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MINIMAL PRESENTATIONS OF FULL SUBSEMIGROUPS OF $\mathbf{N}^2$

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ABSTRACT. We show that the cardinality of a minimal presentation for a two-dimensional full affine subsemigroup of $\mathbf{N}^2$ minimally generated by p elements is $\binom{p-1}{2}$.

A subsemigroup S of $\mathbf{N}^2$ is full if $S = \mathbf{G}(S) \cap \mathbf{N}^2$, where $\mathbf{G}(S)$ denotes the subgroup of $\mathbf{Z}^2$ spanned by S. In this paper we are going to assume that S is a full subsemigroup of $\mathbf{N}^2$ such that $\operatorname{rank}(\mathbf{G}(S)) = 2$. (The case when $\operatorname{rank}(\mathbf{G}(S)) \leq 1$ has no interest, because under this assumption $S = \{(0,0)\}$ or $S \cong \mathbf{N}$.) Note that if $a, b \in S$ and $a - b \in \mathbf{N}^2$, then $a - b \in \mathbf{G}(S) \cap \mathbf{N}^2 = S$. As a consequence, if $M = \{(a_1, b_1), \dots, (a_p, b_p)\}$ is the set of minimal elements of $S \setminus \{0\}$ with respect to the ordering $a \leq b$ if and only if $b - a \in \mathbf{N}^2$, then S is minimally generated by M. Furthermore, we can assume that the elements in M are ordered so that $a_1 < a_2 < \cdots < a_p$ and $b_1 > b_2 > \cdots > b_p$.

We define the map

$$\varphi : \mathbf{N}^p \longrightarrow S$$
$$\varphi(\lambda_1, \dots, \lambda_p) = \sum_{i=1}^{p} \lambda_i (a_i, b_i)$$

and denote its kernel congruence by σ. Clearly, $S \cong \mathbf{N}^p/\sigma$. We say that ρ is a minimal system of generators for σ if ρ generates σ and ρ has minimal cardinality among the generating systems of σ. It can be shown that $\#\rho \geq p - 2$ (see [**5**]).

Given $s \in S \setminus \{0\}$, we define the graph G_s as the graph whose vertices are $V(G_s) = V_s = \{(a_i, b_i) \in M \mid s - (a_i, b_i) \in S\}$ and whose edges are $E(G_s) = E_s = \{[(a_i, b_i), (a_j, b_j)] \mid s - ((a_i, b_i) + (a_j, b_j)) \in S, 1 \leq i, j \leq p\}$.

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It can be shown (see [**7**] and [**2**]) that a minimal system of generators, ρ, of σ can be constructed as follows. For any $s \in S$, define ρ_s in the following way

1. If G_s is connected, then $\rho_s = \varnothing$.

2. If G_s is not connected and $G_s^1, \dots, G_s^t$ are the connected components of G_s, then choose a vertex $(a_{j_i}, b_{j_i}) \in V(G_s^i)$ and an element $\alpha_i^s = (\lambda_1^i, \dots, \lambda_p^i) \in \mathbf{N}^p$ such that $\varphi(\alpha_i^s) = s$ and $\lambda_{j_i}^i \neq 0$; define

$$\rho_s = \{(\alpha_2^s, \alpha_1^s), \dots, (\alpha_t^s, \alpha_1^s)\}.$$

Take $\rho = \cup_{s \in S} \rho_s$.

Our purpose is to count the number of elements belonging to ρ under the assumption that S is full and two-dimensional. In terms of semigroup rings, the problem translates to the following. We can construct the semigroup ring $K[S]$ associated to S as the set $\oplus_{s \in S} K y^s$, where addition is defined componentwise and multiplication is determined by the rule $y^s y^{s'} = y^{s+s'}$. The morphism φ now can be viewed as the ring homomorphism defined as

$$\varphi : K[x_1, \dots, x_p] \longrightarrow K[S]$$
$$\varphi(x_i) = y^{(a_i, b_i)}.$$

The kernel I_S of this morphism has a tight relationship with σ. As a matter of fact, the cardinality of a minimal system of generators for I_S equals the cardinality of a minimal system of generators for σ (see [**5**] for more details).

Note also that $K[S]$ can be viewed as a subring of $K[s,t]$ using the injective map $y^{(a,b)} \mapsto s^a t^b$. In this way, $K[S]$ becomes $K[s^{a_1} t^{b_1}, \dots, s^{a_p} t^{b_p}] \cong K[x_1, \dots, x_p]/I_S$, which is the ring of coordinates of the curve $C_S = \{(s^{a_1} t^{b_1}, \dots, s^{a_p} t^{b_p}) \mid t, s \in K\}$. Since S is cancellative and torsion free, $K[S]$ is an integral domain (see [**3**]) and therefore I_S is a prime ideal. As a consequence of this fact, I_S and its radical are the same ideal and, therefore, our problem translates to the problem of the minimum number of implicit equations required to define C_S.

We are going to show that the number of elements in a minimal system of generators for σ, and therefore for I_S, is $\binom{p-1}{2}$ and therefore depends exclusively on the number of generators of S. This contrasts with what happens for numerical semigroups, where it can be shown that no bound for the cardinality of minimal systems of generators can be found in terms of the number of generators (see [**1**]).

In order to prove that $\#\rho = \binom{p-1}{2}$, we need some technical lemmas which will tell us which elements s in S fulfill the assumption that G_s is not connected, and how many connected components these graphs can have. We start with a result that tells us how the sets of vertices of the graphs G_s are.

Lemma 1. *Given $s \in s$, there exist $i, j \in \mathbf{N}$ such that*

$$V_s = \{(a_{i+1}, b_{i+1}), \ldots, (a_{i+j}, b_{i+j})\}.$$

Proof. Let us assume that $s = (x, y)$. Take $\{a_1, \ldots, a_t\} = \{a_i \mid a_i \leq x\}$. Since $(x, y) \in S$, there must exist an element $(a_k, b_k) \leq (x, y)$ and therefore $1 \leq k \leq t$. Hence, $b_t \leq b_k \leq y$. Take i such that $b_t < b_{t-1} < \cdots < b_{i+1} \leq y < b_i < \cdots < b_1$. Clearly, $\{(a_{i+1}, b_{i+1}), \ldots, (a_t, b_t)\} = \{(a_k, b_k) \in M \mid (a_i, b_i) \leq (x, y)\} = V_s$. $\square$

The following result, at first glance, seems to have no connection with our problem. But we will see later that with it we can use a property of semigroups fulfilling the notion that their semigroup ring is Cohen-Macaulay. A subsemigroup T of $\mathbf{N}^2$ is normal if $T = \mathbf{G}(T) \cap \mathbf{L}_{\mathbf{Q}_+}(T)$, where $\mathbf{L}_{\mathbf{Q}^+}(A) = \{\sum_{i=1}^{n} q_i a_i \mid n \in \mathbf{N}, a_i \in A, q_i \in \mathbf{Q}^+\}$. The use of the word "normal" to refer to this type of semigroups comes from the fact that T is normal if and only if $K[T]$ is normal. Furthermore, Hochster shows in [**6**] that if $K[T]$ is normal, then $K[T]$ is Cohen-Macaulay. We will say, under this setting, that T is Cohen-Macaulay.

Lemma 2. *Let $i \neq j \in \{1, \ldots, p\}$ and $\overline{S} = S \cap \mathbf{L}_{\mathbf{Q}_+}(\{(a_i, b_i), (a_j, b_j)\})$. Then $\overline{S}$ is a normal subsemigroup of $\mathbf{N}^2$.*

Proof. Since $\overline{S}$ is always a subset of $\mathbf{G}(\overline{S}) \cap \mathbf{L}_{\mathbf{Q}_+}(\overline{S})$, it is enough to show that $\mathbf{G}(\overline{S}) \cap \mathbf{L}_{\mathbf{Q}_+}(\overline{S}) \subseteq \overline{S}$. Take $g \in \mathbf{G}(\overline{S}) \cap \mathbf{L}_{\mathbf{Q}_+}(\overline{S})$. By the definition of $\overline{S}$, it is clear that $\mathbf{G}(\overline{S}) \subseteq \mathbf{G}(S)$, and therefore

$g \in \mathbf{G}(S) \cap \mathbf{L}_{\mathbf{Q}_+}(\overline{S}) \subseteq \mathbf{G}(S) \cap \mathbf{N}^2 = S$. From the hypothesis that $g \in \mathbf{L}_{\mathbf{Q}_+}(\overline{S}) \subseteq \mathbf{L}_{\mathbf{Q}_+}(\{(a_i, b_i), (a_j, b_j)\})$, we get that $g \in S \cap \mathbf{L}_{\mathbf{Q}_+}(\{(a_i, b_i), (a_j, b_j)\}) = S$. $\square$

Note that since $\overline{S}$ normal, it is Cohen-Macaulay. The fact that a semigroup is Cohen-Macaulay has been characterized in several papers. It can be shown that since $\overline{S}$ is Cohen-Macaulay, if $s - (a_i, b_i)$ and $s - (a_j, b_j) \in \overline{S}$ then $s - ((a_i, b_i) + (a_j, b_j)) \in \overline{S}$, because (a_i, b_i) and (a_j, b_j) are the extremal rays of $\overline{S}$ (see Theorem 1.1 in [**8**] for a proof of this fact, which is a straightforward consequence of a characterization appearing in [**4**]). This idea yields the following two results which enables us to know how s must be in order to ensure that G_s is not connected.

Lemma 3. *If $s - (a_i, b_i)$ and $s - (a_j, b_j)$, with $i < j$, are in $\langle (a_i, b_i), \ldots, (a_j, b_j) \rangle$, then $s - ((a_i, b_i) + (a_j, b_j)) \in S$.*

Proof. Put $\overline{S} = S \cap \mathbf{L}_{\mathbf{Q}_+}(\{(a_i, b_i), (a_j, b_j)\}$. We already know that $\overline{S}$ is Cohen-Macaulay with extremal rays (a_i, b_i) and (a_j, b_j). Since $(a_i, b_i), (a_{i+1}, b_{i+1}), \ldots, (a_j, b_j) \in S \cap \mathbf{L}_{\mathbf{Q}_+}(\{(a_i, b_i), (a_j, b_j)\}) = \overline{S}$, if $s - (a_i, b_i)$ and $s - (a_j, b_j)$ are in $\langle (a_i, b_i), \ldots, (a_j, b_j) \rangle$ then $s - (a_i, b_i)$ and $s - (a_j, b_j)$ must belong to $\overline{S}$. Hence $s - ((a_i, b_i) + (a_j, b_j)) \in \overline{S} \subseteq S$. $\square$

Lemma 4. *Let $s \in S$. Then G_s is not connected if and only if $s = (a_i, b_i) + (a_j, b_j)$ with $|i - j| \geq 2$.*

Proof. Let us assume that G_s is not connected. Take $i = \min\{k \in \{1, \ldots, p\} \mid (a_k, b_k) \in V_s\}$ and $j = \max\{k \in \{1, \ldots, p\} \mid (a_k, b_k) \in V_s\}$. Since $(a_k, b_k) \in V_s$ if and only if $s - (a_k, b_k) \in S$, and this holds if and only if $s - (a_k, b_k) \in \mathbf{N}^2$, then $V_s = \{(a_i, b_i), \ldots, (a_j, b_j)\}$. Hence, $s - (a_i, b_i)$ and $s - (a_j, b_j)$ are in $\langle (a_i, b_i), \ldots, (a_j, b_j) \rangle$, which, by the previous lemma, implies that $s = (a_i, b_i) + (a_j, b_j) + s'$ for some $s' \in S$. Let us show that s' must be equal to zero. If this is not the case, then there exist $(a_k, b_k) \in M$ such that $s' - (a_k, b_k) \in S$. Note that this implies that $s - (a_k, b_k) \in S$ and therefore $(a_k, b_k) \in V_s$. Besides, $(a_i, b_i) + (a_j, b_j) - (a_l, b_l) \in \mathbf{N}^2$ for all $l \in \{i, \ldots, j\}$ and therefore

$(a_i, b_i)+(a_j, b_j)-(a_l, b_l) \in S$. This means that $s-((a_k, b_k)+(a_l, b_l)) = (a_i+b_i)+(a_j-b_j)-(a_l, b_l)+s'-(a_k, b_k) \in S$ for all $l \in \{i, \dots, j\}$ and therefore G_s is connected, a contradiction. Hence s' must be equal to zero, and consequently $s = (a_i, b_i) + (a_j, b_j)$. Observe that $|i-j|$ must be greater than one, because otherwise $s = (a_i, b_i) + (a_{i+1}, b_{i+1})$ and $V_s = \{(a_i, b_i), (a_{i+1}, b_{i+1})\}$ which would imply that G_s is connected.

Now, let us suppose that $s = (a_i, b_i) + (a_j, b_j)$ for some $i < j$ such that $j-i \geq 2$. Clearly, $(a_i, b_i)+(a_j, b_j)-(a_{i+1}, b_{i+1}) \in \mathbf{G}(S) \cap \mathbf{N}^2 = S$, and therefore $(a_i, b_i) + (a_j, b_j) = (a_{i+1}, b_{i+1}) + s$ for some $s \in S$. Since $\{(a_i, b_i), (a_j, b_j)\}$ is the set of vertices of a connected component of G_s, the last equality implies that G_s has another connected component containing (a_{i+1}, b_{i+1}). □

Lemma 5. *Let s be an element of S such that G_s is nonconnected. Then G_s has only one connected component whose vertices are not of the form $\{(a_i, b_i), (a_j, b_j)\}$, with $|i-j| \geq 2$.*

Proof. Since G_s is not connected, we already know that there exists $i < j \in \{1, \dots, p\}$ such that $V_s = \{(a_i, b_i), \dots, (a_j, b_j)\}$ and $j-i \geq 2$. We also know that $\{(a_i, b_i), (a_j, b_j)\}$ is the set of vertices of a connected component of G_s.

If G_s only has another connected component, then it must be $\{(a_{i+1}, b_{i+1}), \dots (a_{j-1}, b_{j-1})\}$ and this one is not of the form $\{(a_k, b_k), (a_l, b_l)\}$, with $|k-l| \geq 2$.

Suppose G_s has more than two connected components. Since $s - (a_{i+1}, b_{i+1})$ and $s-(a_{j-1}, b_{j-1})$ are both in S and (a_{i+1}, b_{i+1}) is not connected with (a_i, b_i), we get that $s-(a_{i+1}, b_{i+1})$ and $s-(a_{j-1}, b_{j-1})$ are both in $\langle (a_{i+1}, b_{i+1}), \dots, (a_{j-1}, b_{j-1}) \rangle$. Using Lemma 3, we get that $s - ((a_{i+1}, b_{i+1}) + (a_{j-1}, b_{j-1})) \in S$, and therefore $s = ((a_{i+1}, b_{i+1}) + (a_{j-1}, b_{j-1})) + s'$ with $s' \in S$. Since there are at least two connected components in G_s different from those with vertices $\{(a_i, b_i), (a_j, b_j)\}$, we can deduce, using similar reasoning to that used in the previous lemma, that $s' = 0$ and therefore $s = (a_{i+1}, b_{i+1}) + (a_{j-1}, b_{j-1}) = (a_i, b_i) + (a_j, b_j)$. Repeating this process, we obtain that there is exactly one connected component in G_s which is not of the form $\{(a_k, b_k), (a_l, b_l)\}$ with $|k-l| \geq 2$. □

With this result, we do not need to count how many connected components each graph has in order to know the cardinality of ρ, because this result ensures that there is exactly one connected component in each nonconnected graph with a special shape, and we can choose this special connected component to be G_s^1 in the construction of ρ_s. Note that we get a new element in ρ_s for each connected component of G_s different from G_s^1, and these connected components have their set of vertices of the form $\{(a_i, b_i), (a_j, b_j)\}$ with $|i-j| \geq 2$. Hence, we only have to count how many expressions there are of the form $(a_i, b_i) + (a_j, b_j)$ with $|i-j| \geq 2$. Clearly we get $\binom{p-1}{2}$, and this proves the following result.

Theorem 6. *Let S be a full subsemigroup of $\mathbf{N}^2$ minimally generated by $\{(a_1, b_1), \ldots, (a_p, b_p)\}$ such that* $\operatorname{rank}(\mathbf{G}(S)) = 2$. *Then the cardinality of a minimal presentation for S is $\binom{p-1}{2}$.*

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