ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 3, Fall 2000

ON FUNCTIONAL REPRESENTATION OF COMMUTATIVE LOCALLY A-CONVEX ALGEBRAS

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ABSTRACT. We shall give a Gelfand type of representation of commutative locally A-convex algebras by using a certain family of seminorms defined on the carrier space of this algebra. By using this representation we give a generalization of locally convex uniform algebras.

1. Introduction. Let (A, T) be a commutative algebra over the complex numbers equipped with a topology T. If A has unit element it will be denoted by e. In this paper we assume that the topology T on A has been given by means of a family $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ of seminorms on A. This topology will be denoted by $T(\mathcal{P})$. We assume that $T(\mathcal{P})$ is a Hausdorff topology (i.e., from the condition $p_\lambda(x) = 0$, $x \in A$, for all $\lambda \in \Lambda$ it follows that $x = 0$). Suppose further that $\mathcal{P}$ has the following property. If λ and $\mu \in \Lambda$ then $\max\{p_\lambda, p_\mu\} \in \mathcal{P}$, i.e., $\mathcal{P}$ is directed. This property is needed in some place, but it is not necessary in general. We shall say that $(A, T(\mathcal{P}))$ is a locally A-convex algebra if for each $x \in A$ and $\lambda \in \Lambda$ there is some constant $M_{(x,\lambda)} > 0$ (depending on x and λ) such that

$$(1) \qquad p_\lambda(xy) \le M_{(x,y)} p_\lambda(y) \quad \text{for all } y \in A.$$

If the above $M_{(x,\lambda)}$ does not depend on λ, i.e., (1) holds for all $\lambda \in \Lambda$ for some constant $M_x > 0$ depending only on x, then we say that $(A, T(\mathcal{P}))$ is a locally uniformly A-convex algebra. Furthermore, we say that $(A, T(\mathcal{P}))$ is locally m-convex if $p_\lambda(xy) \le p_\lambda(x)p_\lambda(y)$ for all x and $y \in A$ and $\lambda \in \Lambda$. Obviously a locally m-convex algebra is locally A-convex. Note that the multiplication in locally A-convex algebra is in general only separately continuous and in locally m-convex algebra jointly continuous. The concepts of A-convex and uniformly A-convex algebras were introduced in [**13**], [**14**] and [**15**]. See also [**9**], [**21**], [**22**], [**23**] and [**24**]. A standard example of uniformly locally A-convex algebra is an algebra of bounded continuous complex-valued functions

Received by the editors on November 21, 1998.

defined on some completely regular space equipped with so-called strict topology. See [**14**]. So we can say that in the theory of locally A-convex algebras we give an abstract form for the function algebras equipped with a certain type of weighted topology. Note that there are also noncommutative locally A-convex algebras.

For a topological space X denote by $C(X)$ (correspondingly $C_b(X)$) the set of all continuous (correspondingly bounded) complex valued functions on X. If X is locally compact we denote by $C_0(X)$ the set of all continuous complex-valued functions vanishing at infinity.

Let $\Delta(A, T(\mathcal{P}))$ be the set of all nontrivial continuous complex homomorphisms on A. Note that it may happen that the set $\Delta(A, T(\mathcal{P}))$ is empty even if A has unit. Let $(A, T(\mathcal{P}))$ be a locally A-convex algebra with a nonempty set of continuous complex homomorphisms. Let $x \in A$ be given. Then we can define a complex-valued function $\hat{x}$ on $\Delta(A, T(\mathcal{P}))$ by $\hat{x}(\tau) = \tau(x)$, $\tau \in \Delta(A, T(\mathcal{P}))$. We shall equip $\Delta(A, T(\mathcal{P}))$ with the weak topology generated by the functions $\hat{A} = \{\hat{x} \mid x \in A\}$. We shall call $\Delta(A, T(\mathcal{P}))$ with this topology the carrier space of $(A, T(\mathcal{P}))$. So we have $\hat{A} \subset C(\Delta(A, T(\mathcal{P}))$. Clearly the carrier space $\Delta(A, T(\mathcal{P}))$ is completely regular.

The history of functional representation (Gelfand representation) of locally m-convex algebras dates back to the papers of Arens [**3**] and Michael [**19**]. Since then this subject has been considered in many papers. See, for example, [**5**], [**10**], [**11**], [**17**], [**19**] and [**20**]. Since a locally A-convex algebra is a generalization of a locally m-convex algebra it is a natural problem to study the Gelfand representation of such algebras. For uniformly locally A-convex algebras can be given so-called Gelfand representation by using a type of weight functions defined on the corresponding carrier space of this algebra. This has been done in [**9**], [**14**] and [**24**]. For locally A-convex algebras a corresponding representation has been studied in [**23**]. In this paper we shall study this type of representation for locally A-convex algebras and by using this representation we shall define and study a type of locally A-convex algebra which appears to be a generalization of locally convex uniform algebra. As was mentioned earlier each locally m-convex algebra is locally A-convex, and for locally m-convex algebras the corresponding representation has been studied in [**5**] and [**10**] (without using weight functions). Note that also in a locally m-convex case the use of weight functions sometimes gives a more exact description of this algebra by using Gelfand representation.

2. Basic results. Let $(A, T(\mathcal{P}))$ be a commutative A-convex algebra. Let $x \in A$ and $\lambda \in \Lambda$ be given. Now for each $y \in A$ there is some constant $M_{(x,\lambda)}$ such that

$$p_\lambda(xy) \leq M_{(x,\lambda)} p_\lambda(y). \tag{2}$$

Let $\tilde{p}_\lambda(x)$ be infimum of all numbers $M_{(x,\lambda)}$ satisfying the inequality (2) for all $y \in A$. Now it can be shown that $\tilde{p}_\lambda(x)$, $x \in A$, defines a submultiplicative seminorm on A. See [**9**] and [**13**]. Let $T(\tilde{P})$ be a topology defined on A by the family $\tilde{\mathcal{P}} = \{\tilde{p}_\lambda \mid \lambda \in \Lambda\}$. It can also be shown that

$$\tilde{p}_\lambda(x) = \sup\{p_\lambda(xy) \mid p_\lambda(y) \leq 1\}. \tag{3}$$

$\tilde{p}_\lambda$ is sometimes called the operator seminorm connected with p_λ. From the definition we immediately get that in the case A has unit $p_\lambda(x) \leq p_\lambda(e)\tilde{p}_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$. Thus if A has unit and we denote $\tilde{\mathcal{P}} = \{\tilde{p}_\lambda \mid \lambda \in \Lambda\}$, then we have $\Delta(A, T(\mathcal{P})) \subset \Delta(A, T(\tilde{\mathcal{P}}))$. It can be shown that $T(\tilde{\mathcal{P}})$ is the weakest locally m-convex topology on A which is stronger than $T(\mathcal{P})$. See [**21**]. If $p_\lambda(e) \neq 1$ then the normalized seminorm $q_\lambda = (1/p_\lambda(e))p_\lambda$ is an A-convex seminorm on A which satisfies $q_\lambda(e) = 1$. Note that q_λ is not necessarily submultiplicative even if p_λ is. However the topologies $T(\mathcal{P})$ and $T(\mathcal{Q})$ are always equivalent. In general we also have $q_\lambda(x) \leq \tilde{p}_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$. Furthermore, if p_λ is submultiplicative, then $\tilde{p}_\lambda(x) \leq p_\lambda(x)$ for all $x \in A$.

If $(A, T(\mathcal{P}))$ is a locally A-convex algebra, then for all $\lambda \in \Lambda$ the set $N_\lambda = \{x \in A \mid p_\lambda(x) = 0\}$ is a closed ideal of $(A, T(\mathcal{P}))$. Let $h(N_\lambda) = \{\tau \in \Delta(A, T(\mathcal{P})) \mid \hat{x}(\tau) = 0 \text{ for all } x \in N_\lambda\}$, the hull of N_λ. Note that $h(N_\lambda)$ is just the set of all p_λ-continuous elements of $\Delta(A, T(\mathcal{P}))$. The quotient algebra A/N_λ is an A-normed algebra with the A-norm defined by $\dot{p}_\lambda(x + N_\lambda) = p_\lambda(x)$ for $x + N_\lambda \in A/N_\lambda$. Note that $\Delta(A/N_\lambda) = \Delta(A/N_\lambda, T(\{\dot{p}_\lambda\}))$ is homeomorphic to $h(N_\lambda)$. See [**18**, p. 339]. If p_λ is an A-norm for some $\lambda \in \Lambda$, then $h(N_\lambda)$ is homeomorphic to $\Delta(A, T(\{p_\lambda\}))$, since A/N_λ is in this case isometrically isomorphic to $(A, T(\{p_\lambda\}))$.

We shall now give an example of locally A-convex algebra which is not uniformly A-convex and not m-convex.

Example 1. Let X = the set of real numbers equipped with its usual topology. We shall define a family $\{g_n \mid n \in \mathbf{N}\}$ of functions on X as follows

$$g_1(t) = \begin{cases} 2|t| & \text{if } |t| \leq \frac{1}{2}, \\ 2 - 2|t| & \text{if } \frac{1}{2} \leq |t| \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and recursively if $n \in \mathbf{N}$, $n \geq 2$, then

$$g_n(t) = \begin{cases} g_{n-1}(t) & \text{if } t \in [-n+1, n-1], \\ 2|t| - 2n + 2 & \text{if } n-1 \leq |t| \leq \frac{2n-1}{2}, \\ 2n - 2|t| & \text{if } \frac{2n-1}{2} \leq |t| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

For $n \in \mathbf{N}$, let p_n be a seminorm on $C(X)$ defined by

$$p_n(x) = \sup_{t \in [-n,n]} g_n(t)|x(t)|, \quad x \in C(X).$$

Denote by $\mathcal{P} = \{p_n \mid n \in \mathbf{N}\}$ and $T(\mathcal{P})$ the topology on $C(X)$ defined by the seminorms of $\mathcal{P}$. Then $(C(X), T(\mathcal{P}))$ is a locally A-convex algebra which is not m-convex and not uniformly A-convex. Note that $(C(X), T(\mathcal{P}))$ is not complete. Furthermore, for $(C(X), T(\mathcal{P}))$, we have the following. For each $n \in \mathbf{N}$, $\tilde{p}_n(x) = \sup_{t \in [-n,n]} |x(t)|$, $x \in C(X)$. Denote by $\tilde{\mathcal{P}} = \{\tilde{p}_n \mid n \in \mathbf{N}\}$. Since we have $p_n(e) = 1$ for all $n \in \mathbf{N}$ there is no locally m-convex topology on $C(X)$ between $T(\mathcal{P})$ and $T(\tilde{\mathcal{P}})$. See [**16**]. Furthermore, $\Delta(C(X), T(\mathcal{P})) = \{\tau_t \mid t \in \mathbf{R} \backslash \mathbf{Z}\}$ and $\Delta(C(X), T(\tilde{\mathcal{P}})) = \{\tau_t \mid t \in \mathbf{R}\}$ where τ_t is a point evaluation at t. Obviously $h(N_n) = \{\tau_t \mid t \in [-n,n] \backslash \{0, \pm 1, \pm 2, \dots, \pm n\}\}$ and $h(\tilde{N}_n) = \{\tau_t \mid t \in [-n,n]\}$. (Here $N_n = \ker p_n$ and $\tilde{N}_n = \ker \tilde{p}_n$, $n \in \mathbf{N}$.) Note that the multiplication in this algebra is not jointly continuous.

It may also happen that $(A, T(\mathcal{P}))$ is locally m-convex algebra even if each $p_\lambda \in \mathcal{P}$ is A-convex but not submultiplicative as the following example shows.

Example 2. Let A be as in Example 1. For each $n \in \mathbf{N}$, let g_n be a function defined by

$$g_n(t) = \begin{cases} -\dfrac{1}{n}|t| + 1 & \text{if } |t| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $T(\mathcal{P})$ be a topology on A defined by the seminorms p_n, $n \in \mathbf{N}$, where $p_n(x) = \sup_{t\in[-n,n]} g_n(t)|x(t)|$, $x \in A$. Then $\tilde{p}_n = \sup_{t\in[-n,n]} |x(t)|$ for all $x \in A$. Now each seminorm p_n, $n \in \mathbf{N}$, is A-convex, but not submultiplicative. However, since $\tilde{p}_n(x)/2 \leq p_{2n}(x) \leq \tilde{p}_{2n}(x)$ for all $x \in A$ and $n \in \mathbf{N}$ we can see that the topologies $T(\mathcal{P})$ and $T(\tilde{\mathcal{P}})$ are equivalent. So $(A, T(\mathcal{P}))$ is a locally m-convex algebra.

Example 3. The above-mentioned examples are special cases for the more general case where X is a completely regular space. $\mathcal{K}$ is a compact cover of X and $A = C(X)$. Let $\mathcal{B} \subset C(X)$ where $g(t) \geq 0$ for all $g \in \mathcal{B}$ and $t \in X$. For $K \in \mathcal{K}$ and $g \in \mathcal{B}$ we shall define a seminorm $p_{(K,g)}$ on $C(X)$ by

$$p_{(K,g)}(x) = \sup_{t\in K} g(t)|x(t)|, \quad x \in C(X).$$

Denote by $\mathcal{P}$ this family of seminorms. Now $(C(X), T(\mathcal{P}))$ is a locally A-convex algebra. This type of algebra has been studied in [**16**]. There is also a rich literature of so-called Nachbin algebras (see, e.g. [**22**] where this type of structure has been considered). Now if $\tilde{p}_{(K,g)}$ is an operator-seminorm seminorm that corresponds with $p_{(K,g)}$, then we clearly have $\tilde{p}_{(K,g)}(x) = \sup_{t\in K\setminus Z(g)} |x(t)|$ for all $x \in C(X)$ where $Z(g)$ is the zeroset of g so that $\tilde{p}_{(K,g)}$ does not depend on the function g. If $g(t) > 0$ for all $t \in K$, then it is easy to see that $p_{(K,g)}$ and $\tilde{p}_{(K,g)}$ are equivalent seminorms. Thus if for all $K \in \mathcal{K}$ there is a $g \in \mathcal{B}$ such that $g(t) > 0$ for all $t \in K$, then the topologies $T(\mathcal{P})$ and $T(\tilde{P})$ are equivalent and thus $(A, T(\mathcal{P}))$ is a locally m-convex algebra, see [**16**]. Furthermore, it is easy to see that $h(N_{(K,g)}) = \{\tau_t \mid t \in K\setminus Z(g)\}$ and $Z(g)$ is the zeroset of g and $\Delta(A, T(\mathcal{P})) = \{\tau_t \mid t \in X\setminus Z(\mathcal{B})\}$ where $Z(\mathcal{B})$ is the zeroset of $\mathcal{B}$. On the other hand, if we assume that $T(\mathcal{P})$ is a Hausdorff-topology (as we always do) then this condition is valid only if $\operatorname{cl}(X\setminus Z(\mathcal{B})) = X$. If $K \in \mathcal{K}$ and $g \in \mathcal{B}$ then $\tilde{p}_{(K,g)}(x) = \sup_{t\in K} |x(t)|$, $\Delta((C(X), T(\mathcal{P})) = \{\tau_t \mid t \in X\setminus Z(\mathcal{B})\}$, $\Delta(C(X), T(\tilde{\mathcal{P}})) = \{\tau_t \mid t \in X\}$,

$h(N_{(K,g)}) = \{\tau_t \mid t \in K \backslash Z(g)\}$ and $h(\tilde{N}_{(K,g)}) = \{\tau_t \mid t \in K\}$. The multiplication in this algebra is jointly continuous if the set $\mathcal{B}$ is closed under taking square roots of its elements. Note that the weight functions do not necessarily have to be continuous.

Example 4. Let $A = C(X)$ where X is the set of reals with its usual topology. For $x \in A$ and $n \in \mathbf{N}$ define a seminorm $p_n(x) = \int_{-n}^{n} |x(t)|$. Then $(A, T(\mathcal{P}))$ is a locally A-convex algebra for which $\Delta(A, T(\mathcal{P}))$ is empty and $\tilde{p}_n(x) = \sup_{t \in [-n,n]} |x(t)|$ for all $x \in A$ and $n \in \mathbf{N}$. Thus $\Delta(A, T(\tilde{\mathcal{P}})) = \{\tau_t \mid t \in X\}$.

Next we shall give some general properties of A-convex seminorms. Let p be an A-convex seminorm on A. We shall say that p is weakly regular if there is a constant $m > 0$ such that

$$p(x) \leq m\tilde{p}(x) \quad \text{for all } x \in A. \tag{4}$$

Let m_p be the infimum of all constants satisfying (4). If $p = p_\lambda$ for some index λ we shall denote $m_{p_\lambda} = m_\lambda$. Now we also have $p(x) \leq m_p \tilde{p}(x)$ for all $x \in A$. We shall say that p is regular if $\tilde{p} = p$. Clearly p can be regular only if it is submultiplicative. We shall say that a topology $T(\mathcal{P})$ is weakly regular if p_λ is weakly regular for all $\lambda \in \Lambda$. For weakly regular $T(\mathcal{P})$ we naturally have $\Delta(A, T(\mathcal{P})) \subset \Delta(A, T(\tilde{\mathcal{P}}))$ and it also holds that $T(\tilde{P})$ is the weakest locally m-convex topology for A which is stronger than $T(\mathcal{P})$. See [**21**].

In the following lemmas A is a commutative algebra and p is an A-convex seminorm on A.

Lemma 1. *If A has unit, then $p(e) = m_p$.*

Proof. Since A has unit the inequality $p(x) \leq p(e)\tilde{p}(x)$ holds for all $x \in A$. So we have $m_p \leq p(e)$. On the other hand, $p(e) \leq m_p \tilde{p}(e) = m_p$. So $p(e) = m_p$.

Lemma 2. *If $m > 0$, then $(mp)^\sim = \tilde{p}$.*

Proof. This follows directly from the definition of $\tilde{p}$.

Lemma 3. *p is regular if and only if it is weakly regular, submultiplicative and $m_p = 1$.*

Lemma 4. *If A has unit or if p is regular, then $\tilde{\tilde{p}} = \tilde{p}$.*

Proof. If p is regular, then the result is obvious. Suppose that A has unit. Since $\tilde{p}$ is submultiplicative, we have $\tilde{\tilde{p}}(x) \leq \tilde{p}(x)$ for all $x \in A$. On the other hand $\tilde{p}(x) \leq \tilde{p}(e)\tilde{\tilde{p}}(x) = \tilde{\tilde{p}}(x)$ for all $x \in A$ from which the result follows.

Lemma 5. *If p is weakly regular, then $\ker p = \ker \tilde{p}$.*

Proof. From the assumption we get that $\ker \tilde{p} \subset \ker p$. The inclusion to the other direction follows from the fact that $\ker p$ is an ideal of A.

In some cases there may be in A such a nonzero element x for which the annihilator is A. For such an x we clearly have $\tilde{p}(x) = 0$. So in such a case there can be an element $x \in A$ for which $\tilde{p}(x) < p(x)$. To avoid this we can define a new seminorm p' on A by defining $p'(x) = \max\{p(x), \tilde{p}(x)\}$, $x \in A$, see [**21**]. This seminorm is submultiplicative, and we have $p(x) \leq p'(x)$ for all $x \in A$. If p is regular, then $p' = \tilde{p}$. Furthermore, if p is weakly regular, then p' and $\tilde{p}$ are equivalent.

Example 5. A seminorm p is called subquadrative if $p(x^2) \leq p(x)^2$ for all $x \in A$. (This type of seminorm has been studied in [**4**].) We shall show that a weakly regular subquadrative seminorm p is equivalent with $\tilde{p}$. Namely, if p is subquadrative, then $p(xy) \leq 2p(x)p(y)$ for all x and y in A. Thus if we take $q = 2p$, then q is a submultiplicative weakly regular seminorm on A. By Lemma 2.2 $\tilde{q} = \tilde{p}$. Thus we have

$$\tilde{p}(x) = \tilde{q}(x) \leq q(x) \leq m_q\tilde{q}(x) = 2m_p\tilde{p}(x) \quad \text{for all } x \in A$$

so that $(\tilde{p}(x)/2) \leq p(x) \leq m_p\tilde{p}(x)$ for all $x \in A$.

3. Gelfand representation of $(A, T(\mathcal{P}))$. Next we shall give some properties of the carrier space $\Delta(A, T(\mathcal{P}))$.

If we assume that $T(\mathcal{P})$ is weakly regular, then $p_\lambda(x) \le m_\lambda \tilde{p}_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$. Thus we have by Lemma 5 $\tilde{N}_\lambda = N_\lambda$. If we now define $h(\tilde{N}_\lambda) = \{\tau \in \Delta(A, T(\tilde{\mathcal{P}})) \mid \hat{x}(\tau) = 0 \text{ for all } x \in \tilde{N}_\lambda\}$, then $h(N_\lambda) \subset h(\tilde{N}_\lambda)$ for all $\lambda \in \Lambda$. Since each $\tilde{p}_\lambda$ is submultiplicative it follows that $h(\tilde{N})$ is locally compact for all $\lambda \in \Lambda$. See [**10**]. Furthermore the restricted function $\hat{x}|_{h(\tilde{N}_\lambda)} \in C_0(h(\tilde{N}_\lambda))$ for all $\lambda \in \Lambda$. We also have $\Delta(A, T(\tilde{P})) = \cup_{\lambda \in \Lambda} h(\tilde{N}_\lambda)$ (see [**10**]). If p_λ is submultiplicative, then obviously $p_\lambda = \tilde{p}_\lambda$ and in this case $h(N_\lambda)$ is locally compact.

Lemma 6. *Let $(A, T(\mathcal{P}))$ be a locally A-convex algebra. Suppose that $T(\mathcal{P})$ is weakly regular. Then*

(i) $\Delta(A, T(\mathcal{P})) = \cup_{\lambda \in \Lambda} h(N_\lambda)$

(ii) $\hat{x}|_{h(N_\lambda)} \in C_b(h(N_\lambda))$.

Proof. Let $\tau \in \Delta(A, T(\mathcal{P}))$ be given. By the continuity of τ there is $\lambda \in \Lambda$ and some constant $M > 0$ such that $|\tau(x)| \le M p_\lambda(x)$ for all $x \in A$. From this we get part (i). Part (ii) follows from the facts that $h(N_\lambda) \subset h(\tilde{N})$ for all $\lambda \in \Lambda$ and $\hat{x}|_{h(\tilde{N}_\lambda)} \in C_0(h(\tilde{N}_\lambda))$.

So we can see that each $x \in A$ determines a continuous complex-valued function $\hat{x}$ both on $\Delta(A, T(\mathcal{P}))$ and $\Delta(A, T(\tilde{\mathcal{P}}))$. Furthermore, for each $\hat{x}$ we have $\hat{x}|_{h(N_\lambda)} \in C_b(h(N_\lambda))$ and $\hat{x}|_{h(\tilde{N}_\lambda)} \in C_0(h(\tilde{N}_\lambda))$. Since $\Delta(A, T(\mathcal{P})) \subset \Delta(A, T(\tilde{P}))$ each $\hat{x} \in C(\Delta(A, T(\tilde{P})))$ is an extension of the corresponding $\hat{x} \in C(\Delta(A, T(\mathcal{P})))$. We shall now give a type of Gelfand representation of locally A-convex algebras by using the weight functions defined on the carrier space of this algebra. First we shall construct these weight functions. So let $(A, T(\mathcal{P}))$ be a locally A-convex algebra. Let $\tau \in \Delta(A, T(\mathcal{P}))$ be given. By the continuity of τ there is some $\lambda \in \Lambda$ and constant $M_{(\tau,\lambda)} > 0$ (depending on τ and λ) such that

$$|\hat{x}(\tau)| \le M_{(\tau,\lambda)} p_\lambda(x) \quad \text{for all } x \in A. \tag{5}$$

Let $\mathcal{M}_{(\tau,\lambda)}$ be the set of all constants satisfying inequality (5). We shall now define a function g_λ on $\Delta(A, T(\mathcal{P}))$ by

$$g_\lambda(\tau) = [\inf\{M \mid M \in \mathcal{M}_{(\tau,\lambda)}\}]^{-1}, \tag{6}$$

if $\tau \in h(N_\lambda)$ and otherwise we define $g_\lambda(\tau) = 0$. Suppose that A has unit. If $\tau \in h(N_\lambda)$, then $1 = |\tau(e)| \le Mp_\lambda(e)$ for all $M \in \mathcal{M}_{(\tau,\lambda)}$ which implies that g_λ is bounded on $h(N_\lambda)$. If $\tau \in \Delta(A, T(\mathcal{P}))\backslash h(N_\lambda)$, then τ is not p_λ continuous, and thus in this case $\mathcal{M}_{(\tau,\lambda)}$ is empty and, as is known for emptyset, we can define $\inf \varnothing = \infty$. Thus we can define $g_\lambda(\tau) = 0$ if $\tau \in \Delta(A, T(\mathcal{P}))\backslash h(N_\lambda)$. It is easy to see that g_λ can also be defined as

$$(7) \qquad g_\lambda(\tau) = \sup\{M \mid M \ge 0 \text{ and } M|\hat{x}(\tau)| \le p_\lambda(x) \text{ for all } x \in A\}.$$

There is still another form for g_λ. Namely it can be shown that

$$g_\lambda(\tau) = \Big[\sup_{p_\lambda(x) \le 1} |\hat{x}(\tau)| \Big]^{-1}, \quad \tau \in \Delta(A, T(\mathcal{P})),$$

if this supremum is finite and otherwise $g_\lambda(\tau) = 0$. For this form of g_λ, see [**9**] and [**14**]. From the definition we immediately get that

$$g_\lambda(\tau)|\hat{x}(\tau)| \le p_\lambda(x) \quad \text{for all } x \in A \quad \text{and } \tau \in h(N_\lambda).$$

Lemma 7. *Suppose that* λ *and* $\mu \in \Lambda$. *Let* $p_m = \max\{p_\lambda, p_\mu\}$. *Then* $g_m = \max\{g_\lambda, g_\mu\}$ *and* $h(N_m) = h(N_\lambda) \cup h(N_\mu)$. *Further, if* $p_\lambda(x) \le p_\mu(x)$ *for all* $x \in A$, *then* $g_\lambda(\tau) \le g_\mu(\tau)$ *for all* $\tau \in \Delta(A, T(\mathcal{P}))$.

Proof. This follows directly from the definition of g_λ and g_μ.

For $\lambda \in \Lambda$, denote by $\tilde{g}_\lambda$ the weight function corresponding to the seminorm $\tilde{p}_\lambda$.

Lemma 8. *Suppose that* A *has unit. Then* $\tilde{g}_\lambda = \chi_{h(\tilde{N}_\lambda)}$ *is a characteristic function of* $h(\tilde{N}_\lambda)$.

Proof. Now for any $\tau \in h(\tilde{N}_\lambda)$ we have $|\hat{x}(\tau)| \le \tilde{p}_\lambda(x)$ for all $x \in A$. So we have $\tilde{g}_\lambda(\tau) \ge 1$ for all $\tau \in h(\tilde{N}_\lambda)$. On the other hand, if in the definition (6) of g_λ we take $x = e$, we can see that $M \le \tilde{p}_\lambda(e) = 1$ and thus also the supremum of those constants must be less than or equal to 1. So $\tilde{g}_\lambda(\tau) \le 1$ for all $\tau \in h(\tilde{N}_\lambda)$. Thus $\tilde{g}_\lambda = \chi_{h(\tilde{N}_\lambda)}$.

Note that if $T(\mathcal{P})$ is weakly regular then each g_λ is bounded and since it vanishes outside $h(N_\lambda)$ we can see that $g_{\lambda|h(N_\lambda)}$ is a bounded positive-valued function.

Lemma 9. *Suppose that $T(\mathcal{P})$ is weakly regular. Then $0 < g_\lambda(\tau) \le m_\lambda$ for all $\tau \in h(N_\lambda)$. In particular, if A has unit, then $0 < g_\lambda(\tau) \le p_\lambda(e)$ for all $\tau \in h(N_\lambda)$.*

Proof. This result follows directly from the definition of g_λ.

We shall now define a family $\mathcal{P} = \{\hat{p}_\lambda \mid \lambda \in \Lambda\}$ of seminorms on $\hat{A}$ by

$$\hat{p}_\lambda(\hat{x}) = \sup_{\tau \in h(N_\lambda)} g_\lambda(\tau)|\hat{x}(\tau)|, \quad x \in A.$$

Denote by $T(\hat{\mathcal{P}})$ the topology on $\hat{A}$ defined by these seminorms. From the definition it follows that $\hat{p}_\lambda(\hat{x}) \le p_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$. Furthermore,

$$\hat{p}_\lambda(\hat{x}\hat{y}) \le \Big(\sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)|\Big)\, \hat{p}_\lambda(\hat{y}) \quad \text{for all } x \text{ and } y \in A \quad \text{and } \lambda \in \Lambda.$$

This shows that each $\hat{p}_\lambda$ is an A-convex seminorm on $\hat{A}$ and thus $T(\hat{\mathcal{P}})$ is a locally A-convex topology on $\hat{A}$.

As was pointed out in [**23**], the necessary condition which guarantees that A is imbedded algebraically into $C(\Delta(A, T(\mathcal{P}))$ is the assumption that $(A, T(\mathcal{P}))$ is strongly semi-simple; i.e., from the condition $\hat{x}(\tau) = 0$ for all $\tau \in \Delta(A, T(\mathcal{P}))$ it follows that $x = 0$. This condition implies that A is necessarily commutative as was noted in [**24**].

Theorem 1. *Let $(A, T(\mathcal{P}))$ be a strong semi-simple locally A-convex algebra. Suppose that $T(\mathcal{P})$ is weakly regular. Then the mapping (Gelfand mapping) $x \mapsto \hat{x}$, $x \in A$, is a continuous homomorphism from $(A, T(\mathcal{P}))$ onto $(\hat{A}, T(\hat{P}))$.*

The advantage we get here (compared with a standard use of the compact-open topology) is that the topology $T(\hat{\mathcal{P}})$ is of the same type as the original topology $T(\mathcal{P})$ and the Gelfand mapping is automatically continuous. Next we shall study an m-convex case.

Theorem 2. *Let $(A, T(\mathcal{P}))$ be a locally m-convex algebra with unit and suppose that $p_\lambda(e) = 1$ for all $\lambda \in \Lambda$. Then $g_\lambda = \chi_{h(N_\lambda)}$ for all $\lambda \in \Lambda$.*

Proof. If $(A, T(\mathcal{P}))$ is locally m-convex and with unit, then each p_λ is regular, by Lemmas 1 and 3. Thus, by Lemma 8, $g_\lambda = \chi_{h(N_\lambda)}$ for all $\lambda \in \Lambda$. $g_\lambda = \chi_{h(N_\lambda)}$.

Note that Theorem 2 is valid without an assumption of strong semisimplicity.

Corollary 1. *If $(A, T(\mathcal{P}))$ is a locally m-convex algebra with unit and $p_\lambda(e) = 1$ for all $\lambda \in \Lambda$, then $\hat{p}_\lambda(\hat{x}) = \sup_{\lambda \in h(N_\lambda)} |\hat{x}(\tau)|$ for all $x \in A$ and $\lambda \in \Lambda$.*

Theorem 3. *Suppose that $(A, T(\mathcal{P}))$ is a locally m-convex algebra. If $T(\mathcal{P})$ is weakly regular, then $1 \le g_\lambda(\tau) \le m_\lambda$ for all $\tau \in h(N_\lambda) = h(\tilde{N}_\lambda)$.*

Proof. Since each p_λ is by the assumption submultiplicative, we clearly have $h(N_\lambda) = h(\tilde{N}_\lambda)$ for all $\lambda \in \Lambda$, and we also have $|\tau(x)| \le p_\lambda(x)$ for all $\tau \in h(N_\lambda)$ and $x \in A$. This implies that $g_\lambda(\tau) \ge 1$ for all $\tau \in h(N_\lambda)$. From the weak regularity of each p_λ it follows that $g_\lambda(\tau)|\hat{x}(\tau)| \le p_\lambda(x) \le m_\lambda \tilde{p}_\lambda(x)$ for all $x \in A$ and $\tau \in h(N_\lambda)$. So we can see that $1 \le g_\lambda(\tau) \le m_\lambda$ for all $\tau \in h(N_\lambda)$.

Corollary 2. *If $(A, T(\mathcal{P}))$ is m-convex with $T(\mathcal{P})$ weakly regular, then also $T(\hat{\mathcal{P}})$ is m-convex topology on $\hat{A}$.*

Example 5. Consider the algebra $(C(X), T(\mathcal{P}))$ of Example 1. Let $\hat{g}_n$ be the weight function on $\Delta(C(X), T(\mathcal{P}))$ that corresponds to the weight function g_n. If $\tau \in \Delta(C(X), T(\mathcal{P}))$, then τ is of the form $\tau = \tau_t$ for some $t \in \mathbf{R}\backslash\mathbf{Z}$. Suppose that $t \in \mathbf{R}\backslash\mathbf{Z}$. If $t \in \mathbf{R}\backslash[-n, n]$, then $\tau_t \notin h(N_n)$ which implies that $\hat{g}_n(\tau_t) = 0$. If $t \in [-n, n]$, then $\tau_t \in h(N_n)$ and $\hat{g}_n(\tau_t) = \sup\{M \mid M \geq 0, M|x(t)| \leq p_n(x)$ for all $x \in C(X)\}$. Since $g_n(t)|x(t)| \leq p_n(x)$ for all $x \in C(X)$ we can see that $g_n(t) \leq \hat{g}_n(\tau_t)$. Suppose that $\hat{g}_n(\tau_t) > g_n(t)$. Now there is some $x \in C(X)$ for which $g_n(t)|x(t)| = p_n(x)$. But for this x we have $\hat{p}_n(\hat{x}) \geq \hat{g}_n(\tau_t)|\hat{x}(\tau_t)| > g_n(t)|x(t)| = p_n(x)$, which is a contradiction. Thus we must have $\hat{g}_n(\tau_t) = g_n(t)$. So we can see that $\hat{g}_n(\tau_t) = g_n(t)$ for all $t \in \mathbf{R}\backslash\mathbf{Z}$. We can also see that the functions of $C(X)^\wedge$ can be considered as a continuous function on $\mathbf{R}\backslash\mathbf{Z}$ and clearly $C(X)^\wedge$ can be identified with the algebra $C(Y)$ where $Y = \mathbf{R}\backslash\mathbf{Z}$.

It is known from the theory of Banach algebra that for each algebra norm $\|\ \|$ on an algebra A with unit there is an equivalent norm $\|\ \|_0$ such that $\|e\|_0 = 1$. As a matter of fact, we can choose $\|x\|_0 = \sup_{\|y\| \leq 1} \|xy\|$. This is actually a regularization of the original norm, see [**25**]. Therefore it can always be assumed that the original norm of the algebra has this property. If we look at what influence this assumption has on the Gelfand representation of the algebra $(A, \|\ \|)$, we can see that the weight function defined above is in this case the characteristic function on $\Delta(A, \|\ \|)$. If this regularization assumption is not valid, then we know that the weight function has property $1 \leq g(\tau) \leq \|e\|$ for all $\tau \in \Delta(A, \|\ \|)$. But in case A has no unit, we do not even know whether this weight function is bounded or not. This has, of course, the influence on the Gelfand representation of $(A, \|\ \|)$. In the first place, in order that the operator norm $\|\ \|_0$ could be equivalent to $\|\ \|$ we must assume that $\|\ \|$ is weakly regular and then the assumption $\|e\| = 1$ in case A has unit corresponding with the condition $m_{\|\ \|} = 1$ where $m_{\|\ \|} = \inf\{m \mid \|x\| \leq m\|x\|_0$ for all $x \in A\}$. It looks, however, that in the literature of normed algebras these considerations have been omitted.

Remark. If $(A, T(\mathcal{P}))$ is a commutative locally convex algebra for which $p_\lambda(x^2) = p_\lambda(x)^2$ for all $x \in A$ and $\lambda \in \Lambda$, then it is automatically locally m-convex and thus also locally A-convex algebra. An algebra with this property will be called a locally convex square algebra or a locally convex uniform algebra, see [**18**]. It can be shown that $(A, T(\mathcal{P}))$ is a locally convex square algebra if and only if $p_\lambda(x) = \sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)|$ for all $x \in A$ and $\lambda \in \Lambda$. See [**6**], [**7**] and [**8**]. So, for a locally m-convex case, the assumption

$$\hat{p}_\lambda(\hat{x}) = p_\lambda(x) \quad \text{for all } x \in A \quad \text{and } \lambda \in \Lambda \tag{8}$$

is equivalent to the assumption that $(A, T(\mathcal{P}))$ is a locally convex square algebra. We shall now study an algebra with the property defined in (8). It must be noted that algebras of this type are generalizations of locally uniform algebras. Namely, for locally uniform algebra $(A, T(\mathcal{P}))$ we have $g_\lambda = \chi_{h(N_\lambda)}$ for all $\lambda \in \Lambda$. We shall call an algebra with the property (8) a locally A-uniform algebra. It follows from the Hausdorff condition that a locally A-uniform algebra is automatically strongly semi-simple. If $(A, T(\mathcal{P}))$ is a locally A-uniform algebra, then the algebras $(A, T(\mathcal{P}))$ and $(\hat{A}, T(\hat{\mathcal{P}}))$ can be identified and thus $(A, T(\mathcal{P}))$ is just a function algebra with a topology given by means of some family of weight functions defined on $\Delta(A, T(\mathcal{P}))$, and these weight functions depend on some cover of the space $\Delta(A, T(\mathcal{P}))$.

Theorem 4. *Let $(A, T(\mathcal{P}))$ be a locally A-uniform algebra. Then*

$$\tilde{p}_\lambda(x) = \sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)| = \sup_{\tau \in h(\tilde{N}_\lambda)} |\hat{x}(\tau)|$$

for all $x \in A$ and $\lambda \in \Lambda$.

Proof. Let $x \in A$ and $\lambda \in \Lambda$ be given. Now, for all $y \in A$, we have

$$\begin{aligned} p_\lambda(xy) &= \hat{p}_\lambda((xy)^\wedge) = \hat{p}_\lambda(\hat{x}\hat{y}) = \sup_{\tau \in h(N_\lambda)} g_\lambda(\tau)|\hat{x}(\tau)||\hat{y}(\tau)| \\ &\le \sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)| \sup_{\tau \in h(N_\lambda)} g_\lambda(\tau)|\hat{y}(\tau)| = \Big(\sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)|\Big)\hat{p}_\lambda(\hat{y}) \\ &\le \sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)| p_\lambda(y). \end{aligned}$$

It follows from this inequality that $\tilde{p}_\lambda(x) \le \sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)|$. Thus, we have

$$\tilde{p}_\lambda(x) \le \sup_{\tau \in h(N_\lambda)} |\hat{x}(\tau)| \le \sup_{\tau \in h(\tilde{N}_\lambda)} |\hat{x}(\tau)| \le \tilde{p}_\lambda(x)$$

which yields the result.

The following result is easy to see.

Theorem 5. *Suppose that* $(A, T(\mathcal{P}))$ *is a locally* A*-uniform algebra. Then* $m_\lambda = \sup_{\tau \in h(N_\lambda)} g_\lambda(\tau)$ *for all* $\lambda \in \Lambda$.

Let $(A, T(\mathcal{P}))$ be a locally A-uniform algebra. We shall call $\Lambda_0 \subset \Lambda$ a directed subset if, for each λ and $\mu \in \Lambda_0$, $\lambda \neq \mu$, either $p_\lambda(x) \leq p_\mu(x)$ or $p_\mu(x) \leq p_\lambda(x)$ for all $x \in A$ (denoted by $p_\lambda \leq p_\mu$ and $p_\mu \leq p_\lambda$), and if $p_\lambda \leq p_\mu$, then $g_{\mu|h(N_\lambda)} = g_\lambda$. Now each directed subset Λ_0 defines a function g_{Λ_0} from $\Delta(A, T(\mathcal{P}))$ into $(0, \infty)$ such that $g_{\Lambda_0|h(N_\lambda)} = g_\lambda$ for all $\lambda \in \Lambda_0$. If A has unit and $p_\lambda(e) = 1$ for all $\lambda \in \Lambda$, then $g_{\Lambda_0}(\Delta(A, T(\mathcal{P}))) \subset (0, 1]$ for each directed subset Λ_0. Note that g_{Λ_0} is not in general bounded. The functions g_{Λ_0} correspond to the weight functions of Examples 1–3 and the sets $h(N_\lambda)$ correspondingly to the members of the covers of X.

Lemma 10. *Suppose that* $(A, T(\mathcal{P}))$ *is a locally* A*-uniform algebra. If* A *is symmetric, i.e., for each* $x \in A$ *there is a* $y \in A$ *such that* $\bar{\hat{x}} = \hat{y}$ (*here the bar denotes complex conjugation*)*, then* $\hat{A}$ *is* $T(\hat{\tilde{\mathcal{P}}})$ *dense in* $C_\infty(\Delta(A, T(\tilde{P})) = \{g_{|\Delta(A, T(\tilde{\mathcal{P}})} \mid g \in C(\Delta(A, T(\tilde{P}) \cup \{\tau_\infty\}), g(\tau_\infty) = 0\}$*, where* τ_∞ *is a complex homomorphism on* A_e (*which is the algebra obtained from* A *by adjoining the unit*) *satisfying* $\tau_\infty(x, \alpha) = \alpha$ *for all* $(x, \alpha) \in A_e$.

Proof. If $\hat{p}_\lambda(\hat{x}) = p_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$, it follows from Theorem 4 that $(A, T(\tilde{\mathcal{P}}))$ is a locally convex square algebra which yields the result. For a detailed proof, see [**10**].

Theorem 6. *Suppose that* $(A, T(\mathcal{P}))$ *is a locally* A*-uniform algebra. If* A *is symmetric, then* $\Delta(A, T(\mathcal{P}))$ *is dense in* $\Delta(A, T(\tilde{\mathcal{P}}))$.

Proof. Suppose first that A has unit. Now, if $\Delta(A, T(\mathcal{P}))$ is not dense in $\Delta(A, T(\tilde{\mathcal{P}}))$, then there is a point τ_0 in $\Delta(A, T(\tilde{\mathcal{P}}))$ and an open neighborhood $V(\tau_0) \subset \Delta(A, T(\tilde{\mathcal{P}}))$ of τ_0 such that $V(\tau_0) \cap \Delta(A, T(\mathcal{P})) = \varnothing$. Since $\Delta(A, T(\tilde{\mathcal{P}}))$ is completely regular there must be a function $g \in C(\Delta(A, T(\tilde{\mathcal{P}})))$ such that $g(\tau_0) = 1$ and $g(\tau) =$

0 for all $\tau \in \Delta(A, T(\tilde{\mathcal{P}}))\backslash V(\tau_0)$. By Lemma 2 $\hat{A}$ is $T(\hat{\tilde{\mathcal{P}}})$ dense in $C(\Delta(A, T(\tilde{\mathcal{P}})))$. By the assumption $\Delta(A, T(\mathcal{P}))$ is a subset of $\Delta(A, T(\tilde{\mathcal{P}}))\backslash V(\tau_0)$. Thus there is an element $x \in A$ for which $\hat{x}(\tau_0) \neq 0$ and $\hat{x}(\tau) = 0$ for all $\tau \in \Delta(A, T(\mathcal{P}))$. But this latter condition implies that $p_\lambda(x) = \hat{p}_\lambda(\hat{x}) = 0$ for all $\lambda \in \Lambda$ which in turn implies that $x = 0$ and we get a contradiction. Thus $\Delta(A, T(\mathcal{P}))$ must be dense in $\Delta(A, T(\tilde{\mathcal{P}}))$. If A is without unit we can apply the results of [**10**] to the algebra $(A_e, T(\tilde{\mathcal{P}}_e))$ where A_e is the algebra obtained from A by adjoining the unit and $\tilde{\mathcal{P}}_e = \{\tilde{\mathcal{P}}_\lambda \mid \lambda \in \Lambda\}$ where the seminorms $\tilde{P}_\lambda$ are defined by $\tilde{P}_\lambda(x, \alpha) = \sup_{\tilde{p}_\lambda(y) \leq 1} \tilde{p}_\lambda(xy + \alpha y)$, $(x, \alpha) \in A_e$. Now by using the same methods as above, we can see that $\Delta(A, T(\mathcal{P}))$ is dense in $\Delta(A, T(\tilde{\mathcal{P}}))$.

The following result is obvious.

Corollary 3. *Suppose that the assumptions of Theorem* 5 *are valid. Then $h(N_\lambda)$ is dense in $h(\tilde{N}_\lambda)$ for all $\lambda \in \Lambda$.*

Note that adjoining the unit to a locally A-convex algebra $(A, T(\mathcal{P}))$ without unit is not so easy as for the locally m-convex case. For the locally m-convex case this type of problem has been considered in [**10**].

We shall say that $(A, T(\tilde{\mathcal{P}}))$ is full if $\hat{A} = C_\infty(\Delta(A, T(\tilde{\mathcal{P}}))$. The assumption that $(A, T(\tilde{\mathcal{P}}))$ is full does not necessarily imply that $\Delta(A, T(\mathcal{P}))$ is dense in $\Delta(A, T(\tilde{\mathcal{P}}))$. Namely, if we take an algebra $(A, T(\mathcal{P}))$ of Example 4, then for this algebra $(A, T(\tilde{\mathcal{P}}))$ is full in $C(\Delta(A, T(\tilde{\mathcal{P}})))$, but $\Delta(A, T, (\mathcal{P}))$ is empty and thus not dense in $\Delta(A, T(\tilde{\mathcal{P}}))$.

Corollary 4. *Let $(A, T(\mathcal{P}))$ be a locally A-uniform algebra for which $\hat{A}$ is $T(\hat{\tilde{\mathcal{P}}})$-dense in $C(\Delta(A), T(\tilde{\mathcal{P}}))$. If each p_λ is subquadrative, then p_λ is submultiplicative for all $\lambda \in \Lambda$ and in particular if $p_\lambda(e) = 1$ for all $\lambda \in \Lambda$, then $g_\lambda = \chi_h(N_\lambda)$ for all $\lambda \in \Lambda$.*

Proof. Now, if $p_\lambda(x^2) \leq p_\lambda(x)^2$, then it is easy to see that $g_\lambda(\tau) \geq 1$ for all $\tau \in h(N_\lambda)$ from which it follows that p_λ is submultiplicative, see [**1**]. If $p_\lambda(e) = 1$, then it follows from Theorem 2 that $g_\lambda = \chi_{h(N_\lambda)}$ for all $\lambda \in \Lambda$.

In the case when $\Delta(A, T(\mathcal{P}))$ is dense in $\Delta(A, T(\tilde{\mathcal{P}}))$ we can use a type of extended representation of $(A, T(\mathcal{P}))$. Namely, if the assumptions of Lemma 10 are valid, then clearly also each $h(N_\lambda)$ is dense in $h(\tilde{N}_\lambda)$, and thus we can extend each g_λ into a larger set $h(\tilde{N}_\lambda)$ by defining $g_\lambda(\tau) = 0$ if $\tau \in h(\tilde{N}_\lambda)\backslash h(N_\lambda)$. Now we can study the Gelfand function $\hat{x}$ of the element x of $(A, T(\mathcal{P}))$ as a function on larger set $\Delta(A, T(\tilde{P}))$. Then $\hat{A} \subset C(\Delta(A, T(\tilde{\mathcal{P}})))$ and we can define a seminorm $\hat{p}_\lambda$ by $\hat{p}_\lambda(\hat{x}) = \sup_{\tau \in h(\tilde{N}_\lambda)} g_\lambda(\tau)|\hat{x}(\tau)|$ for all $x \in A$ and $\lambda \in \Lambda$. If we apply this representation to Example 5 we naturally get $\hat{A} = C(\Delta(A, T(\tilde{\mathcal{P}})))$, and for the extended weight functions we have $\hat{g}_\lambda(\tau_t) = g(t)$ for all $t \in X$ and $\hat{x}(\tau_t) = x(t)$ for all $x \in A$ and $t \in X$.

4. Open problems and questions. It is known that if p is a square preserving (uniform) seminorm on an (associative) algebra A, then p is automatically regular, submultiplicative, and the quotient algebra $A/\ker p$ is commutative. See [**8**] or [**12**]. Thus all topological algebras $(A, T(\mathcal{P}))$ where $T(\mathcal{P})$ is a uniform Hausdorff-topology are commutative and locally m-convex. Also algebras of this type can always be considered as function algebras. Similarly, each C^*-seminorm on an algebra A is automatically submultiplicative, see [**26**]. A seminorm on an algebra A with an involution is called a C^*-seminorm if $p(xx^*) = p(x)^2$ for all $x \in A$. An algebra $(A, T(\mathcal{P}))$ with a topology $T(\mathcal{P})$ is called a locally C^*-algebra if the seminorms of $\mathcal{P}$ are all C^*-seminorms. It would be interesting to know whether the analogical results will also hold for locally A-convex algebras. So does there exist some property for the seminorms of $\mathcal{P}$ which would imply that the algebra $(A, T(\mathcal{P}))$ is a locally A-convex algebra and which in the commutative case is a locally A-uniform algebra (and thus a function algebra). In a commutative case some results along this line have been represented in [**1**], but all topologies studied in this paper are equivalent with a locally m-convex topology. An interesting property for the topology $T(\mathcal{P})$ is the following. Suppose that A is an algebra with an involution. Let $T(\mathcal{P})$ be a topology on A defined by the family of seminorms $\mathcal{P}$ where each $p \in \mathcal{P}$ satisfies the conditions

$$p(xx^*) = p(x^2) \quad \text{and} \quad p(x^*) = p(x) \quad \text{for all } x \in A.$$

Does it follow from these two properties that $(A, T(\mathcal{P}))$ is a locally

A-convex algebra for which in the commutative case $\tilde{p}$ is a square preserving for all $p \in \mathcal{P}$. Note that all algebras presented in Examples 1–4 (complex conjugation as an involution) satisfy the conditions given above.

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