more, we want to study families of vector bundles on U "parametrized" (not one-to-one and usually not even generically finite to one) by integral varieties or irreducible and reduced complex spaces. In some cases a natural topological structure appears which allows us to say that a family of vector bundles is in the closure of another set of vector bundles. We give an easy example. Let $\pi : W \to \mathbf{P}^2$ be the blowing-up of the complex plane at a point. Let U be a small open Euclidean neighborhood of the exceptional divisor C on W. Consider a rank two holomorphic bundle E over W with $E \mid C \cong \mathbf{O}_C(2) \oplus \mathbf{O}_C(-2)$. A simple application of [**3**, Theorem 2.1] tells us that $E \mid U$ can be given by a 2×2 transition matrix of the form (g_{ij}) with $g_{11} = z^2$, $g_{22} = z^{-2}$, $g_{21} = 0$ and $g_{12} \in \mathbf{C}[z, u]$ with g_{12} of the form $g_{12} = (p_{10} + p_{11}z)u + p_{21}zu^2$

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with p_{10}, p_{11} and p_{21} complex numbers. By [**3**, Theorem 3.4] we have the following three strata S_1, S_2 and S_3:

(i) a generic stratum S_1 corresponding to the case where the bundle is a nontrivial extension on the first infinitesimal neighborhood, i.e., to the case in which $p_{10} + p_{11}z \neq 0$; this stratum is parametrized by $\mathbf{P}^1$;

(ii) a stratum S_2 consisting of bundles that are trivial extensions to the first infinitesimal neighborhood of C but not to the second infinitesimal neighborhood; here, up to isomorphisms, there is a unique bundle;

(iii) a stratum S_3 corresponding to bundles whose restriction to the second infinitesimal neighborhood of C splits; there is a unique such bundle and it corresponds to the split bundle on the formal neighborhood of C.

It is obvious that S_2 and S_3 are limits of bundles in the stratum S_1. In this paper we want to study further these situations. However, for bundles E on W with $E \mid C \cong \mathbf{O}_C(a) \oplus \mathbf{O}_C(b)$ and $a - b \geq 6$ the situation (i.e., the dependence of $E \mid U$ on the polynomial g_{12} appearing in its transition matrix) is very complicated. Therefore, in this paper we will look for a stratification given by numerical invariants which are simple to identify. Our numerical invariants will be associated to each vector bundle using certain transformations, called elementary transformations, described below. Essentially we choose a particular curve on a surface and perform elementary transformations until we get to bundles which are trivial on the chosen curve. Then we list numerical invariants which represent the sequence of elementary transformations we performed. We use such lists of invariants to stratify parameter spaces of bundles and then we give a naive definition of when certain strata are in the closure of another stratum. This program was started in [**1**] for rank two vector bundles when C is an exceptional divisor, i.e., when $q = 0$ and the normal bundle, N, of C in W has degree -1. In this paper we again study the same situation (see Section 1) and then consider more general cases: the case $q = 0$ but arbitrary rank in Section 2, the case $q = 0$ and $\deg(N) = -2$ in Section 3 and the case $q > 0$ (but only for rank two vector bundles) in Section 4. In Section 5 we give several easy technical lemmas and remarks which are used in the first four sections. As in [**1**] we use the notion of elementary transformation of a vector bundle along a Cartier divisor of the ambient variety to study vector bundles on U and, when $q = 0$ and $\deg(N) = -1$ or $\deg(N) = -2$, to put them in a normal form in an algorithmic way. The invariants associated to each bundle E on U using this procedure give a stratification of the set of all vector bundles on U. If $q = 0$, $\deg(N) = -1$, and we consider only rank two vector bundles. We prove that these strata are parametrized by an irreducible variety (Theorem 1.2) and we show when bundles in one stratum are in the closure of the set of all bundles in another stratum (Theorem 1.4). These two theorems are the main results of this paper, but the main aim of this paper is to give the general approach and show how far it can lead in more general situations. For rank two vector bundles when $q > 0$ we analyze several different cases according to the triple of integers $(q, e, s(E))$ where $e := \deg(N)$ and $s(E)$ is the stability (or instability) degree of the bundle E in the sense of [**8**]. We were stimulated to study vector bundles on U by [**7**, Problem 17]. It is often important to study vector bundles on formal spaces and on the formal completion of a compact subspace of a complex space. In particular, this is essential for the study of coherent modules over germs of isolated singularities. Another interesting case is when we take the compact exceptional set, D, of a strongly 1-convex space X, i.e., when there is a proper holomorphic map $\pi : X \to Y$ with Y a Stein space, $\dim(D) = 0$, D has no isolated point and $\pi \mid (X \backslash D) : X \backslash D \to Y \backslash \pi(D)$ is biholomorphic; in this case, the cohomological properties of a vector bundle on X are determined by its restriction to the formal completion of D in X [**11**, Proposition 2.16]. In the algebraic case (in arbitrary dimension) elementary transformations are known to be a powerful tool; for instance, a theorem of Maruyama [**9**, Theorem 1.12] shows that every vector bundle is obtained from the trivial bundle making suitable elementary transformations along Cartier divisors. In our two-dimensional set-up we use only one Cartier divisor, C. If $C \cong \mathbf{P}^1$ and C is an exceptional divisor in [**1**] for rank 2 and here in Section 3 for arbitrary rank it was shown how to use elementary transformations along C to compute "the drop of c_2 along C," i.e., a suitable integral on the boundary of a tubular neighborhood of C.

1. C **rational, deg** $(N) = -1$**, rank two.** In this paper we will use the following notations. W is a smooth surface and C is a smooth curve of genus q on W. U is either the formal neighborhood of C in W or, if we work over $\mathbf{C}$, a small tubular neighborhood of C in W in the Euclidean topology. N is the normal bundle of C in W. Set $e := \deg(N)$. Let $\mathbf{I}$ be the ideal sheaf of C in U. For every integer $n \geq 0$ let $C^{(n)}$ be the nth infinitesimal neighborhood of C in U. Hence $C^{(n)}$ is the closed subscheme of U with $\mathbf{I}^{n+1}$ as ideal sheaf. In particular, $C^{(0)} = C$ and $C^{(n)}_{\mathrm{red}} = C$ for every $n \geq 0$. Hence for every integer $n \geq 0$ we have the following exact sequences

$$0 \longrightarrow \mathbf{I}^n/\mathbf{I}^{n+1} \longrightarrow \mathbf{O}_U/\mathbf{I}^{n+1} \longrightarrow \mathbf{O}_U/\mathbf{I}^n \longrightarrow 0. \tag{1}$$

In this section we will study the stratification of "admissible types" for rank 2 vector bundles in a formal neighborhood, or a Euclidean tubular neighborhood if we are in the complex analytic category, of an exceptional curve of the first kind. We assume that W contains an exceptional divisor C, i.e., a smooth curve $C \cong \mathbf{P}^1$ with $\mathbf{O}_C(-1)$ as a normal bundle.

Remark 1.1. Let E be a vector bundle on U. It is known and follows easily from the exact sequences (1.1) that if $E \mid C$ is trivial, then E is trivial.

Let E be a rank two vector bundle on U and (a, b) the splitting type of $U \mid C$, i.e., let a, b be the integers with $a \geq b$ and such that $E \mid C \cong \mathbf{O}_C(a) \oplus \mathbf{O}_C(b)$. In the paper [**1**] there was associated to E an integer $t \geq 1$, a finite sequence $\{(a_i, b_i)\}_{1 \leq i \leq t}$ of pairs of integers with $a_1 = a$, $b_1 = b$, $a_t = b_t = (a+b+t-1)/2$ and a finite number of bundles E_i, $1 \leq i \leq t$, with $E_1 = E$, $E_i \mid C$ with splitting type (a_i, b_i) in the following way. Fix a line bundle R on C and see R as a coherent torsion sheaf on U and on W supported on C. Fix a surjection $\mathbf{r} : E \to R$ induced by a surjection $\rho : E \mid C \to \mathcal{R}$; in our situation $\mathrm{Pic}(U) \cong \mathrm{Pic}(C)$ (Lemma 5.1) but for the map $\mathbf{r}$ we view R a sheaf supported on C, not as a vector bundle on U. There exists such a surjection if and only if $\deg(R) \geq b$. If $\deg(R) = b < a$, then ρ is unique, up to a multiplicative constant. Hence if $\deg(R) = b < a$ the sheaf $\mathrm{Ker}(\mathbf{r})$ is uniquely determined, up to an isomorphism. Since C is a Cartier divisor, the sheaf $\mathrm{Ker}(\mathbf{r})$ is a vector bundle on U. We will say that $\mathrm{Ker}(\mathbf{r})$ is the bundle obtained from E making the negative elementary transformation induced by $\mathbf{r}$. Note that $\mathrm{Ker}(\rho)$ is a line bundle on C with $\deg(\mathrm{Ker}(\rho)) = a + b - \deg(R)$. Since $\deg(\mathbf{I}/\mathbf{I}^2) = 1$ it is easy to check that $\deg(\mathrm{Ker}(\mathbf{r}) \mid C) = a + b + 1$ and that we have

an exact sequence on C:
(2)
$$0 \longrightarrow \mathbf{O}_C(a+b+1-\deg(\mathrm{Ker}(\rho))) \longrightarrow \mathrm{Ker}(\mathbf{r}) \mid C \longrightarrow \mathrm{Ker}(\rho) \longrightarrow 0.$$

Furthermore, using (2), we obtain a surjection $\mathbf{t} : \mathrm{Ker}(\mathbf{r}) \to \mathrm{Ker}(\rho)$ such that $\mathrm{Ker}(\mathbf{t}) \cong E(-C)$. In particular, $\mathrm{Ker}(\mathbf{t}) \mid C \cong \mathbf{O}_C(a+1) \oplus \mathbf{O}_C(b+1)$. Thus, up to the twist by the line bundle $\mathbf{O}_U(-C)$, the negative elementary transformation induced by $\mathbf{r}$ has an inverse operation and we will say that E is obtained from $\mathrm{Ker}(\mathbf{r})$ making a positive elementary transformation supported by C. Note that if $\deg(R) = b$, then $\mathrm{Ker}(\mathbf{r}) \mid C$ fits in an exact sequence

(3) $$0 \longrightarrow \mathbf{O}_C(b+1) \to \mathrm{Ker}(\mathbf{r}) \mid C \longrightarrow \mathbf{O}_C(a) \longrightarrow 0.$$

Hence if $b < a$ then $\mathrm{Ker}(\mathbf{r}) \mid C$ is more balanced than $E \mid C$. If $b \le a-3$, then (3) does not determine uniquely $\mathrm{Ker}(\mathbf{r}) \mid C$. If $b \le a-2$ and $\mathrm{Ker}(\mathbf{r}) \mid C$ is not balanced, we iterate the construction starting from $\mathrm{Ker}(\mathbf{r})$ and taking as R' the lowest degree factor of $\mathrm{Ker}(\mathbf{r}) \mid C$ and the unique surjection, up to a multiplicative constant, $\rho' : \mathrm{Ker}(\mathbf{r}) \to R'$. In a finite number of steps, say $t-1$ steps, we send E into a bundle which, up to a twist by $\mathbf{O}_U(-((a+b+t-1)/2)C)$, has trivial restriction to C and hence into a bundle isomorphic to $\mathbf{O}_U(-(a+b+t-1)/2)C)^{\oplus 2}$ by Remark 1.1. Set $E_1 := E$, $a_1 := a$ and $b_1 := b$. If $a_1 = b_1$, set $t := 1$ and stop. Assume $a_1 > b_1$. Hence we have defined the bundle $\mathrm{Ker}(\mathbf{r})$. Set $E_2 := \mathrm{Ker}(\mathbf{r})$. Let (a_2, b_2) be the splitting type of $\mathrm{Ker}(\mathbf{r}) \mid C$. Note that $a_2+b_2 = a_1+b_1+1$ and $b_1 < b_2 \le a_2 \le a_1$. Hence $a_2-b_2 < a_1-b_1$. If $a_2 = b_2$, set $t := 2$ and stop. If $a_2 > b_2$ iterate the construction. In a finite number of steps, say $t-1$ steps, we arrive at a bundle E_t with splitting type (a_t, b_t) with $a_t = b_t$. Call E_i, $2 \le i \le t$, the bundle we obtained after $i-1$ steps and (a_i, b_i) the splitting type of $E_i \mid C$. Call $\{E_i\}_{1 \le i \le t}$ the associated sequence of bundles of E. Note that the finite sequence $\{(a_i, b_i)\}_{1 \le i \le t}$ of pairs obtained in this way has the following properties: $a_i \ge b_i$ for every $i > 0$, $a_i + b_i = a_1 + b_1 + i - 1$ for every $i > 1$, $a_i \ge a_{i+1} \ge b_{i+1} > b_i$ for every $i \ge 1$, $a_t = b_t$. We will call "admissible" any such finite sequence of pairs of integers. We will say that an admissible sequence $\{(a_i, b_i)\}_{1 \le i \le t}$ is the admissible sequence associated to the bundle E if this sequence is created by the algorithm just described. If $\{A_\alpha\}_{\alpha \in T}$ and $\{A'_\beta\}_{\beta \in T'}$ are flat families of bundles on W with the same Chern classes and ranks, it is natural to ask if the first family is in the closure of the second one (in a suitable moduli space or in a local deformation functor or in some other sense). For us, without any stability or simpleness assumption, this would be equivalent to the existence of a flat family $\{A''_x\}_{x\in S}$ with S an integral variety, U' an open subset of S, V a closed subset of S, and dominant maps $T \to V$, $T' \to U'$ such that the families $\{A_\alpha\}_{i\in\Gamma}$ and $\{A'_\beta\}_{\beta\in T'}$ are as pull-backs from the family $\{A''_x\}_{x\in S}$. This is a very naive definition, but at least it gives that the usual semi-continuity statements for all cohomological properties of the "generic" bundle of the two given families are as expected. For bundles on U instead of c_1 we fix $a_1 + b_1$ and then we may use the same naive definition. The aim of this section is the study of this set-theoretic stratification by admissible type of the set of rank 2 vector bundles on U. We will use a key idea contained in [**3**], i.e., the use of explicit transition matrices for bundles on U as in the proof of Lemma 5.7.

Theorem 1.2. *Let W be either a smooth two-dimensional algebraic variety or a smooth two-dimensional complex manifold. Let $C \subset W$ be an exceptional $\mathbf{P}^1$ in W, i.e., $C \cong \mathbf{P}^1$ with normal bundle of degree -1. Let U be the formal completion of W along C. Fix an admissible sequence $\{(a_i, b_i)\}_{1\le i\le t}$ and let E, F be rank 2 vector bundles on U with $\{(a_i, b_i)\}_{1\le i\le t}$ as an associated admissible sequence. Then there exist a flat family $\{E_s\}_{s\in T}$ of rank 2 vector bundles on U parametrized by an integral variety T and $o, o' \in T$ with $E_o \cong E$ and $E_{o'} \cong F$ and such that for every $s \in T$ the bundle E_s has $\{(a_i, b_i)\}_{1\le i\le t}$ as an associated admissible sequence.*

Proof. We use induction on t. If $t = 1$ the result is obvious because if $t = 1$ both E and F are trivial vector bundles by Remark 1.1. Assume $t > 1$ and the result is true for the integer $t-1$. Let E_2 (respectively F_2) be the second associated bundle of E (respectively F). Hence E_2 and F_2 have $\{(a_i, b_i)\}_{2\le i\le t}$ as an associated admissible sequence. By the inductive assumption there is a flat family $\{E'_s\}_{s\in S}$ of rank 2 vector bundles on U and m, $m' \in S$ with $E'_m \cong E_2$, $E'_{m'} \cong F_2$ and such that for every $s \in S$ the bundle E'_s has $\{(a_i, b_i)\}_{2\le i\le t}$ as an associated admissible sequence. By the construction of the sequences of associated bundles of E and F, we have exact sequences

$$0 \longrightarrow E \longrightarrow E_2(C) \longrightarrow \mathbf{O}_C(a_1 - 1) \longrightarrow 0 \tag{4}$$

$$0 \longrightarrow F \longrightarrow F_2(C) \longrightarrow \mathbf{O}_C(a_1 - 1) \longrightarrow 0. \tag{5}$$

For every bundle M on U with $\{(a_i,b_i)\}_{2\le i\le t}$ as an associated admissible sequence, the set of all surjections $M(C)\to \mathbf{O}_C(a_1-1)$ is parametrized by an integral nonempty variety whose dimension depends only on a_1,a_2 and $b_2:=a_1+b_1+1-a_2$. For any such surjection, $\mathbf{t}$, $\mathrm{Ker}\,(\mathbf{t})\mid C$ is an extension of $\mathbf{O}_C(b_1)$ by $\mathbf{O}_C(a_1)$. Since $a_1>b_1$ this extension splits and hence the bundle $\mathrm{Ker}\,(\mathbf{t})$ has $\{(a_i,b_i)\}_{1\le i\le t}$ as an associated admissible sequence. Varying M among the bundles E'_s, $s\in S$, the set of all such surjections are parametrized by an irreducible nonempty variety, T. Furthermore, the family of all such kernels is flat (e.g., use that it is flat its restriction to every infinitesimal neighborhood $C^{(n)}$ of C in W, $n>0$, because for any fixed ample line bundle H on $C^{(n)}$, the bundles in this family have by the exact sequences (1) the same Hilbert polynomials with respect to H). Hence, we conclude the proof. $\square$

1.3. We fix homogeneous coordinates z_0,z_1 on C and set $C_0:=\{x\in C: z_0\neq 0\}$, $C_1:=\{x\in C: z_1\neq 0\}$, $C'':=C_0\cap C_1$. Call U_0 (respectively U_1, respectively U'') the open formal subschemes of U with support C_0 (respectively C_1, respectively C''). Set $z:=z_1/z_0$ as a regular function on U'' and C''. We may take as W the vector bundle $\mathbf{V}(\mathbf{O}_C(1))$. Hence we will take as regular coordinates on $U''z$ and a "fiber coordinate" w such that $C^{(n)}$, $n\ge 0$, is given scheme-theoretically by the equation $w^{n+1}=0$. Note that every vector bundle on U is trivial on U_0 and U_1 (Lemma 5.7). Hence every rank 2 vector bundle E on U is given by an invertible matrix on U'' with coordinates z and w. Note that $\mathbf{O}_C(k)$ is given on C'' by the 1×1 transition matrix z^{-k}. Hence the same transition matrix on U'' defines the line bundle $\mathbf{O}_U(-kC)$. Assume that $E\mid C$ has splitting type (a,b) with $a>b$. It is very easy to check (see [**3**] or the proof of Lemma 5.2) that E has a transition matrix $\mathbf{a}:=a(i,j)_{1\le i\le 2,1\le j\le 2}$ with $a(1,1)=z^{-b}$, $a(2,2)=z^{-a}$, $a(2,1)=0$ and as $a(2,2)$ a polynomial in u,z of the form $a(2,2)(z,u)=z^{-a+1}(\sum_{0\le i\le a-b-2}\sum_{0\le k\le a-b-1-i}p_{ik}z^iu^k)$ with $p_{ik}\in\mathbf{K}$. Vice versa, any such matrix $\mathbf{a}$ defines a rank 2 vector bundle F on U such that $F\mid C$ is an extension of $\mathbf{O}_C(a)$ by $\mathbf{O}_C(b)$. This extension is uniquely determined by $a(2,2)(z,0)$. This extension splits if and only if $F\mid C$ has $\mathbf{O}_C(a)$ as a subbundle. Call $\mathbf{e}\in H^1(C,\mathrm{Hom}\,(\mathbf{O}_C(a),\mathbf{O}_C(b)))$ the extension class of $F\mid C$. For every integer m with $b<m\le a$, we have a cup-product bilinear map

$H^1(C, \mathrm{Hom}\,(\mathbf{O}_C(a), \mathbf{O}_C(b))) \times H^0(C, \mathbf{O}_C(a-m)) \to H^1(C, \mathbf{O}_C(b-m))$ and $H^0(C,(F \mid C)(-m))$ is the right annihilator of the extension class $\mathbf{e}$ for the cup-product bilinear map. The bundle $F \mid C$ has a subsheaf isomorphic to $\mathbf{O}_C(m)$ if and only if $H^0(C,(F \mid C)(-m)) \neq 0$. Writing $\mathbf{e}$ in the form $z^{-a+1}(q(z))$ with $q(z) = \sum_{1 \le i \le a-b-2} q_i z^i$ polynomial of degree $\le a-b-2$ and taking a similar identification for $H^1(C, \mathbf{O}_C(b-m))$ with the space of all polynomials of degree $\le m-b-2$, we obtain that $H^0(C,(F \mid C)(-m)) \neq 0$ with $b < m < a$ if and only if there is an integer $i \ge 2m-a-b-1$ with $q_i \neq 0$. Hence for every integer m with $(a+b)/2 < m < a$ the maximal degree of a subbundle of $F \mid C$ is m if and only if $q_{2m-a-b-1}$ is the first nonzero coefficient of $\mathbf{e}$. Furthermore, the extension of $\mathbf{O}_C(a)$ by $\mathbf{O}_C(b)$ giving $F \mid C$ splits if and only if $q_i = 0$ for every integer i with $1 \le i \le a-b-2$. Hence $E \mid C$ has splitting type (a,b) if and only if $p_{i0} = 0$ for every integer i with $1 \le i \le a-b-2$. Set $E_1 := E$ and $\mathbf{a}_1 := \mathbf{a}$, say $\mathbf{a}_1 := \{a_1(i,j)\}_{1 \le i \le 2, 1 \le j \le 2}$. Call $\mathbf{a}_2 = \{a_2(i,j)\}_{1 \le i \le 2, 1 \le j \le 2}$ the matrix giving the associated bundle E_2 of E as an extension of $\mathbf{O}_U(a)$ by $\mathbf{O}_U(b+1)$. We have $a_2(2,2)(z,u) = z^{-a+1}(\sum_{0 \le i \le a-b-3} \sum_{1 \le k \le a-b-1-i} p_{ik} z^i u^{k-1})$. Hence the splitting type of $E_2 \restriction C$, i.e., the pair (a_2, b_2) is associated to $\sum_{0 \le i \le a-b-3} \sum_{1 \le k \le a-b-1-i} p_{ik} z^i u^{k-1}$. If $b_2 < a_2$ to obtain the pair (a_3, b_3), it is sufficient to express E_2 as an extension of $\mathbf{O}_U(-a_2 C)$ by $\mathbf{O}_U(-b_2 C)$ and then iterate the computations just made. If it is possible to carry out this step in a very explicit way we could, inductively, read off from $a(2,2)$ the admissible sequence $\{(a_i,b_i)\}_{1 \le i \le t}$ of E. We are unable to carry out this step. To avoid this problem in the last part of the proof of Theorem 1.4 we will use a trick. However, the computations just made show that if E, F are rank 2 bundles with $E \mid C^{(1)} \cong F \mid C^{(1)}$, then $E_2 \mid C \cong F_2 \mid C$. Inductively we obtain that if E, F are rank 2 bundles with $E \mid C^{(n)} \cong F \mid C^{(n)}$ for some integer $n \ge 1$, then for all integers i with $2 \le i \le n$, we have $E_i \mid C^{(n-i+1)} \cong F_i \mid C^{(n-i+1)}$.

Theorem 1.4. *Fix admissible types $\{(a_i,b_i)\}_{1 \le i \le t}$ and $\{(c_i,d_i)\}_{1 \le i \le t'}$ with $a_1 + b_1 = c_1 + d_1$, $t' \ge t$ and $a_i \le c_i$ for every $i \le t$. Let E be a rank 2 vector bundle on U with admissible type $\{(c_i,d_i)\}_{1 \le i \le t'}$. Then there exist an integral variety T (or a reduced and irreducible complex space T) $o \in T$, and a flat family $\{E_x\}_{x \in T}$ of vector bundles on U parametrized by T with $E_o \cong E$ and E_x of admissible type*

$\{(a_i, b_i)\}_{1\leq i\leq t}$ *for every* $x \in (T\backslash\{o\})$.

Proof. The following partial order on the set of all types for rank two vector bundles on U was introduced at the end of [**1**]. Take two admissible sequences $\{(a_i, b_i)\}_{1\leq i\leq t}$ and $\{(c_i, d_i)\}_{1\leq i\leq t'}$. We will write $\{(a_i, b_i)\}_{1\leq i\leq t} \geq \{(c_i, d_i)\}_{1\leq i\leq t'}$ if $a_1 + b_1 = c_1 + d_1$ and for all integers i with $1 \leq i \leq \min\{t, t'\}$ we have $a_i \leq c_i$. Note that if $\{(a_i, b_i)\}_{1\leq i\leq t} \geq \{(c_i, d_i)\}_{1\leq i\leq t'}$ we have $t \leq t'$. This partial order is generated by a "proximity relation." We will say that $\{(a_i, b_i)\}_{1\leq i\leq t}$ is larger and proximal to $\{(c_i, d_i)\}_{1\leq i\leq t'}$ if $a_1 + b_1 = c_1 + d_1$ and either $t' = t$ and there is an integer $s < t$ with $a_i = c_i$ for $i \neq s$, $a_s = c_s - 1$, or $t' = t + 2$, $a_i = c_i$ for $i < t$ and $c_t = d_t + 2$, i.e., $c_t = a_t + 1$. A deformation of a deformation is a deformation (even for formal schemes); for formal schemes taking $C^{(n)}$ for some fixed large n (e.g., $n = a_1 - b_1$) and using Lemmas 5.7 and 1.8, we may reduce this assertion to the case of proper schemes or analytic space. Hence we reduce to the case in which $\{(a_i, b_i)\}_{1\leq i\leq t}$ and $\{(c_i, d_i)\}_{1\leq i\leq t'}$ are proximal. Note that for every $i \leq t$ we have $a_i + b_i = a_1 + b_1 + i - 1 = c_1 + d_1 + i - 1 = c_i + d_i$. We use induction on the integer t', the case $t' \leq 2$ being trivial. Assume $t' > 2$. First we consider the case $a_1 = c_1$. By the inductive assumption we may find an integral variety S, a flat family $\{H_j\}_{j\in S}$ of rank 2 vector bundles on U parametrized by S and $m \in S$ with $H_m \cong E_2$ and with H_y of type $\{(a_i, b_i)\}_{2\leq i\leq t}$ for every $y \in S\backslash\{m\}$. Recall that $E \cong \mathrm{Ker}\,(\mathbf{u})$, where $\mathbf{u} : E_2(-C) \to \mathbf{O}_C(a)$ is a surjection. Since $d_2 \leq a_2 \leq a_1$, for every integer $n \geq 0$ and every $y \in T$ we have $h^0(C^{(n)}, \mathrm{Hom}\,(H_y(-C) \mid C^{(n)}, \mathbf{O}_C(a_1))) = h^0(C, \mathrm{Hom}\,(H_y(-C) \mid C, \mathbf{O}_C(a_1))) = h^0(C, \mathrm{Hom}\,(E_2(-C) \mid C, \mathbf{O}_C(a_1))) = h^0(C^{(n)}, \mathrm{Hom}\,(E_2(-C) \mid C^{(n)}, \mathbf{O}_C(a_1)))$. Hence by the Formal Function Theorem [**6**], we may extend $\mathbf{u}$ to a family of surjections $\mathbf{u}_y : H_y \to \mathbf{O}_C(a)$ with y near m. Since $b_i \leq d_i \leq a_1$ for every integer $n \geq 0$ and for every $y \in S$ we have $h^1(C, \mathrm{Hom}\,(H_y(-C) \mid C^{(n)}, \mathbf{O}_C(a_1)) \otimes \mathbf{I}^n/\mathbf{I}^{n+1}) = 0$. Hence, by the Formal Function Theorem and the exact sequences (1) for every $y \in S$ the restriction map $H^0(U, \mathrm{Hom}\,(H_y(-C), \mathbf{O}_C(a_1))) \to H^0(C, \mathrm{Hom}\,(H_y(-C) \mid C, \mathbf{O}_C(a_1)))$ is surjective. Note that the elementary transformation of each H_y induced by each such surjection has splitting type (a_1, b_1) on C. Since $a_1 = c_1$, we conclude in this subcase. Now we assume $a_1 \neq c_1$ and hence since the two admissible sequences are proximal $t' = t$ and $a_i = c_i$ for $2 \leq i \leq t$. We fix the bundle E_2 and we express it as an extension of $\mathbf{O}_U(-a_1C)$ by $\mathbf{O}_U(-b_1C)$ (Lemma 5.2). We use an explicit matrix of transition functions $\mathbf{a}_2 := a_2(i,j)\}_{1\le i\le 2, 1\le j\le 2}$. In (1.3) we listed all possible matrices of transition functions with E_2 as the associated bundle. Any such matrix is uniquely determined by a polynomial $\sum_{1\le i\le a-b-2} q_i z^i$ with $q_i \in \mathbf{K}$. Since $E \mid C$ has splitting type (a_1, b_1), the associated polynomial of E has $q_i = 0$ for every i. Taking a polynomial λz^{a-b-2} with $\lambda \in \mathbf{K}\backslash\{0\}$ we obtain a bundle E_λ with $E_\lambda \mid C$ of splitting type $(a_1-1, b_1+1) = (a_2, b_2)$, see (2.3). Hence, sending λ to 0 we obtain the flat family parametrized by the affine line we were looking for. □

2. C **rational,** $\mathbf{deg}\,(N) = \mathbf{-1}$**.** In this section we study the stratification of "admissible types" for rank r vector bundles in a formal neighborhood, or a Euclidean tubular neighborhood if we are in the complex analytic category, of an exceptional curve of the first kind, extending the case $r = 2$ considered in [**1**] and studied again here in Section 1. We assume $C \cong \mathbf{P}^1$ and that $\mathbf{O}_C(-1)$ is the normal bundle of C in W. Fix an integer $r \ge 2$. Let E be a rank r vector bundle on U and call $a_1 \ge \cdots \ge a_r$ the splitting type of $E \mid C$. We will associate to E an integer $t \ge 1$, a finite sequence $\{(a(i,j)\}_{1\le i\le t, 1\le j\le r}$ of r-tuples of nonincreasing integers with $a(1,j) = a_j$ for $1 \le j \le r$, $\sum_{1\le j\le r} a(i,j) = \sum_{1\le i\le r} a(1,j) + i - 1$ for $2 \le i \le t$, $a(t,j) = a(t,1)$ for $2 \le i \le r$, and a finite number of bundles E_i, $1 \le i \le t$, with $E_1 = E$, $E_i \mid C$ with splitting type $a(i,1) \ge \cdots \ge a(i,r)$ in the following way. Set $E_1 := E$ and $a(1,j) = a_j$ for $1 \le j \le t$. If $a(1,1) = a(1,r)$, set $t := 1$ and stop. Assume $a(1,1) > a(1,r)$. Choose a surjection $\rho : E \mid C \to \mathbf{O}_C(a_r)$ and make the corresponding elementary transformation $\mathbf{r} : E \to \mathbf{O}_C(a_r)$. Set $E_2 := \mathrm{Ker}\,(\mathbf{r})$. Since $a_r \le a_{r-1}$, we have $\mathrm{Ker}\,(\rho) \cong \oplus_{1\le i\le r-1}\mathbf{O}_C(a_i)$ and $\mathrm{Ker}\,(\mathbf{r}) \mid C$ fits in the exact sequence

$$(6) \qquad 0 \longrightarrow \mathbf{O}_C(a_r+1) \longrightarrow \mathrm{Ker}\,(\mathbf{r}) \mid C \longrightarrow \bigoplus_{1\le i\le r-1} \mathbf{O}_C(a_i) \longrightarrow 0.$$

Call $a(2,1) \ge \cdots \ge a(2,r)$ the splitting type of E_2. In particular, $\sum_{1\le j\le r} a(i,j) = \sum_{1\le i\le r} a(1,j) + 1$. Note that if $i < t$ the bundle E_2 is more balanced than the bundle E_1 in the following sense. We have $a(1,1) \ge a(2,1)$, $a(1,r) \le a(2,r)$, the number of integers j with $a(2,j) = a(1,r)$ is exactly one less than the number of integers m with

$a(1,m) = a(1,r)$ and the bundle $\mathrm{Ker}\,(\mathbf{r}) \mid D$ is more balanced (in the sense of the Harder-Narasimhan polygon of bundles of fixed degree) than the bundle $\oplus_{1\le i\le r-1}\mathbf{O}_C(a_i) \oplus \mathbf{O}_C(a_r+1)$. If $a(2,1) = a(2,r)$, set $t = 2$ and stop. Otherwise, we repeat the construction using E_2 instead of E_1. In a finite number of steps, say $t-1$ steps, we arrive at a balanced bundle $E_t := \oplus_{1\le i\le r}\mathbf{O}_C(a(t,i))$ with $a(t,i) = a(t,1)$ for every i. Note that this relation gives a strong restriction on the integer t because we have $\sum_{1\le i\le r} a(1,j) + t - 1 \equiv 0$ modulo r. This construction stops after at most $(r-1)a_1 - \sum_{2\le i\le r} a_i$ steps and hence $1 \le t \le 1 + (r-1)a_1 - \sum_{2\le i\le r} a_i$. Note that if $i < t$ the bundle E_{i+1} is more balanced than the bundle E_i. If W is complete or (in the analytic case) compact and $E \cong A \mid U$ with A the vector bundle on W, we claim that the admissible splitting type of E determines uniquely the so-called drop of c_2 of A related to C, i.e., the integer $d_2(A)$ defined in [**1**]. Indeed, the associated bundles E_i are of the form $U_i \mid C$ with U_i bundles on W. If $t = 1$, we have $d_2(A) = r(r-1)a_1/2$. If $t > 1$ we leave to the interested reader the task of using the $t-1$ exact sequences defining the bundles E_i, $1 \le i \le t$, and the relation $d_2(E_t) = r(r-1)a(t,1)$ to obtain $d_2(A_{t-1})$ and then $d_2(A_{t-2})$ and so on, mimicking the proof of [**1**, Theorem 0.3].

3. C **rational,** $\mathbf{deg}\,(N) = -\mathbf{2}$. In this section we consider the case of a smooth curve $C \cong \mathbf{P}^1$ with normal bundle of degree -2. By assumption $\mathbf{O}_C(-2)$ is the normal bundle of C in W. Fix an integer $r \ge 2$. Let E be a rank r vector bundle on U, and let $a_1 \ge \cdots \ge a_r$ be the splitting type of $E \mid C$. Now we will associate to E an integer $t \ge 1$, a finite sequence $\{(a(i,j)\}_{1\le i\le t, 1\le j\le r}$ of r-tuples of nonincreasing integers with $a(1,j) = a_j$ for $1 \le i \le j$, $\sum_{1\le j\le r} a(i,j) = \sum_{1\le i\le r} a(1,j) + 2i - 2$ for $2 \le i \le t$, $a(t,r) \le a(t,1)+1$ for $2 \le i \le r$, and a finite number of bundles E_i, $1 \le i \le t$, with $E_1 = E$, $E_i \mid C$ with splitting type $a(i,1) \ge \cdots \ge a(i,r)$. Set $E_1 := E$ and $a(1,j) = a_j$ for $1 \le j \le t$. If $a(1,1) \le a(1,r)+1$, set $t := 1$ and stop. Assume $a(1,1) \ge a(1,r)+2$. Choose a surjection $\rho : E \mid C \to \mathbf{O}_C(a_r)$ and make the corresponding elementary transformation $\mathbf{r} : E \to \mathbf{O}_C(a_r)$. Set $E_2 := \mathrm{Ker}\,(\mathbf{r})$. Since $a_r \le a_{r-1}$, we have $\mathrm{Ker}\,(\rho) \cong \oplus_{1\le i\le r-1}\mathbf{O}_C(a_i)$ and $\mathrm{Ker}\,(\mathbf{r}) \mid C$ fits in the exact sequence

$$(7) \qquad 0 \longrightarrow \mathbf{O}_C(a_r+2) \longrightarrow \mathrm{Ker}\,(\mathbf{r}) \mid C \longrightarrow \bigoplus_{1\le i\le r-1} \mathbf{O}_C(a_i) \longrightarrow 0.$$

Call $a(2,1) \ge \cdots \ge a(2,r)$ the splitting type of E_2. In particular, $\sum_{1\le j\le r} a(i,j) = \sum_{1\le i\le r} a(1,j)+2$. Note that if $i<t$ the bundle E_2 is more balanced than the bundle E_1 in the following sense. We have $a(1,1) \ge a(2,1)$, $a(1,r) \le a(2,r)$, the number of integers j with $a(2,j) \le a(1,r)+1$ is exactly one less than the number of integers m with $a(1,m) \le a(1,r)+1$ and the bundle $\mathrm{Ker}\,(\mathbf{r}) \mid C$ is more balanced (in the sense of the Harder-Narasimhan polygon of bundles of fixed degree) than the bundle $\oplus_{1\le i\le r-1}\mathbf{O}_C(a_i)\oplus\mathbf{O}_C(a_r+2)$. Furthermore, we have $h^1(C,\mathrm{End}\,(E_2 \mid C)) < h^1(C,\mathrm{End}\,(E_1 \mid C))$ and $h^0(C,\mathrm{End}\,(E_2 \mid C)) < h^0(C,\mathrm{End}\,(E_1 \mid C))$. If $a(2,1) \le a(2,r)+1$, set $t=2$ and stop. Otherwise, we repeat the construction using E_2 instead of E_1. In a finite number of steps, say $t-1$ steps, we arrive at a bundle $E_t := \oplus_{1\le i\le r}\mathbf{O}_C(a(t,i))$ with $a(t,1) \le a(t,i) \le a(t,1)+1$ for every i. Note that if $r=2$ this is equivalent to the fact that the bundle $E_t \mid C$ is rigid. This construction stops after at most $((r-1)a_1 - \sum_{2\le i\le r} a_i)/2$ steps and hence $1 \le t \le 1 + ((r-1)a_1 - \sum_{2\le i\le r} a_i)/2$. Note that if $i<t$ the bundle $E_{i+1} \mid C$ is more balanced than the bundle $E_i \mid C$, that $h^1(C,\mathrm{End}\,(E_{i+1} \mid C)) < h^1(C,\mathrm{End}\,(E_i \mid C))$ and $h^0(C,\mathrm{End}\,(E_{i+1} \mid C)) < h^0(C,\mathrm{End}\,(E_i \mid C))$ and that no bundle E_i with $i<t$ is rigid. Of course, a similar construction may be done for every smooth rational curve C embedded with normal bundle $\mathbf{O}_C(e)$ with $e \le -3$, just considering the condition "$a_1 - a_r \le -e$" instead of the condition "$a_1 = a_r \le -2$," but the final bundle, E_t, we obtain for $e \le -3$ seems not to have any geometric property.

4. C **arbitrary, rank two.** In this section we consider the case of rank 2 vector bundles when the curve C has arbitrary genus, q. Let N be the normal bundle of C in W (or U). Set $e := \deg\,(N)$.

Remark 4.1. If $e > 4q+5$ by [**4**] U is uniquely determined by C and N and in the algebraic case even a Zariski open neighborhood of C in W is uniquely determined by C and N and is isomorphic to a Zariski open neighborhood of the zero-section of the vector bundle $\mathbf{V}(N^*)$ on C.

Let E be a rank 2 vector bundle on U. We distinguish several different cases according to the triple of integers $(q, e, s(E))$ where $s(E)$ is the stability degree of the bundle E in the sense of [**8**], which will be described below. At the end of the paper we will give a more precise description when $e = -1$. We assume that $E \mid C$ fits in an exact sequence

$$0 \longrightarrow A \longrightarrow E \mid C \longrightarrow B \longrightarrow 0 \tag{8}$$

with A and B line bundles on C. We fix $R \in \mathrm{Pic}\,(C)$ and a surjection $\rho : E \mid C \to R$. Composing the restriction map $E \to E \mid C$ with the surjection ρ we obtain a surjection $\mathbf{r} : E \to R$, where R is seen as a torsion coherent sheaf on U. Since C is a Cartier divisor in U, $\mathrm{Ker}\,(\mathbf{r})$ is a rank 2 vector bundle on U (see, e.g., [**9**] or [**10**]). The vector bundle $\mathrm{Ker}\,(\mathbf{r})$ is said to be obtained from E making a negative elementary transformation supported by C. We have $\det(\mathrm{Ker}\,(\mathbf{r}) \mid C) \cong \det(E \mid C) \otimes N^*$ and $\mathrm{Ker}\,(\mathbf{r}) \mid C$ fits in the following exact sequence on C:

$$0 \longrightarrow R \otimes N^* \longrightarrow \mathrm{Ker}\,(\mathbf{r}) \mid C \longrightarrow \det(E \mid C) \otimes R^* \longrightarrow 0. \tag{9}$$

Now we distinguish several cases and subcases according to the stability or instability of $E \mid C$ (and its order) the degree of R and the value e. If E fits in an exact sequence (8) with $\deg(A)$ maximal, we will set $s(E) := \deg(B) - \deg(A)$ and call $s(E)$ the degree of stability of E. By definition of stability and semi-stability E is stable if and only if $s(E) > 0$, E is semi-stable but not stable if and only if $s(E) = 0$, while E is called unstable if and only if $s(E) < 0$.

Case 1. Assume that $E \mid C$ is stable, i.e., $s(E \mid C) > 0$. In particular, we have $\deg(B) > \deg(A)$. We assume that A is a subbundle of $E \mid C$ of maximal degree. Set $s := \deg(B) - \deg(A) = s(E)$. Note that $s \equiv \deg(E \mid C)$ modulo 2. The variety of rank 2 stable vector bundles on C with fixed determinant was studied in detail in [**8**]. By an old theorem of Segre and Nagata we have $0 < s \leq q$ if $q \geq 2$. This case does not occur for $q = 0$, while if $q = 1$ we have $s = 1$. Since we assume that (9) does not split, we have $\deg(R) \geq \deg(B)$. We distinguish two subcases according to the value of $\deg(R)$.

Subcase 1.1. Assume $\deg(R) > \deg(B)$. In this case there are surjections ρ if and only if $H^0(C, \mathrm{Hom}\,(A, R)) \neq 0$, $H^0(C, \mathrm{Hom}\,(B, R)) \neq 0$

and the base locus of $|R \otimes A^*|$ does not intersect the base locus of $|R \otimes B^*|$. For instance this condition is always satisfied if $\deg(R) \geq 2q + \deg(A)$ because $0 \leq s \leq q$. If (A, B, R) are a general triple of $\mathrm{Pic}^{\deg(A)}(C) \times \mathrm{Pic}^{\deg(B)}(C) \times \mathrm{Pic}^{\deg(R)}(C)$, then this condition is satisfied if and only if $\deg(R) \geq \deg(B) + q$. The exact sequence (9) shows that $\mathrm{Ker}(\mathbf{r}) \mid C$ is an extension of a line bundle of degree $2\deg(A) + s - \deg(R)$ by a line bundle of degree $\deg(R) - e$. Note that $2\deg(A) + s + e < 2\deg(R) + s$ for $e \leq 0$ and in almost all interesting cases with $e > 0$ (see Remark 4.1). Hence, if $e \leq 0$ and in many other cases $\mathrm{Ker}(\mathbf{r}) \mid C$ is "less stable" than $E \mid C$. If $2\deg(A) + s + e < 2\deg(R)$, then $\mathrm{Ker}(\mathbf{r}) \mid C$ is unstable and (9) is a destabilizing exact sequence for $\mathrm{Ker}(\mathbf{r}) \mid C$, i.e., its Harder-Narasimhan filtration.

Subcase 1.2. Here we assume $\deg(R) = \deg(B)$. For many bundles $E \mid C$ the exact sequence (9) with $\deg(A)$ maximal is not uniquely determined (see [**8**]) and for such bundles the isomorphism class of R is not uniquely determined by the existence of the surjection ρ. If $e - s < 0$ the bundle $\mathrm{Ker}(\mathbf{r}) \mid C$ is always unstable.

Case 2. Here we assume that $E \mid C$ is semi-stable but not stable, i.e., that $s(E \mid C) = 0$. We take as exact sequence (8) a sequence with $\deg(A) = \deg(B)$, i.e., with $\deg(A)$ maximal. Such a sequence is unique if $E \mid C$ is indecomposable. We repeat the discussion of Case 1 taking just $s := 0$ and obtain the two subcases 2.1) (with $\deg(R) > \deg(A)$) and 2.2) (with $\deg(R) = \deg(A)$).

Case 3. Here we assume that $E \mid C$ is properly unstable, i.e., that $s(E \mid C) < 0$. We take as exact sequence (8) the Harder-Narasimhan filtration of $E \mid C$ and set $s := \deg(B) - \deg(A)$. Hence $s < 0$ and for some bundle $E \mid C$ every negative value of s may occur, but if $s < -2q + 2$ we have $E \mid C \cong A \oplus B$. To have a surjection ρ we need $\deg(R) \geq \deg(B)$. Again we distinguish two subcases.

Subcase 3.1. Here we assume $\deg(R) > \deg(B)$. Note that the existence of the surjection ρ implies $\deg(R) \geq \deg(A)$. $\mathrm{Ker}(\mathbf{r}) \mid C$ is an extension of a line bundle of degree $\deg(A) + \deg(B) - \deg(R)$ by a line bundle of degree $\deg(R) + e$ and, hence, unless e is very negative, $\operatorname{Ker}(\mathbf{r}) \mid C$ is unstable and (9) is the Harder-Narasimhan filtration of $\operatorname{Ker}(\mathbf{r}) \mid C$.

Subcase 3.2. Here we assume $\deg(R) = \deg(B)$. Hence $R \cong B$ and, up to a multiplicative constant, ρ is unique. $\operatorname{Ker}(\mathbf{r}) \mid C$ is an extension of a line bundle of degree $\deg(A) = \deg(B) - s$ by a line bundle of degree $\deg(B) - e$.

From now on in this section we assume $e = -1$ and show an "algorithm" to transform any bundle E into a semi-stable but not stable bundle E_t, i.e., into a bundle E_t with associated integer $s(E_t) = 0$. Since the case $q = 0$ was done in [**1**], we may assume $q \geq 1$. We will define an integer $t \geq 1$, a family of bundles E_i, $1 \leq i \leq t$, with degree of stability $s_i := s(E_i) \in \mathbf{Z}$, such that $E_1 = E$, $s_1 = s(E)$, $t = 1$ if and only if E is semi-stable but $\deg(E_{i+1}) = \deg(E_i) + 1$ if $s_t = 0$ (i.e., E_t is semi-stable but not stable) and such that $s_i < 0$ for $2 \leq i < t$, $s_{i+1} > s_i$ for $2 \leq i < t$, $s_2 > s_1$ if $s_1 < 0$, $s_2 = -s_1 + 1$ if $s_1 > 0$. Set $E_1 = E$ and $s_1 := s(E)$. If E is semi-stable but not stable, set $t = 1$ and stop. Now assume E is stable. We apply the construction of subcase 1.2. Now assume E is unstable and apply the construction of subcase 3.2. Both in the stable and unstable case set $E_2 := \operatorname{Ker}(\mathbf{r})$ and $s_2 := s(E_2)$. Note that E_2 is unstable if E is stable, while E_2 is semi-stable or unstable if E is unstable and in this case we have $s_2 > s_1$. If E_2 is not semi-stable, we continue taking E_2 instead of E. In a finite number of steps, say $t - 1$ steps, we arrive at a semi-stable bundle E_t and stop. Note that if E is stable we have $3 \leq t \leq s(E)$, while if E is unstable we have $2 \leq t \leq -s(E) + 1$.

5. Technical lemmas. In this section we collect several technical results which were used in the previous sections.

Lemma 5.1. *The restriction map* $\operatorname{Pic}(U) \to \operatorname{Pic}(C)$ *is surjective.*

Proof. Since $\dim(C) = 1$, we have $h^2(C, \mathbf{I}^n/\mathbf{I}^{n+1}) = 0$ for every $n \geq 0$. To prove the result even for singular and/or reducible C it is sufficient to copy [**5**, p. 179]. If C is smooth we may also give the following proof. Since C is smooth it is sufficient to show that, for every $P \in C$ there is a Cartier divisor $Z(P)$ on U with $Z(P) \mid C = \{P\}$ (as Cartier divisors). This is obvious in several cases. In the analytic set-up this is obvious not just for formal neighborhoods but also for "good" (i.e., locally trivial in the transcendental topology) tubular neighborhoods. The existence of "good" tubular neighborhoods follows from the smoothness of C. To obtain an extension to tubular neighborhoods in the analytic case for singular curves, C', just use that every line bundle on C' has a meromorphic section with zeros and poles on C_{reg} and use a fundamental system of neighborhoods of C' which are locally trivial at every point of C'_{reg}. $\square$

For related results, see Lemmas 5.9 and 5.10.

Lemma 5.2. *Assume $C \cong \mathbf{P}^1$ and $e := \deg(N) < 0$. Let E be a rank r vector bundle on U. Then there is an increasing filtration $\{E_i\}_{0\le i\le r}$ of E with E_i a saturated subbundle of E, $\mathrm{rank}\,(E_i) = i$, $E_0 = \{0\}$, $E_r = E$ and $E_{i+1}/E_i \in \mathrm{Pic}\,(U)$ for $0 \le i < r$. Let $a_1 \ge \cdots \ge a_r$ be the splitting type of $E \mid C$. Fix integers $b_1, \dots, b_r$ such that $\sum_{1\le i\le r} b_i = \sum_{1\le i\le r} a_i$ and such that for every integer u with $1 < u \le r$ we have $\sum_{u\le i\le r} b_i \le \sum_{u\le i\le r} a_i$. Then there exists such a filtration $\{E_i\}_{0\le i\le r}$ of E with $\deg(E_{i+1}/E_i \mid C) = b_{r-i}$ for every $i < r$, i.e., such that $E_{i+1}/E_i \cong \mathbf{O}_U(-b_{r-i}C)$ for every $i < r$.*

Proof. We will use induction on r. First we will show that E has a subbundle $E_1 \cong \mathbf{O}_U(-b_rC)$ with E/E_1 locally free. Set $F := E(b_rC)$. Since $F \mid C$ has splitting type $a_1 - b_r \ge \cdots \ge a_r - b_r \ge 0$, $F \mid C$ has a saturated rank 1 trivial subbundle. Tensoring by F the exact sequences (1.1) and using that $\deg(\mathbf{I}^n/\mathbf{I}^{n+1}) = -ne > 0$, we obtain $H^1(U, F) = 0$ and that the chosen section of $F \mid C$ lifts to a section of F. By Nakayama's lemma we obtain that the associated map $\mathbf{O}_U(-b_rC) \to E$ has a locally free quotient. This implies the lemma for $r = 2$. Assume $r \ge 3$. We use induction on r. The proof of the existence of the rank 1 subbundle E_1 shows that we may take as $(E/E_1) \mid C$ any rank $r-1$ bundle which is the quotient of $E \mid C$ by $\mathbf{O}_C(b_r)$. Since $\oplus_{1\le i<r}\mathbf{O}_C(b_i)$ is such a quotient, we conclude by the inductive assumption. $\square$

Since every vector bundle on a smooth projective curve is obtained by taking a sequence of $r-1$ extensions by line bundles, the proof of Lemma 5.2, the surjectivity of the map $\mathrm{Pic}\,(U) \to \mathrm{Pic}\,(C)$ (Lemma 5.1) and Riemann-Roch on C give the following result.

Lemma 5.3. *Assume C smooth of genus q and $e := \deg(N) < 0$. Let E be a rank r vector bundle on U. Then there is an increasing filtration $\{E_i\}_{0 \le i \le r}$ of E with E_i a saturated subbundle of E, $\deg(E_i) = i$, $E_0 = \{0\}$, $E_r = E$ and $E_{i+1}/E_i \in \mathrm{Pic}\,(U)$ for $0 \le i < r$. Assume that $E \mid C$ has an increasing filtration $\{F_i\}_{0 \le i \le r}$ with F_i a subbundle of $E \mid C$, $\mathrm{rank}\,(F_i) = i$, $F_0 = \{0\}$, $F_r = E \mid C$ and $F_{i+1}/F_i \in \mathrm{Pic}\,(C)$ for $0 \le i < r$; set $a_i := \deg(F_{i+1}/F_i)$ and fix integers $b_1, \dots, b_r$ with $\sum_{1 \le i \le r} b_i = \sum_{1 \le i \le r} a_i$ and such that for every integer u with $1 < u \le r$ we have $\sum_{u \le i \le r} b_i + 2q(r-u+1) \le \sum_{u \le i \le r} a_i$. Fix $L_i \in \mathrm{Pic}\,(C)$, $2 \le i \le r$, with $\deg(L_i) = b_i$ and $M_i \in \mathrm{Pic}\,(U)$ with $M_i \mid C \cong L_i$. Then there exists such a filtration $\{E_i\}_{0 \le i \le r}$ of E with $E_{i+1}/E_i \cong M_{r-i}$ for every $i < r$.*

Lemma 5.4. *Let Y be either an affine variety or a Stein analytic space and E, F and G vector bundles on Y. Assume $E \mid Y_{\mathrm{red}}$ trivial. Then E is trivial. Furthermore, every isomorphism $u : F \mid Y_{\mathrm{red}} \to G \mid Y_{\mathrm{red}}$ is the restriction of an isomorphism between F and G and every homomorphism $v : F \mid Y \to G \mid Y$ with $v \mid Y_{\mathrm{red}} = u$ is an isomorphism.*

Proof. Taking $E := F$ and $G := \mathbf{O}_Y^{\oplus r}$, $r := \mathrm{rank}\,(F)$ (or trivial bundles of perhaps different ranks in different connected components of Y) we reduce the first assertion to the second one. Let $u : F \mid Y_{\mathrm{red}} \to G \mid Y_{\mathrm{red}}$ be an isomorphism. By Cartan-Serre Theorem A the map u extends to a homomorphism $v : F \mid Y \to G \mid Y$. Since F and G are vector bundles, and for every $x \in Y_{\mathrm{red}}$ the map $u \mid \{x\}$ is an isomorphism, by Nakayama's lemma v is an isomorphism. □

Remark 5.5 By the proof of [**2**, Theorem 30] every complex analytic vector bundle over a one-dimensional complex Stein space is trivial (as a complex analytic vector bundle).

Remark 5.6. It is very easy and very well-known that every algebraic vector bundle over the affine line is trivial.

Lemma 5.7. *Let Y be either a one-dimensional projective scheme over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$ or a one-dimensional quasi-projective scheme with $Y_{\mathrm{red}} \cong \mathbf{P}^1$ or a compact one-dimensional complex space. Then every vector bundle on Y_{red} is the restriction of a vector bundle on Y.*

Proof. Let E be a vector bundle on Y_{red}. If Y is algebraic and $\mathrm{char}(\mathbf{K}) = 0$ by the Lefschetz principle, we reduce to the case $\mathbf{K} = \mathbf{C}$ and then by GAGA we reduce to proving the extension of an analytic vector bundle over the analytic space $(Y_{\mathrm{red}})_{\mathrm{an}}$ to an analytic vector bundle over the analytic space Y_{an}. In all cases by Remarks 5.5 and 5.6 there are two open sets U_1 and U_2 with $Y = U_1 \cup U_2$ and with $E \mid (U_1)_{\mathrm{red}}$ and $E \mid (U_2)_{\mathrm{red}}$ trivial. Hence E is defined by an invertible matrix of functions g_{12} on $(U_1)_{\mathrm{red}} \cap (U_2)_{\mathrm{red}}$ and there is no cocycle condition. Since $U_1 \cap U_2$ is Stein (or affine if we work in the algebraic set-up with $Y_{\mathrm{red}} \cong \mathbf{P}^1$) g extends to a matrix h_{12} of regular functions on $U_1 \cap U_2$. By Nakayama's lemma this matrix h_{12} is invertible and hence it gives a vector bundle on Y which extends E. $\square$

Lemma 5.8. *Let Y be either a one-dimensional projective scheme over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$ or a one-dimensional quasi-projective scheme with $Y_{\mathrm{red}} \cong \mathbf{P}^1$. Then every flat family $\{\mathbf{E}_s\}_{s \in T}$ of vector bundles on Y_{red} parametrized by a reduced scheme T, or a reduced complex space T, is the restriction of a flat family of vector bundles on Y parametrized by T.*

Proof. We follow the proof of Lemma 5.7 and obtain an "algebraic family" of vector bundles on Y parametrized by T. We need to check the flatness of this family. We fix an ample line bundle $\mathbf{O}_Y(1)$ on Y. Let $\mathbf{J}$ be the nilradical of $\mathbf{O}_Y$, i.e., the ideal sheaf of Y. Fix an integer $k > 0$ with $\mathbf{J}^k = 0$. For every integer $n > 0$ we have an exact sequence

$$0 \longrightarrow \mathbf{J}^n/\mathbf{J}^{n+1} \longrightarrow \mathbf{O}_Y/\mathbf{J}^{n+1} \longrightarrow \mathbf{O}_Y/\mathbf{J}^n \longrightarrow 0 \tag{10}$$

and we may tensor (10) by any product $M \otimes \mathbf{O}_Y(z)$ with M vector bundle on Y and $z \in \mathbf{Z}$. We obtain that the Hilbert polynomial of M

with respect to $\mathbf{O}_Y(1)$ depends only on the Hilbert polynomials of all tensor powers $(M \mid Y_{\mathrm{red}}) \oplus \mathbf{J}^n/\mathbf{J}^{n+1}$, $1 \leq n < k$. Hence all vector bundles appearing in our "algebraic family" have the same Hilbert polynomials. The flatness of our "algebraic family" follows from the proof of [**6**, Theorem III.9.9]. □

The definition of vector bundle on a formal space and the proof of Lemma 5.7 give the following two results.

Lemma 5.9. *Assume either* $\mathrm{char}\,(\mathbf{K}) = 0$ *or* $C \cong \mathbf{P}^1$. *Then every vector bundle on* C *is the restriction of a vector bundle on* U.

Lemma 5.10. *Assume either* $\mathrm{char}\,(\mathbf{K}) = 0$ *or* $C \cong \mathbf{P}^1$. *Then every flat family of vector bundles on* C *parametrized by an integral variety* (*or reduced and irreducibly complex space*) T *is the restriction to* C *of a flat family of vector bundles on* U *parametrized by* T.

Remark 5.11. Let A and E be vector bundles on U. Since $\dim(C) = 1$ the Formal Function Theorem [**6**, III.11.1] implies that $H^2(U, \mathbf{I} \otimes A) = 0$. In the analytic case in which U is a tubular neighborhood of C in W this is true because $\dim(U) = 2$ and U contains no compact two-dimensional component. Hence the restriction map $H^1(U, A) \to H^1(C, A \mid C)$ is surjective. Applying this remark to the case $A = \mathrm{End}\,(E)$ and using the vanishing of the obstruction spaces $H^2(C, \mathrm{End}\,(E)/\mathbf{I}^{t+1}) \cong H^2(C^{(t)}, \mathrm{End}\,(E \mid C^{(t)}))$ for every $t \geq 0$, we see that the local deformation space of E as bundle on U surjects onto the local deformation space of $E \mid C$ even if $E \mid C$ is not simple. The same proof works taking W instead of U if we assume that there is a contraction $\pi : W \to Z$ of C with $Z = \pi(W)$ and Z affine variety or, in the analytic case, a Stein space. There is a small problem to define the local deformation functor for E on U and on W because $H^0(U, \mathrm{End}\,(E))$ and $H^0(W, \mathrm{End}\,(E))$ have infinite dimension in the cases we are interested in, but in these cases, for a fixed bundle $E \mid C$, we may replace U or W with a high order infinitesimal neighborhood of C in U; here "surjectivity of the local deformation functors" just means existence of lifting to arbitrary high order infinitesimal neighborhoods of C in U of every flat deformation of $E \mid C$, as a bundle on C. In particular, the surjectivity of the local deformation functors gives that $d_2(B)$, the drop of c_2, defined in [**1**] for any vector bundle B on U (or on W if Z is affine or Stein) is not invariant under flat deformations.

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Department of Mathematics, University of Trento, 38050 Povo (TN), Italy
E-mail address: ballico@science.unitn.it

Departamento de Matematica Universidade Federal de Pernambuco, Cidade Universitaria, Recife, PE 50670-901, Brazil
E-mail address: gasparim@dmat.ufpe.br