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POLYNOMIAL FIRST INTEGRALS OF QUADRATIC SYSTEMS

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ABSTRACT. The main purpose of this paper is to give the classification and the topological phase portraits of all quadratic systems having minimal polynomial first integrals of degree less than 5, and to prove the existence of minimal polynomial first integrals of any degree for quadratic systems. Moreover, we prove that quadratic systems with minimal polynomial first integrals of degree larger than 1 have at most three invariant straight lines, and under convenient assumptions we give the greatest degree of the irreducible polynomial first integrals.

1. Introduction. By definition, a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where the dependent variables x and y and the independent variable (the *time*) t are all real, and $P, Q \in \mathbf{R}[x,y]$, as usual $\mathbf{R}[x,y]$ denotes the ring of polynomials in the variables x and y with real coefficients. In what follows, all mentioned functions are in $\mathbf{R}[x,y]$ and all constants are real. We say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system. The polynomial systems of degree 2 will be called *quadratic systems*.

Quadratic systems have been investigated intensively, and nearly 1000 papers have been published about these systems (see, for instance, [**23**], [**26**] and [**27**]). But it is an open problem to know what are the integrable quadratic systems. We must define what it means that a polynomial system is integrable because this notation changes with the authors, see, for instance, [**8**].

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The search of first integrals is a classical tool in the classification of all trajectories of a polynomial system. In 1878, Darboux [**10**] showed how the first integrals of planar polynomial systems possessing sufficient invariant algebraic curves can be constructed. The best improvements to Darboux's results for planar polynomial systems are due to Jouanolou [**13**] in 1979, to Prelle and Singer [**22**] in 1983, and to Singer [**25**] in 1992. Some recent interesting results related with Darboux theory of integrability have been made by many authors (see, for instance, Kooij and Christopher [**14**], Zholadek [**29**], Chavarriga, Giacomini, Gine and Llibre [**8**], etc.). In fact, the Darboux theory of integrability extends to polynomial differential systems in arbitrary finite dimension, see, for instance, Christopher and Llibre [**9**].

This work investigates the existence of polynomial first integrals for quadratic systems (1) and gives the corresponding topological phase portraits.

Moulin-Ollagnier [**19**] and Labrunie [**16**] have characterized the polynomial first integrals of a special three-dimensional Lotka-Volterra system, the so-called *abc* system, i.e.,

$$\dot{x} = x(cy + z), \qquad \dot{y} = y(x + az), \qquad \dot{z} = z(bx + y).$$

Cairó and Llibre [**7**] gave the polynomial first integrals of any degree for the two-dimensional Lotka-Volterra quadratic system of the form

$$\dot{x} = x(a_1 + b_{11}x + b_{12}y), \qquad \dot{y} = y(a_2 + b_{21}x + b_{22}y).$$

The paper is organized as follows. In Section 2 we give some basic definitions that we will need later on. In Section 3 we state our seven main theorems, in which Theorems A, B, C and F characterize all topological phase portraits of quadratic systems having minimal polynomial first integrals of degree 1, 2, 3 and 4, respectively. Theorems D and E give the classification of quadratic systems having a minimal polynomial first integral of degree 4. Theorem G shows that quadratic systems having more than three invariant straight lines have no minimal polynomial first integrals of degree larger than 1. In Section 4 we prove that there are quadratic systems having minimal polynomial first integrals of any degree, and that if a quadratic system has a polynomial first integral $H(x, y)$ such that $H(x, y) + c$ is irreducible in $\mathbf{R}[x, y]$ for all $c \in \mathbf{R}$, then $\deg H = 3$. Furthermore, we give some other basic results about polynomial first integrals that we will need in the proofs of our main theorems. Sections 5–11 establish the proofs of our seven theorems stated in Section 3.

2. Preliminary definitions. In this section we introduce some basic definitions and notations for the investigation of integrability, and for the analysis of topological phase portraits of polynomial systems (1).

2.1 First integral. For convenience, we denote by

$$\mathbf{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \tag{2}$$

the vector field associated with the system (1). An *invariant algebraic curve* of the system (1) is an algebraic curve $f(x,y) = 0$ with $f \in \mathbf{R}[x,y]$ satisfying $\mathbf{X}f = kf$ for some polynomial $k \in \mathbf{R}[x,y]$ called the *cofactor* of the invariant algebraic curve $f = 0$.

Here we say that a polynomial $H \in \mathbf{R}[x,y] \setminus \mathbf{R}$ is a *first integral* of system (1) on $\mathbf{R}^2$, if it is constant on all solution curves $(x(t), y(t))$ of system (1) on $\mathbf{R}^2$, i.e., $H(x(t), y(t)) \equiv$ *constant* for all values of t for which the solution $(x(t), y(t))$ is defined on $\mathbf{R}^2$. Obviously H is a first integral if and only if $\mathbf{X}H \equiv 0$ on $\mathbf{R}^2$.

We say that a polynomial first integral H of system (1) is *minimal* if every other polynomial first integral F of system (1) satisfies that $\deg F \geq \deg H$.

Let $R : \mathbf{R}^2 \to \mathbf{R}$ be a function which is not constant. The function R is an *integrating factor* of system (1) if the following three equivalent conditions hold

$$\begin{gathered} \frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \qquad \operatorname{div}(RP, RQ) = 0, \\ \mathbf{X}P = -R\operatorname{div}(P, Q). \end{gathered} \tag{3}$$

The first integral H associated to the integrating factor R is given by

$$H(x,y) = \int R(x,y)P(x,y)\,dy + h(x),$$

satisfying $(\partial H/\partial x) = -\,RQ$. Then

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \qquad \dot{y} = RQ = -\,\frac{\partial H}{\partial x}.$$

Conversely, given a first integral H of system (1), we always can find an integrating factor R for which these last two equations hold.

2.2 Singular points. Let $\mathbf{X}$ be the vector field given in (2). A point $q \in \mathbf{R}^2$ is a *singular point* of the vector field $\mathbf{X}$ if $P(q) = Q(q) = 0$.

If $D = P_x(q)Q_y(q) - P_y(q)Q_x(q)$ and $T = P_x(q) + Q_y(q)$, then a singular point q is *nondegenerate* if $D \neq 0$. Then the singular point must be isolated. Furthermore, q is a *saddle* if $D < 0$, a *node* if $T^2 \geq 4D > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), a focus if $T^2 < 4D$ and $T \neq 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), and either a *weak focus* or a *center* if $T = 0 < D$; for more details, see [**5**, p. 183].

A singular point q is *elementary* if $D = 0$ and $T \neq 0$, and then q is also isolated in the set of all singular points. The results on elementary singular points are summarized in Theorem 65 of [**5**](see also Theorem 7.1 of [**28**]).

A singular point q is *nilpotent* if $D = T = 0$ and the Jacobian matrix at q is not the zero matrix and q is isolated in the set of all singular points. Theorems 66 and 67 of [**5**] summarize the results on nilpotent singular points (see also [**1**], and Theorems 7.2 and 7.3 of [**28**]).

If the Jacobian matrix at the singular point q is identically zero and q is isolated in the set of all singular points, we say that q is *linearly zero*. Then the study of its local phase portraits needs a particular treatment (directional blow-ups), see [**2**] and [**24**] for more details.

We denote by $\mathbf{P}_n(\mathbf{R}^2)$ the set of all planar real vector fields of degree n. For $\mathbf{X} \in \mathbf{P}_n(\mathbf{R}^2)$ the *Poincaré compactified vector field* $p(\mathbf{X})$ corresponding to $\mathbf{X}$ is a vector field induced on $\mathbf{S}^2$ as follows (see, for instance, [**11**] and [**5**]). Let $\mathbf{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbf{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (called the *Poincaré sphere*) and $T_y\mathbf{S}^2$ be the tangent space to $\mathbf{S}^2$ at point y. Consider the central projections $f_+ : T_{(0,0,1)}\mathbf{S}^2 \to \mathbf{S}^2_+ = \{y \in \mathbf{S}^2 : y_3 > 0\}$ and $f_- : T_{(0,0,1)}\mathbf{S}^2 \to \mathbf{S}^2_- = \{y \in \mathbf{S}^2 : y_3 < 0\}$. These maps define two copies of $\mathbf{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathbf{X}'$ the vector fields $Df_+ \circ \mathbf{X}$ and $Df_- \circ \mathbf{X}$ defined on $\mathbf{S}^2$ except on its equator $\mathbf{S}^1 = \{y \in \mathbf{S}^2 : y_3 = 0\}$.

Obviously $\mathbf{S}^1$ is identified with the circle at infinity $\mathbf{S}^1$ of $\mathbf{R}^2$. In order to extend $\mathbf{X}'$ to an analytic vector field on $\mathbf{S}^2$ (including $\mathbf{S}^1$) it is necessary that $\mathbf{X}$ satisfy suitable hypotheses. In the case that $\mathbf{X} \in \mathbf{P}_n(\mathbf{R}^2)$, the *Poincaré compactification* $p(\mathbf{X})$ is the only analytic extension of $y_3^{n-1}\mathbf{X}'$ to $\mathbf{S}^2$. The set of all compactified vector fields $p(\mathbf{X})$ with $\mathbf{X} \in \mathbf{P}_n(\mathbf{R}^2)$ is denoted by $\mathbf{P}_n(\mathbf{S}^2)$. For the flow of the compactified vector field $p(\mathbf{X})$, the equator $\mathbf{S}^1$ is invariant. On $\mathbf{S}^2 \setminus \mathbf{S}^1$ there are two symmetric copies of $\mathbf{X}$, and knowing the behavior of $p(\mathbf{X})$ around $\mathbf{S}^1$, we know the behavior of $\mathbf{X}$ near infinity. The projection of the closed northern hemisphere of $\mathbf{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc* and it is denoted by $\mathbf{D}^2$.

As $\mathbf{S}^2$ is a differentiable manifold, for computing the expression of $p(\mathbf{X})$, we can consider the six local charts $U_i = \{y \in \mathbf{S}^2 : y_i > 0\}$ and $V_i = \{y \in \mathbf{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$, and the diffeomorphisms $F_i : U_i \to \mathbf{R}^2$ and $G_i : V_i \to \mathbf{R}^2$ defined as the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$, then z represents different things according to the local charts under consideration. Some straightforward calculations give for $p(\mathbf{X})$ the following expressions:

$$
\begin{gathered}
z_2^n \Delta(z)\left[Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right)\right] \quad \text{in } U_1, \\
z_2^n \Delta(z)\left[P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right)\right] \quad \text{in } U_2, \\
\Delta(z)[P(z_1, z_2), Q(z_1, z_2) \quad \text{in } U_3,
\end{gathered}
$$

where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-(n-1)/2}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbf{S}^1$. In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(\mathbf{X})$. Thus we obtain a polynomial vector field in each local chart.

2.3 Topological equivalence. In the rest of this paper, we say that polynomial vector fields $\mathbf{X}$ and $\mathbf{Y}$ on $\mathbf{R}^2$ are *topologically equivalent* if a homeomorphism on $\mathbf{S}^2$ exists preserving the circle at infinity $\mathbf{S}^1$ carrying orbits of the flow induced by $p(\mathbf{X})$ into orbits of the flow induced by $p(\mathbf{Y})$, preserving or reversing simultaneously the sense of all orbits. The definition of topological equivalence is not the usual one, but it is very convenient for study of the phase portraits of polynomial vector fields.

A *separatrix* of $p(\mathbf{X})$ is an orbit which is a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point. If a quadratic system has a polynomial first integral, then it has no limit cycles. This follows easily from the fact that a polynomial first integral is a continuous function defined in the whole plane.

We denote by $\mathbf{S}(p(\mathbf{X}))$ the set formed by all separatrices of $p(\mathbf{X})$. Neumann [**20**] proved that the set $\mathbf{S}(p(\mathbf{X}))$ is closed. Each open connected component of $\mathbf{S}^2 \setminus \mathbf{S}(p(\mathbf{X}))$ is called a *canonical region* of $p(\mathbf{X})$. A *separatrix configuration* is defined as a union of $\mathbf{S}(p(\mathbf{X}))$ plus one representative solution chosen from each canonical region. We say that $\mathbf{S}(p(\mathbf{X}))$ and $\mathbf{S}(p(\mathbf{Y}))$ are equivalent if a homeomorphism on $\mathbf{S}^2$ exists preserving the infinity $\mathbf{S}^1$ carrying orbits of $\mathbf{S}(p(\mathbf{X}))$ into orbits of $\mathbf{S}(p(\mathbf{Y}))$, preserving or reversing simultaneously the sense of all orbits.

The next lemma due to Neumann [**20**] states the characterization of two topologically equivalent Poincaré compactified vector fields. We shall need it later on for the analysis of the global phase portraits of quadratic system (1) having a polynomial first integral.

Neumann's Lemma. *Suppose that $p(\mathbf{X})$ and $p(\mathbf{Y})$ are two continuous flows on $\mathbf{S}^2$ with isolated singular points. Then $p(\mathbf{X})$ and $p(\mathbf{Y})$ are topologically equivalent if and only if their separatrix configurations are equivalent.*

Neumann's Lemma implies that, in order to obtain the global phase portraits of a vector field $p(\mathbf{X})$ with isolated singular points, we essentially need to determine the α- and β-limit sets of all separatrices of $p(\mathbf{X})$.

Neumann's Lemma was obtained under the additional assumption that the flow has no limit separatrices by Markus [**18**]. But this assumption is redundant.

3. Statement of the main results. Our first intention was to find all possible polynomial first integrals of quadratic systems. But, as is shown in Proposition 7 of Section 4 or in [**7**], quadratic systems may have minimal polynomial first integrals of any degree. So we cannot investigate them one by one according to the degree of polynomial first integrals. In the following we will give the necessary and sufficient conditions, and all topological phase portraits, for quadratic systems (1) having polynomial first integrals of degree less than 5. As $P^2(x,y)+Q^2(x,y) \not\equiv 0$ in (1), without loss of generality, let $Q(x,y) \not\equiv 0$. Furthermore, we also denote by $\mathbf{X} = (P,Q) : \mathbf{R}^2 \to \mathbf{R}^2$ the quadratic vector fields associated to system (1).

Our first result gives the classification and the topological phase portraits of quadratic systems (1) having a polynomial first integral of degree 1.

Theorem A. *A quadratic vector field* $\mathbf{X}$ *has a polynomial first integral of degree* 1 *if and only if* $\mathbf{X}$ *is topologically equivalent to the vector field* $\mathbf{X}^1 = (aQ, Q)$, *where* a *is a constant and* Q *is an arbitrary quadratic polynomial. The phase portrait of* $\mathbf{X}^1$ *is topologically equivalent to one of the* 14 *phase portraits given in Figure* 1. *Moreover, each phase portrait of Figure* 1 *is realizable by some* $\mathbf{X}^1$.

We remark that, for any given positive integer n, the quadratic vector field $\mathbf{X}^1$ has a polynomial first integral of the form $H(x,y) = c_1(x-ay)+c_2(x-ay)^2+\cdots+c_n(x-ay)^n$, where c_i, $i = 1,2,\dots,n$, is an arbitrary constant.

The next theorem characterizes the classification and the topological phase portraits of quadratic systems (1) having a minimal polynomial first integral of degree 2.

Theorem B. *A quadratic vector field* x *having a minimal polynomial first integral of degree* 2 *is topologically equivalent to one of the following three quadratic vector fields*:

$$
\begin{aligned}
\mathbf{X}_1^2 &= (-y(ax+by+c), x(ax+by+c)),\\
\mathbf{X}_2^2 &= (-x(ax+by+c), y(ax+by+c)),\\
\mathbf{X}_3^2 &= (ax+by+c, 2x(ax+by+c)),
\end{aligned}
$$

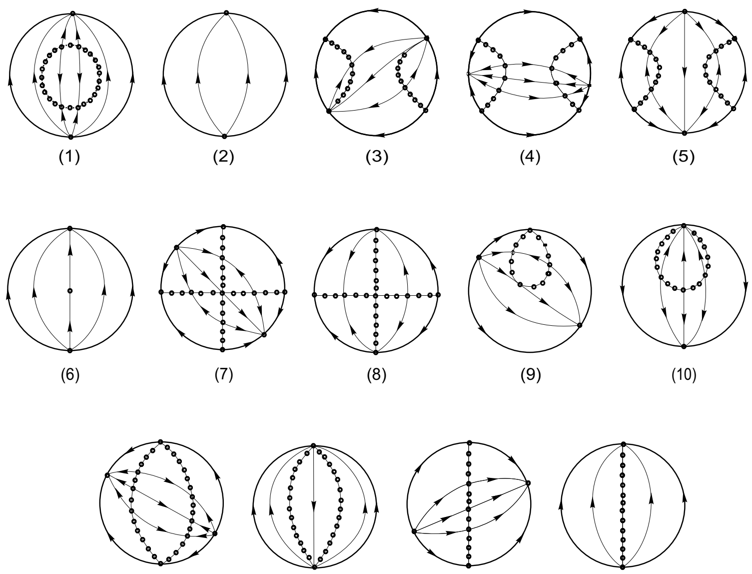

FIGURE 1. Phase portraits of quadratic systems having a polynomial first integral of degree 1.

where $a^2 + b^2 \neq 0$. *Furthermore, the corresponding phase portraits are topologically equivalent to one of the* 8 *phase portraits given in Figure* 2. *Every phase portrait of Figure* 2 *is realized by some* $\mathbf{X}_i^2$ *with* $i \in \{1, 2, 3\}$.

We remark that each quadratic system having a minimal polynomial first integral of degree 2 has a unique straight line formed by singular points.

Our third result characterizes the classification and the topological phase portraits of quadratic systems (1) having a minimal polynomial first integral of degree 3.

Theorem C. *A quadratic vector field* $\mathbf{X}$ *having a minimal polynomial first integral of degree* 3 *is topologically equivalent to one of the following*

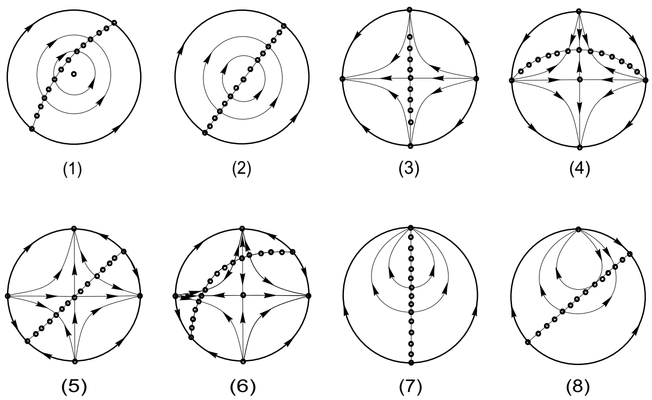

FIGURE 2. Phase portraits of quadratic systems having a minimal polynomial first integral of degree 2.

four quadratic vector fields:

$$
\begin{aligned}
&\mathbf{X}_1^3 = (\alpha + bx + cy, \beta - ax - by - x^2), \quad \alpha^2 + b^2 + c^2 \neq 0,\\
&\mathbf{X}_2^3 = (\alpha + bx + cy + x^2, \beta - ax - by - 2xy), \quad \alpha^2 + \beta^2 + b^2 + c^2 \neq 0,\\
&\mathbf{X}_3^3 = (\alpha + bx + cy - 2xy, \beta - ax - by - 3x^2 + y^2),\\
&\mathbf{X}_4^3 = (\alpha + bx + cy + y^2, \beta - ax - by - x^2).
\end{aligned}
$$

The phase portrait of $\mathbf{X}$ *is topologically equivalent to one of the* 25 *phase portraits given in Figure* 3, *that is, those in Figure* 1.1 *of* [**3**] *except for the phase portraits* 14, 15 *and* 16. *Moreover, each of these phase portraits is realizable by some* $\mathbf{X}_i^3$ *with* $i \in \{1, 2, 3, 4\}$.

The next two results give the classification of quadratic systems (1) having a minimal polynomial first integral of degree 4.

Theorem D. *Assume that* P *and* Q *are relatively prime. Then a quadratic system* (1) *having a minimal polynomial first integral of degree* 4 *can be written* (*after a linear change of variables*) *into one of the following six systems*:

$$
\text{(A)} \qquad \begin{aligned}
\dot{x} &= a_{00} + (b_{02}c - 2b_{01})x + a_{01}y - b_{11}x^2 - 3b_{02}xy + a_{02}y^2,\\
\dot{y} &= b_{00} + b_{11}cx + b_{01}y + b_{11}xy + b_{02}y^2,
\end{aligned}
$$

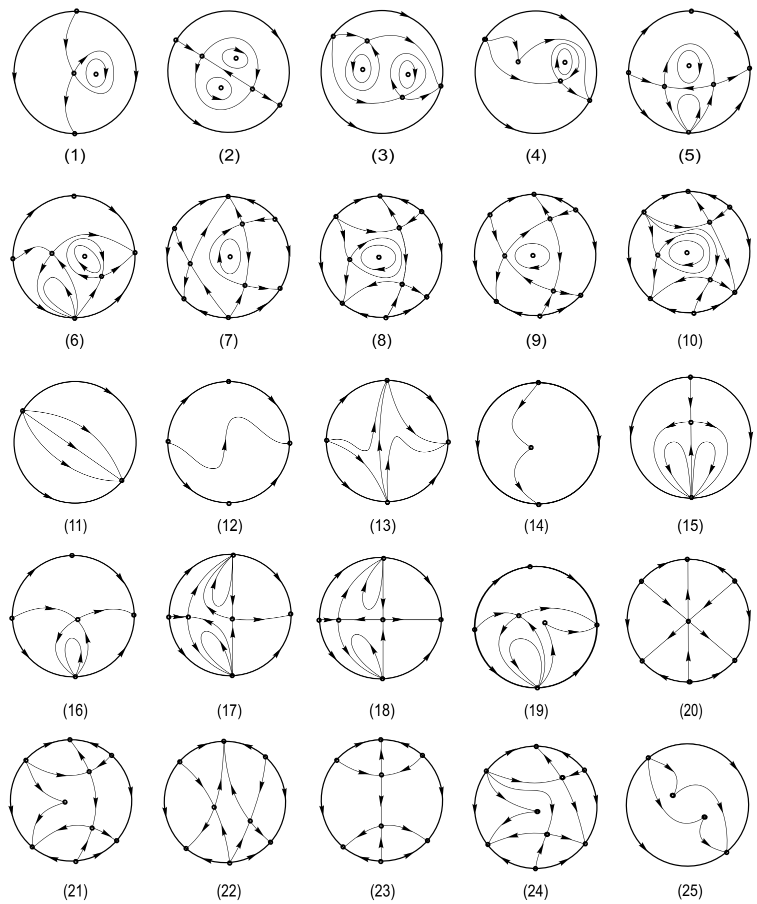

FIGURE 3. Phase portraits of quadratic systems having a minimal polynomial first integral of degree 3.

with a minimal first integral

$$\begin{aligned} H(x,y) = &-b_{00}cx + a_{00}cy - \frac{1}{2}b_{11}c^2x^2 - (b_{01}c + b_{00})xy \\ &+ \frac{1}{2}(a_{01}c + a_{00})y^2 - b_{11}cx^2y - (b_{02}c + b_{01})xy^2 \\ &+ \frac{1}{3}(a_{02}c + a_{01})y^3 - \frac{1}{2}b_{11}x^2y^2 - b_{02}xy^3 + \frac{1}{4}a_{02}y^4, \end{aligned}$$

where c satisfies the equation: $b_{02}c^2 - b_{01}c + b_{00} = 0$.

$$\text{(B)} \qquad \begin{aligned} \dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 - 3b_{02}xy, \\ \dot{y} &= b_{00} + b_{10}x + b_{01}y - 3a_{20}xy + b_{02}y^2, \end{aligned}$$

with a minimal first integral

$$\begin{aligned} H(x,y) = &- b_{00}kx + a_{00}ky - \frac{1}{2}\left(b_{10}k - \frac{a_{20}b_{00}}{b_{02}}\right)x^2 - (b_{01}k + b_{00})xy \\ &+ \frac{1}{2}(a_{01}k + a_{00})y^2 + \frac{1}{3}\frac{b_{10}a_{20}}{b_{02}} + a_{20}\left(k - \frac{a_{10}}{b_{02}}\right)x^2y \\ &+ \left(2a_{10} - \frac{b_{10}b_{02}}{a_{20}}\right)xy^2 + \frac{1}{3}a_{01}y^3 - \frac{a_{20}^2}{b_{02}}x^3y + 2a_{20}x^2y^2 - b_{02}xy^3, \end{aligned}$$

where

$$(4) \qquad \begin{aligned} &a_{20}b_{02} \neq 0, \\ &b_{10}b_{02}^2 + a_{01}a_{20}^2 - 3(a_{10} + b_{01})a_{20}b_{02} = 0, \\ &b_{00}b_{02}^2 - a_{00}a_{20}b_{02} + (a_{10}b_{02} - a_{01}a_{20} + 2b_{01}b_{02})(a_{10} + b_{01}) = 0, \end{aligned}$$

with $k = (b_{10}/a_{20}) - (b_{01} + 2a_{10})/(b_{02}) = k_B$

$$\text{(C)} \qquad \begin{aligned} \dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 - b - 02xy + a_{02}y^2, \\ \dot{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 - a_{20}xy + b_{02}y^2, \end{aligned}$$

with a minimal first integral

$$\begin{aligned} H(x,y) = &- b_{00}kx + a_{00}ky - \frac{1}{2}\left(b_{10}k - 2\frac{b_{00}b_{02}}{a_{02}}\right)x^2 \\ &- (b_{01}k + b_{00})xy + \frac{1}{2}(a_{01}k + a_{00})y^2 \\ &- \frac{1}{3}\left(b_{20}k - \frac{2b_{10}b_{02}}{a_{02}}\right)x^3 + \left(a_{20}k - \frac{2a_{10}b_{02}}{a_{02}}\right)x^2y \\ &- (b_{02}k + b_{01})xy^2 + \frac{1}{3}(a_{01} + a_{20}k)y^3 + \frac{2b_{20}b_{02}}{a_{02}}x^4 \\ &- \frac{1}{3}\left(\frac{2a_{20}b_{02}}{a_{02}} + b_{20}\right)x^3y + \frac{1}{2}\left(a_{20} + 2\frac{b_{02}^2}{a_{02}}\right)x^2y^2 \\ &- b_{02}xy^3 + \frac{1}{4}a_{02}y^4, \end{aligned}$$

where

$$
(5) \qquad \begin{aligned}
&a_{02}b_{02} \neq 0,\\
&4a_{20}b_{02} - a_{02}b_{20} = 0,\\
&a_{02}b_{10} - 2b_{02}(b_{01} + 2a_{10}) + a_{20}a_{02}k = 0,\\
&a_{02}b_{00} - 2a_{00}b_{02} + (a_{10} + b_{01})a_{02}k = 0,
\end{aligned}
$$

with $k = (2a_{01}/a_{02}) - (a_{10} + 2b_{01})/(b_{02}) = k_C$

$$
(\mathrm{D}) \qquad \begin{aligned}
\dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2,\\
\dot{y} &= b_{00} + b_{10}x + b_{01}y - 3a_{20}xy + b_{02}y^2,
\end{aligned}
$$

with a minimal first integral

$$
\begin{aligned}
H(x, y) = &- b_{00}kx + a_{00}ky - \frac{1}{2}\left(b_{10}k - \frac{b_{00}(a_{11} + 3b_{02})}{a_{02}}\right)x^2\\
&- (b_{01}k + b_{00})xy + \frac{1}{2}(a_{01}k + a_{00})y^2\\
&+ \frac{b_{10}(a_{11} + 3b_{02})}{3a_{02}}x^3 + \left(a_{20}k - \frac{a_{10}(a_{11} + 3b_{02})}{a_{02}}\right)x^2y\\
&+ (b_{02}k + b_{01})xy^2 + \frac{1}{3}(a_{02}k + a_{01})y^3\\
&- \frac{a_{20}(a_{11} + 3b_{02})}{a_{02}}x^3y + \frac{1}{2}\left(\frac{b_{02}(a_{11} + 3b_{02})}{a_{02}} + 3a_{20}\right)x^2y^2\\
&- b_{02}xy^3 + \frac{1}{4}a_{02}y^4,
\end{aligned}
$$

where

$$
\begin{aligned}
&a_{20}a_{02}(a_{11} + b_{02})(a_{11} + 3b_{02}) \neq 0,\\
&a_{11}^2 + 4a_{11}b_{02} + 3b_{02}^2 + 2a_{20}a_{02} = 0,\\
&(a_{10} + b_{01})a_{02}k + a_{02}b_{00} - a_{00}(a_{11} + 3b_{02}) = 0,\\
&(a_{11} + 2b_{02})a_{02}k + a_{02}(a_{10} + 2b_{01}) - a_{01}(a_{11} + 3b_{02}) = 0,
\end{aligned}
$$

with $k = (b_{10}/a_{20}) - (b_{01} + 2a_{10})(a_{11} + 3b_{02})/(a_{20}a_{02}) = k_D$.

(E) *System* (1) *with conditions*

$$
\begin{aligned}
&a_{02}(a_{10}+b_{01}) \neq 0,\\
&b_{11}^2+4a_{20}b_{11}+3a_{20}^2-a_{11}b_{20}-b_{20}b_{02}=0,\\
&a_{11}b_{11}+3a_{11}a_{20}+3b_{11}b_{02}+9a_{20}b_{02}-a_{02}b_{20}=0,\\
&(b_{11}+2a_{20})a_{02}k+a_{02}b_{10}-(b_{01}+2a_{10})(a_{11}+3b_{02})=0,\\
&(a_{11}+2b_{02})a_{02}k+a_{02}(a_{10}+2b_{01})-a_{01}(a_{11}+3b_{02})=0,
\end{aligned}
\tag{7}
$$

where $k = (a_{00}(a_{11}+3b_{02}) - a_{02}b_{00})/(a_{02}(a_{10}+b_{01})) = k_E$, *has a minimal first integral*

$$
\begin{aligned}
H(x,y) =& -b_{00}kx + a_{00}ky - \frac{1}{2}\left(b_{10}k - b_{00}\frac{a_{11}+3b_{02}}{a_{02}}\right)x^2\\
&-(b_{01}k+b_{00})xy + \frac{1}{2}(a_{01}k+a_{00})y^2\\
&-\frac{1}{3}\left(b_{20}k - b_{10}\frac{a_{11}+3b_{02}}{a_{02}}\right)x^3 + \left(a_{20}k - a_{10}\frac{a_{11}+3b_{02}}{a_{02}}\right)x^2y\\
&-(b_{02}k+b_{01})xy^2 + \frac{1}{3}(a_{02}k+a_{01})y^3\\
&+\frac{1}{4}b_{20}\frac{a_{11}+3b_{02}}{a_{02}}x^4 + \frac{1}{3}\left(b_{11}\frac{a_{11}+3b02}{a_{02}} - b_{20}\right)x^3y\\
&+\frac{1}{2}\left(b_{02}\frac{a_{11}+3b_{02}}{a_{02}} - b_{11}\right)x^2y^2 - b_{02}xy^3 + \frac{1}{4}a_{02}y^4.
\end{aligned}
$$

(F) *System* (1) *with conditions*

$$
\begin{aligned}
&a_{02}(b_{11}+2a_{20}) \neq 0,\\
&b_{11}^2+4a_{20}b_{11}+3a_{20}^2-a_{11}b_{20}-b_{20}b_{02}=0,\\
&a_{11}b_{11}+3a_{11}a_{20}+3b_{11}b_{02}+9a_{20}b_{02}-a_{02}b_{20}=0,\\
&a_{02}b_{00}-a_{00}(a_{11}+3b_{02})+a_{02}(a_{10}+b_{01})k=0,\\
&(a_{11}+2b_{02})a_{02}k+a_{02}(a_{10}+2b_{01})-a_{01}(a_{11}+3b_{02})=0,
\end{aligned}
$$

where

$$
k = \frac{(b_{01}+2a_{10})(a_{11}+3b_{02}) - a_{02}b_{10}}{a_{02}(b_{11}+2a_{20})} = k_F, \tag{9}
$$

has a minimal first integral given by (8) *with* $k = k_F$.

Theorem E. *Assume that* P *and* Q *have a common factor* A *of degree* 1, *and set* $P = A(a_1x + b_1y + c_1)$, $Q = A(a_2x + b_2y + c_2)$. *Then quadratic system* (1) *has a minimal polynomial first integral* $H(x, y)$ *of degree* 4 *if and only if one of the following four statements holds.*

(a) *If* $a_2 \neq 0$, $a_1 + b_2 \neq 0$ *and* $4a_2b_1 - (a_1 + 3b_2)(3a_1 + b_2) = 0$, *then the minimal first integral is of the form*

$$
\begin{aligned}
H(x,y) = & -c_2ax + c_1ay - \frac{1}{2}(a_2a + c_2b)x^2 + (a_1a + c_1b)xy \\
& + \frac{1}{2}(b_1a + c_1c)y^2 - \frac{1}{3}(a_2b + c_2l)x^3 + (a_1b + c_1l)x^2y \\
& - (b_2c + c_2n)xy^2 + \frac{1}{3}(b_1c + c_1n)y^3 - \frac{1}{4}a_2lx^4 \\
& + a_1lx^3y + \frac{1}{2}(a_1m + b_1l)x^2y^2 - b_2nxy^3 + \frac{1}{4}b_1ny^4,
\end{aligned}
\tag{10}
$$

with

$$
l = 1, \quad m = -\frac{3a_1 + b_2}{a_2}, \quad n = \frac{(3a_1 + b_2)^2}{4a_2^2},
$$

$$
a = \frac{K^2}{(a_1 + b_2)^2}, \quad b = -\frac{2K}{a_1 + b_2}, \quad c = \frac{3a_1 + b_2}{a_2(a_1 + b_2)}K,
$$

where

$$
K = 2c_1 - \frac{3a_1 + b_2}{a_2}c_2.
$$

(b) *If* $a_2 = 0$, $b_1 \neq 0$ *and* $a_1 = -3b_2 \neq 0$, *then the minimal first integral is of the form* (10) *with*

$$
l = m - 0, \quad n = 1, \quad a = \frac{c_2^2}{b_2^2}, \quad b = 0, \quad c = \frac{2c_2}{b_2}.
$$

(c) *If* $a_2 = 0$, $b_1 \neq 0$ *and* $b_2 = -3a_1 \neq 0$, *then the minimal first integral is of the form* (10) *with*

$$
l = \frac{16a_1^2}{b_1^2}, \quad m = \frac{8a_1}{b_1}, \quad n = 1, \quad a = \left(\frac{4c_1}{b_1} + \frac{c_2}{a_1}\right)^2,
$$

$$
b = \frac{32a_1c_1}{b_1^2} + \frac{8c_2}{b_1}, \quad c = \frac{8c_1}{b_1} + \frac{2c_2}{a_1}.
$$

(d) *If* $a_2 = b_1 = 0$ *and* $b_2 = -3a_1 \neq 0$*, then the minimal first integral is of the form* (10) *with*

$$l = 1, \quad m = n = 0, \quad a = \frac{c_1^2}{a_1^2}, \quad b = \frac{2c_1}{a_1}, \quad c = 0.$$

Our next theorem gives all topological phase portraits of quadratic systems (1) with a minimal polynomial first integral of degree 4.

Theorem F. *Assume that* P *and* Q *have no common factor of degree* 2*. Then the phase portrait of a quadratic system* (1) *having a minimal polynomial first integral of degree* 4 *is topologically equivalent to one of the* 28 *phase portraits in Figure* 4*. Moreover, each of the phase portraits in Figure* 4 *is realizable by a quadratic system* (1) *with a minimal polynomial first integral of degree* 4*.*

Since every quadratic system with a common factor of degree 2 has a minimal polynomial first integral of degree 1, we avoid these systems in the statement of Theorem F.

Finally, the following theorem shows that quadratic systems with more than three invariant straight lines have no minimal polynomial first integrals of degree larger than 1.

Theorem G. *Quadratic systems having a minimal polynomial first integral of degree larger than* 1 *cannot have more than three invariant straight lines.*

We remark that, while Theorem G shows that quadratic systems having a polynomial first integral of degree larger than one have at most three invariant straight lines, Theorem F shows that there are quadratic systems having a minimal polynomial first integral of degree 4 and three invariant straight lines.

We note that quadratic systems with more than three invariant straight lines may have a Darboux first integral, see, for instance, Theorem 26(b) of [**6**], and that the quadratic vector field $(0, Q)$ has a minimal polynomial first integral $H(x, y) = x$ of degree 1 and infinitely many invariant straight lines $x =$ constant.

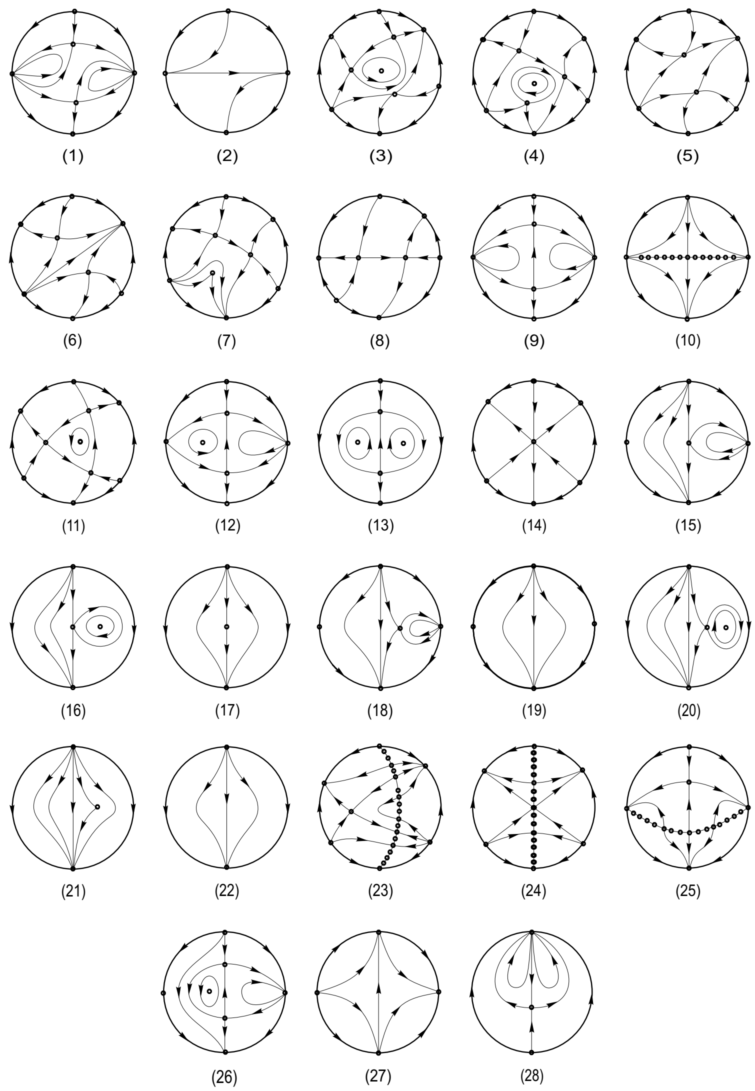

FIGURE 4. Phase portraits of quadratic systems having a minimal polynomial first integral of degree 4.

4. Some basic properties of polynomial first integrals. The next proposition gives the relationship between a polynomial first integral and its integrating factor.

Proposition 1. *Assume that P and Q are relatively prime. Then a quadratic system* (1) *with a polynomial first integral $H(x,y)$ satisfies that* $\deg H \geq 3$ *and that if* $\deg H > 3$, *then system* (1) *has a polynomial integrating factor.*

Proof. Let $H(x,y)$ be a polynomial first integral of degree n. Then we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} \equiv 0,$$

that is,

$$-\frac{(\partial H/\partial y)}{(\partial H/\partial x)} = \frac{dx}{dy} = \frac{P}{Q}.$$

Since P and Q are relatively prime, it is easy to see that $\max\{\deg(\partial H/\partial y), \deg(\partial H/\partial x)\} = n-1 \geq 2$. Hence, $\deg H \geq 3$ and a polynomial denoted by $R(x,y)$ of degree $n-3$ exists such that

$$\frac{\partial H}{\partial y} = PR, \qquad \frac{\partial H}{\partial x} = -QR.$$

If $\deg H > 3$, then $R(x,y)$ is a polynomial integrating factor. When $\deg H = 3$, then R is a constant, and consequently R is not an integrating factor. $\square$

We remark that if $R(x,y)$ is a polynomial integrating factor, then $R = 0$ is an invariant algebraic curve with cofactor $k(x,y) = -\mathrm{div}\,(P,Q)$. Moreover, if P and Q have a common factor, then $\deg H$ may be less than 3, and $R(x,y)$ can be non-polynomial. For instance, if $P = cQ$ with c a constant, then $H(x,y) = x - cy$ and $R(x,y) = 1/Q$ are a polynomial first integral and a rational integrating factor of system (1), respectively.

The next theorem gives the necessary and sufficient conditions in order that a quadratic system (1) has a polynomial first integral.

Theorem 2. *Assume that P and Q are relatively prime, and that*

$$
\begin{aligned}
&P(x,y)=\sum_{i+j=0}^{2} a_{ij}x^iy^j, \qquad Q(x,y)=\sum_{i+j=0}^{2} b_{ij}x^iy^j,\\
&H(x,y)=\sum_{i+j=1}^{n} c_{ij}x^iy^j, \qquad R(x,y)=\sum_{i+j=0}^{n-3} d_{ij}x^iy^j,
\end{aligned} \tag{11}
$$

where $n>3$, and a_{ij}, b_{ij}, c_{ij} and d_{ij} are real constants. Then $H(x,y)$ is a first integral of system (1) with integrating factor $R(x,y)$ if and only if the coefficients of R satisfy the following linear algebraic equations

$$
\begin{aligned}
&M_{n-1}\cdot D_{n-3}=0,\\
&M_{n-2}\cdot D_{n-4}+N_{n-2}\cdot D_{n-3}=0,\\
&M_j\cdot D_{j-2}+N_j\cdot D_{j-1}+T_j=0, \quad \textit{for } j=2,3,\ldots,n-3\\
&(a_{10}+b_{01})d_{00}+a_{00}d_{10}+b_{00}d_{01}=0,
\end{aligned} \tag{12}
$$

where the dot represents the product of a matrix and a vector, and

$$
M_j=\begin{pmatrix} M_j^1 & M_j^2 & M_j^3 \end{pmatrix}, \qquad N_j=\begin{pmatrix} N_J^1 & N_j^2 & N_j^3 \end{pmatrix},
$$

where

$$
M_j^1=\begin{pmatrix}
ja_{20}+b_{11} & b_{20} & 0\\
(j-1)a_{11}+2b_{02} & (j-1)a_{20}+2b_{11} & 2b_{20}\\
(j-2)a_{02} & (j-2)a_{11}+3b_{02} & (j-2)a_{20}+3b_{11}\\
0 & (j-3)a_{02} & (j-3)a_{11}+4b_{02}\\
0 & 0 & (j-4)a_{02}\\
0 & 0 & 0\\
\vdots & \vdots & \vdots\\
0 & 0 & 0
\end{pmatrix},
$$

$$
M_j^2 = \begin{pmatrix}
0 & 0 & & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
3b_{20} & 0 & \cdots & \cdots & 0 \\
(j-3)a_{20}+4b_{11} & 4b_{20} & \ddots & & \vdots \\
(j-4)a_{11}+5b_{02} & (j-4)a_{20}+5b_{11} & \ddots & \ddots & \vdots \\
(j-5)a_{02} & (j-5)a_{11}+6b_{02} & \ddots & (j-6)b_{20} & 0 \\
0 & (j-6)a_{02} & \ddots & 6a_{20}+(j-5)b_{11} & (j-5)b_{20} \\
\vdots & \ddots & \ddots & 5a_{11}+(j-4)b_{02} & 5a_{20}+(j-4)b_{11} \\
\vdots & & \ddots & 4a_{02} & 4a_{11}+(j-3)b_{02} \\
0 & \cdots & \cdots & 0 & 3a_{02} \\
0 & \cdots & \cdots & 0 & 0 \\
0 & \cdots & \cdots & 0 & 0
\end{pmatrix},
$$

$$
M_j^3 = \begin{pmatrix}
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
(j-4)b_{20} & 0 & 0 \\
4a_{20}+(j-3)b_{11} & (j-3)b_{20} & 0 \\
3a_{11}+(j-2)b_{02} & 3a_{20}+(j-2)b_{11} & (j-2)b_{20} \\
2a_{02} & 2a_{11}+(j-1)b_{02} & 2a_{20}+(j-1)b_{11} \\
0 & a_{02} & a_{11}+jb_{02}
\end{pmatrix},
$$

$$
N_j^1 = \begin{pmatrix}
ja_{10}+b_{01} & b_{10} & 0 \\
(j-1)a_{01} & (j-1)a_{10}+2b_{01} & 2b_{10} \\
0 & (j-2)a_{01} & (j-2)a_{10}+3b_{01} \\
0 & 0 & (j-3)a_{01} \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0
\end{pmatrix},
$$

$$
N_j^2 = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
3b_{10} & 0 & \cdots & \cdots & 0 \\
(j-3)a_{10}+4b_{01} & 4b_{10} & \ddots & & \vdots \\
(j-4)a_{01} & (j-4)a_{10}+5b_{01} & \ddots & \ddots & \vdots \\
0 & (j-5)a_{01} & \ddots & (j-5)b_{10} & 0 \\
\vdots & \ddots & \ddots & 5a_{10}+(j-4)b_{01} & (j-4)b_{01} \\
\vdots & & \ddots & 4a_{01} & 4a_{10}+(j-3)b_{01} \\
0 & \cdots & \cdots & 0 & 3a_{01} \\
0 & \cdots & \cdots & 0 & 0 \\
0 & \cdots & \cdots & 0 & 0
\end{pmatrix},
$$

$$
N_j^3 = \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ (j-3)b_{10} & 0 & 0 \\ 3a_{10}+(j-2)b_{01} & (j-2)b_{10} & 0 \\ 2a_{01} & 2a_{10}+(j-1)b_{01} & (j-1)b_{10} \\ 0 & a_{01} & a_{10}+jb_{01} \end{pmatrix},
$$

$$
T_j = \begin{pmatrix} ja_{00}d_{j,0}+b_{00}d_{j-1,1} \\ (j-1)a_{00}d_{j-1,1}+2b_{00}d_{j-2,2} \\ (j-2)a_{00}d_{j-2,2}+3b_{00}d_{j-3,3} \\ \cdots \\ 2a_{00}d_{2,j-2}+(j-1)b_{00}d_{1,j-1} \\ a_{00}d_{1,j-1}+jb_{00}d_{0,j} \end{pmatrix}, \qquad D_j = \begin{pmatrix} d_{j,0} \\ d_{j-1,1} \\ d_{j-2,2} \\ \cdots \\ d_{1,j-1} \\ d_{0,j} \end{pmatrix},
$$

where M_j and N_j are $j \times (j-1)$ and $j \times j$ matrices, respectively, M_j^1, M_j^2 and M_j^3 are $j \times 3$, $j \times (j-7)$ and $j \times 3$ matrices, respectively; N_j^1, N_j^2 and N_j^3 are $j \times 3$, $j \times (j-6)$ and $j \times 3$ matrices, respectively; and T_j and D_j are vectors of $\mathbf{R}^j$ and $\mathbf{R}^{j+1}$, respectively. Furthermore, under conditions (12) *we have*

$$
\begin{aligned}
c_{i0} &= \frac{1}{i}[b_{00}d_{i-1,0} + b_{10}d_{i-2,0} + b_{20}d_{i-3,0}], \\
c_{0j} &= \frac{1}{j}[a_{00}d_{0,j-1} + a_{01}d_{0,j-2} + a_{02}d_{0,j-3}], \\
c_{ij} &= \frac{1}{j}[a_{00}d_{i,j-1} + a_{10}d_{i-1,j-1} + a_{01}d_{i,j-2} \\
&\qquad + a_{20}d_{i-2,j-1} + a_{11}d_{i-1,j-2} + a_{02}d_{i,j-3}], \\
&= -\frac{1}{i}[b_{00}d_{i-1,j} + b_{10}d_{i-2,j} + b_{01}d_{i-1,j-1} \\
&\qquad + b_{20}d_{i-3,j} + b_{11}d_{i-2,j-1} + b_{02}d_{i-1,j-2}],
\end{aligned}
$$

where $i, j \in \{1, 2, \ldots, n\}$ and $d_{i,j} = 0$ when $i < 0$ or $j < 0$ or $i + j > n - 3$.

Proof. From Proposition 1 and its proof we know that

$$
\frac{\partial H}{\partial y} = PR, \qquad \frac{\partial H}{\partial x} = -QR. \tag{13}
$$

Substituting (11) into (13), we have

$$\sum_{\substack{i+j=1\\ j>0}}^{n} jc_{ij}x^i y^{j-1} = \sum_{i+j=0}^{n-1}\bigg(\sum_{k+l=0}^{2} a_{kl}d_{i-k,j-l}\bigg)x^i y^j,$$
$$\sum_{\substack{i+j=1\\ i>0}}^{n} ic_{ij}x^{i-1} y^{j} = -\sum_{i+j=0}^{n-1}\bigg(\sum_{k+l=0}^{2} b_{kl}d_{i-k,j-l}\bigg)x^i y^j,$$

where $d_{i-k,j-l}=0$ if $i-k<0$ or $j-l<0$. Equating the coefficients of $x^i y^{j-1}$ and of $x^{i-1}y^j$ in these last two equations, respectively, we obtain

$$jc_{ij} = \sum_{k+l=0}^{2} a_{kl}d_{i-k,j-1-l} \quad \text{for } j>0,$$
$$ic_{ij} = -\sum_{k+l=0}^{2} b_{kl}d_{i-1-k,j-l} \quad \text{for } i>0,$$

where $i+j=1,2,\dots,n$. Hence, system (1) has a polynomial first integral $H(x,y)$ if and only if the following equations hold

$$i\sum_{k+l=0}^{2} a_{kl}d_{i-k,j-1-l} = -\,j\sum_{k+l=0}^{2} b_{kl}d_{i-1-k,j-l},$$

for all i and j with $i,j>0$ and $i+j=2,3,\dots,n$.

These last equations can be rewritten as

$$\sum_{k+l=2}[ia_{kl}d_{i-k,j-1-l}+jb_{kl}d_{i-1-k,j-l}] + \sum_{k+l=1}[ia_{kl}d_{i-k,j-1-l}+jb_{kl}d_{i-1-k,j-l}] + ia_{00}d_{i,j-1}+jb_{00}d_{i-1,j}=0,$$

with $i,j\in\{1,2,\dots,n-1\}$, and $i+j=2,3,\dots,n$, we recall that $d_{i,j}=0$ if $i+j>n-3$. For each $K\in\{1,2,\dots,n-1\}$ and $i+j=K+1$, this last equation is equivalent to equations (12) with $j=K$ where the first and second components in this last equation correspond to

$M_K \cdot D_{K-2}$ and $N_K \cdot D_{K-1}$ in (12), respectively. So Theorem 2 follows. $\square$

The following lemma, due to Gasull, Sheng and Llibre [**12**], will be used later on, it gives a general classification of quadratic systems.

Lemma 3. *A quadratic differential system* (1) *is affine equivalent, scaling the variable* t*, if necessary, to one of the following ten systems*

$$\begin{array}{lll}
\text{(I)} & \dot{x} = 1 + xy, & \dot{y} = Q(x,y), \\
\text{(II)} & \dot{x} = xy, & \dot{y} = Q(x,y) \\
\text{(III)} & \dot{x} = y + x^2, & \dot{y} = Q(x,y), \\
\text{(IV)} & \dot{x} = y, & \dot{y} = Q(x,y), \\
\text{(V)} & \dot{x} = -1 + x^2, & \dot{y} = Q(x,y), \\
\text{(VI)} & \dot{x} = 1 + x^2, & \dot{y} = Q(x,y), \\
\text{(VII)} & \dot{x} = x^2, & \dot{y} = Q(x,y), \\
\text{(VIII)} & \dot{x} = x, & \dot{y} = Q(x,y), \\
\text{(IX)} & \dot{x} = 1, & \dot{y} = Q(x,y), \\
\text{(X)} & \dot{x} = 0, & \dot{y} = Q(x,y).
\end{array}$$

The next proposition characterizes the degree of irreducible polynomial first integrals of quadratic systems (1).

Proposition 4. *Assume that* P *and* Q *are relatively prime and that* $H(x,y)$ *is a polynomial first integral of a quadratic system* (1)*. If* $H(x,y)+c$ *is irreducible in* $\mathbf{R}[x,y]$ *for all* $c \in \mathbf{R}$*, then* $\deg H = 3$.

Proof. By Proposition 1 and its proof we know that $\deg H \geq 3$, and that if $\deg H > 3$ then a polynomial integrating factor $R(x,y)$ exists associated to the first integral $H(x,y)$ such that $\deg R = \deg H - 3$. Moreover, $R(x,y) = 0$ is an invariant algebraic curve. If $R(x,y)$ can be factorized in $\mathbf{R}[x,y]$, i.e., $R(x,y) = \prod_{i=1}^{k} r_i^{m_i}(x,y)$ with irreducible $r_i \in \mathbf{R}[x,y]$, then $r_i(x,y) = 0$ for $i = 1,2,\dots,k$, are invariant algebraic curves. By the definitions of invariant algebraic curves and polynomial first integrals, a real constant c exists such that $r_i(x,y)$ divides $H(x,y)+c$ (we note that if $R(x,y)$ is irreducible, then $k=1$, $m_k=1$ and $r_k=R(x,y)$). Since $H(x,y)+c$ is irreducible in $\mathbf{R}[x,y]$ for all $c\in\mathbf{R}$, it follows that $\deg R=0$, in contradiction with the fact that $\deg R=\deg H-3\geq 0$. Hence, $\deg H=3$. $\square$

Similar to the proof of Propositions 1 and 4, we can easily get the following corollary.

Corollary 5. *Assume that P and Q are two relatively prime polynomials of degree m and n, respectively, and that $H(x,y)$ is a polynomial first integral of the vector field (P,Q). If $H(x,y)+c$ is irreducible in $\mathbf{R}[x,y]$ for all $c\in\mathbf{R}$, then* $\deg H=\max\{\deg P,\deg Q\}+1$.

The next result gives the existence of minimal irreducible polynomial first integrals of polynomial systems of any degree.

Proposition 6. *For any positive integer n a polynomial system of degree n exists with a polynomial first integral $H(x,y)$ such that $H(x,y)+c$ is irreducible in $\mathbf{R}[x,y]$ for all $c\in\mathbf{R}$.*

Proof. The following system

$$\dot{x}=(n+1)y^n,\qquad \dot{y}=-1,$$

has the polynomial first integral $H(x,y)=x+y^{n+1}$. But $x+y^{n+1}+c$ is irreducible in $\mathbf{R}[x,y]$ for all $c\in\mathbf{R}$. $\square$

Proposition 7. *There are quadratic systems* (1) *having a minimal polynomial first integral of any degree.*

Proof. Theorems A, B and C show that there are quadratic systems having minimal polynomial first integrals of degree 1, 2 and 3. For any positive integer $m > 3$, the polynomial function

$$F(x,y) = x^{m-2}\left[\left(\frac{m}{2} - \frac{m-2}{2}x\right)^2 - \frac{m-2}{m}y^2\right],$$

is a first integral of degree m for the system

$$\dot{x} = xy, \qquad \dot{y} = \frac{m^3}{8} - \frac{m^2(m-1)}{4}x + \frac{m^2(m-2)}{8}x^2 - \frac{m-2}{2}y^2.$$

Obviously the two functions in this last system are relatively prime. By Theorems A, B and C, or by straightforward calculations we know that this last system has no polynomial first integrals of degree less than or equal to 3. If the above system has another polynomial first integral, denoted by $H(x,y)$, of degree $n > 3$, then by Theorem 2 its integrating factor $R(x,y)$ must satisfy the linear algebraic equations in [**13**]. Corresponding to this last quadratic system, the coefficient matrix M_j in [**13**] for $j = 2, 3, \dots, n-1$, can be written as

$$M_j = \begin{pmatrix} * & * & 0 & \cdots & 0 & 0 \\ j+1-m & * & * & \cdots & 0 & 0 \\ 0 & j+1-\frac{3}{2}m & * & \ddots & \vdots & \vdots \\ 0 & 0 & j+1-2m & \ddots & * & 0 \\ \vdots & \vdots & \ddots & \ddots & * & * \\ 0 & 0 & 0 & \ddots & j+1-\frac{j-1}{2}m & * \\ 0 & 0 & 0 & \cdots & 0 & j+1-\frac{j}{2}m \end{pmatrix}$$

where the stars denote the possibly nonzero constants.

If $n < m$. As $j \le n-1$, then $j \le m-2$. So we have that $j + 1 - (km/2) < 0$ for $k = 2, 3, \dots, j$. Then it follows easily that the linear algebraic equations in (12) only have zero solutions. Therefore, by Theorem 2 the above system has no polynomial first integrals of degree less than m. That is to say, the polynomial first integral F is minimal. $\square$

We note that Cairó and Llibre [**7**] recently obtained the necessary and sufficient conditions in order that the two-dimensional Lotka-Volterra quadratic systems has a polynomial first integral of degree $n+3$ for any given nonnegative integer n and gave the expression for this integral (for details, see [**7**]).

We remark that quadratic systems have infinitely many minimal polynomial first integrals, and that quadratic systems having a polynomial first integral have finitely many different topological phase portraits. The first statement is obvious by Proposition 7. The second statement follows from Neumann's lemma. First assume that the compactified quadratic system has finitely many singular points. Thus the quadratic vector field with a polynomial first integral has at most seven singular points (in the finite plane and at infinity), and finitely many separatrices, but has no limit cycles, so the topologically different separatrix configurations of quadratic vector field are also finite. If the compactified quadratic system has infinitely many singular points. we remove such points changing the time variable, and the proof follows as in the previous case.

Finally, we give necessary and sufficient conditions for a quadratic system (1), with P and Q having a common factor of degree 1, to have a polynomial first integral of degree 4.

Proposition 8. *Assume that P and Q have a common factor A of degree* 1, *and set $P = A(a_1x + b_1y + c_1)$, $Q = A(a_2x + b_2y + c_2)$. Then $H(x, y)$ is a polynomial first integral of degree* 4 *for quadratic systems* (1) *such that*

$$f(x,y) = a + bx + cy + lx^2 + my^2 + ny^2,$$

is the greatest common factor of $\partial H/\partial y$ and $\partial H/\partial x$ if and only if the following conditions hold:

$$\begin{aligned}
&(3a_1 + b_2)l + a_2m = 0,\\
&b_1l + (a_1 + b_2)m + a_2n = 0, \qquad (14)\\
&b_1m + (a_1 + 3b_2)n = 0,
\end{aligned}$$

and

$$
\begin{aligned}
&(a_1+b_2)a + c_1 b + c_2 c = 0,\\
&(2a_1+b_2)b + a_2 c + 2c_1 l + c_2 m = 0,\\
&b_1 b + (a_1+2b_2)c + c_1 m + 2c_2 n = 0.
\end{aligned} \tag{15}
$$

Under these conditions, the polynomial first integral $H(x,y)$ *is of the form* (10)*.*

Proof. Since H is a first integral, we have

$$\frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q = 0.$$

Consequently,

$$-\frac{(\partial H/\partial y)}{(\partial H/\partial x)} = \frac{P}{Q} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}.$$

Therefore, if $f(x,y)$ is the greatest common factor of $\partial H/\partial y$ and $\partial H/\partial x$, we have that

$$\frac{\partial H}{\partial y} = (a_1x+b_1y+c_1)f(x,y), \qquad \frac{\partial H}{\partial x} = -(a_2x+b_2y+c_2)f(x,y).$$

Similar to the proof of Theorem 2, by substituting $H(x,y)$ and $f(x,y)$ into these last two equations and equating the coefficients of x^iy^j, the proposition follows. $\square$

We note that in the proof of Proposition 8 the integrating factor is f/A.

5. Proof of Theorem A. Assume that the quadratic vector field $\mathbf{X}$ has a polynomial first integral, denoted by $H(x,y)$ of degree 1. Set

$$H(x,y) = Ax + By.$$

We have, by the definition of the first integral, that $AP + BQ \equiv 0$. As we can assume without loss of generality that $Q(x,y) \not\equiv 0$, then $A \neq 0$, otherwise $A = B = 0$, which contradicts the fact of $\deg H = 1$. Hence, the first part of the theorem follows taking $a = -B/A$.

The quadratic curve $Q = 0$ is a conic. The conics in $\mathbf{R}^2$, after a suitable affine change of coordinates (if necessary), have equations:

$$x^2 + y^2 - 1 = 0, \quad x^2 + y^2 + 1 = 0, \quad x^2 - y^2 - 1 = 0, \quad x^2 + y^2 = 0,$$
$$xy = 0, \quad y - x^2 = 0, \quad x^2 - 1 = 0, \quad x^2 + 1 = 0, \quad x^2 = 0.$$

In what follows, since $(\dot{x}, \dot{y}) = (aQ, Q)$, without loss of generality, we can consider the topological phase portraits of the vector field $\mathbf{X}^1$ with Q one of the nine functions in the last equations. The compactified vector field is $(-az_2^3 Q((1/z_2), (z_1/z_2)), z_2^2 Q((1/z_2), (z_1/z_2))(1 - az_1))$ and $(-z_2^3 Q((z_1/z_2), (1/z_2)), z_2^2 Q((z_1/z_2), (1/z_2))(a - z_1))$ in the local charts U_1 and U_2, respectively.

When $Q = x^2 + y^2 - 1$, the vector field $\mathbf{X}^1$ has the unit circle formed by finite singular points and a unique pair of infinite singular points $(a, 0) \in U^2$. Outside the unit circle all the trajectories of $\mathbf{X}^1$ are contained in parallel straight lines. So the topological phase portrait of $\mathbf{X}^1$ with $Q = x^2 + y^2 - 1$ is given in Figure 1(1).

The fact that the vector field $\mathbf{X}^1$ with $Q = x^2 + y^2 + 1$ or $Q = x^2 + 1$ has a unique pair of infinite singular points, and no finite singular points guarantees that $\mathbf{X}^1$ has topological phase portrait given in Figure 1(2).

When $Q = x^2 - y^2 - 1$, the vector field $\mathbf{X}^1$ has a hyperbola formed by singular points, and three pairs of infinite singular points $(1, 0), (-1, 0)$ and $(a, 0)$ in U_2 if $|a| \neq 1$, or two pairs $(1, 0)$ and $(-1, 0)$ in U_2 if $|a| = 1$. Outside the hyperbola the vector field $\mathbf{X}^1$ is equivalent to the vector field $(a, 1)$. So all trajectories of $\mathbf{X}^1$ are contained in parallel straight lines passing through the infinite singular point $(a, 0)$ in U_2. Therefore, the vector field $\mathbf{X}^1$ has topological phase portraits shown in Figures 1(3), 1(4) and 1(5) depending on $|a| = 1$, $|a| > 1$ and $|a| < 1$, respectively.

When $Q = x^2 + y^2$, the vector field $\mathbf{X}^1$ has a unique finite singular point $(0, 0)$ and a unique pair of infinite singular points $(a, 0)$ in U_2. Similar to the above analysis, we have the topological phase portrait given in Figure 1(6).

The vector field $\mathbf{X}^1$ with $Q = xy$ has two straight lines $x = 0$ and $y = 0$ formed by singular points, and two pairs of infinite singular points (0,0) in U_2 if $a = 0$ and (0,0) in U_1 or three pairs $(a, 0)$ and (0,0) in U_2 if $a \neq 0$ and (0,0) in U_1. Moreover, all trajectories of $\mathbf{X}^1$ are contained in parallel straight lines passing through the infinite singular points $(a, 0)$ in U_2. Hence we have Figures 1(7) and 1(8) showing the topological phase portraits of the vector field $\mathbf{X}^1$ for $a \neq 0$ and $a = 0$, respectively.

The vector field $\mathbf{X}^1$ with $Q = y - x^2$ has a parabola formed by singular points. Similar to the previous analysis, we can get that the topological phase portrait is equivalent to Figure 1(9), respectively Figure 1(10), if $a \neq 0$, respectively $a = 0$.

The vector field $\mathbf{X}^1$ with $Q = x^2 - 1$ has two straight lines $x = \pm 1$ formed by singular points. All the trajectories of $\mathbf{X}^1$ outside the lines $x = \pm 1$ are contained in straight lines, which are all parallel to the line $x = ay$. Therefore, we have the topological phase portraits given in Figures 1(11) and 1(12) if $a \neq 0$ and $a = 0$, respectively.

When $Q = x^2$, by using the similar analysis as that for $Q = x^2 - 1$, we obtain that the topological phase portrait of $\mathbf{X}^1$ is equivalent to Figure 1(13), respectively Figure 1(14), if $a \neq 0$, respectively $a = 0$. This completes the proof of the theorem. $\square$

6. Proof of Theorem B. Let $H(x, y)$ be a polynomial first integral of degree 2. As $H(x, y) = 0$ is a conic, by using some linear changes of coordinates $H(x, y)$ is equivalent to one of the following four polynomials: $x^2 + y^2, xy, y - x^2$ and x^2; for more details, see the proof of Theorem A, in which by changing $x + y$ to $\bar{x}$ and $x - y$ to $\bar{y}$, then $x^2 - y^2$ is transformed to $\bar{x}\bar{y}$. If $H = x^2$ is a first integral, then $H = x$ is also a first integral. Thus x^2 is not a minimal polynomial first integral. Therefore, if $H(x, y)$ is minimal, then it must be either $x^2 + y^2$, xy or $y - x^2$.

By using

$$P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} \equiv 0,$$

i.e.,

$$-\frac{\partial H/\partial y}{\partial H/\partial x} = \frac{P}{Q},$$

we have that there exists a function $A(x, y) = ax + by + c$ with $a^2 + b^2 \neq 0$ such that if $H = x^2 + y^2$, then $P = -yA$ and $Q = xA$; and that if

$H = xy$, then $P = -xA$ and $Q = yA$; and that if $H = y - x^2$, then $P = A$ and $Q = 2xA$. Hence, we get the three vector fields $\mathbf{X}_i^2$ for $i = 1, 2, 3$, of Theorem B. From Theorem A we know that the vector field $\mathbf{X}_i^2$, $i = 1, 2, 3$, has no polynomial first integrals of degree 1. Otherwise, P and Q would have a common factor of degree 2 which will contradict the above facts. Therefore, the functions $x^2 + y^2$, xy and $y - x^2$ are minimal polynomial first integrals of degree 2 for the vector fields $\mathbf{X}_1^2, \mathbf{X}_2^2$ and $\mathbf{X}_3^2$, respectively. So the first part of Theorem B is proved.

For the vector field $\mathbf{X}_1^2$, if $c = 0$, all its finite singular points coincide with the points of the straight line $ax + by + c = 0$ denoted by l. If $c \neq 0$, the vector field $\mathbf{X}_1^2$ has a unique isolated finite singular point $(0, 0)$ outside the finite singular points of l. The compactified vector field is $(z_1z_2(a+bz_1+cz_2), (a+bz_1+cz_2)(1+z_1^2))$ and $(-z_1z_2(az_1+b+cz_2), -(az_1+b+cz_2)(1+z_1^2))$ in the local charts U_1 and U_2, respectively, and so has a unique pair of infinite singular points $[-(a/b), 0]$ in U_1 if $b \neq 0$ or $[-(b/a), 0]$ in U_2 if $a \neq 0$. Moreover, the vector field $\mathbf{X}_1^2$ outside the straight line l is equivalent to the vector field $(-y, x)$ for which all the trajectories are circles with same center (0,0). So, by distinguishing whether the straight line l passes through the center or not, we get the topological phase portraits given in Figures 2(1) and 2(2).

The vector field $\mathbf{X}_2^2$ has the straight line $ax + by + c = 0$ denoted by l, which is full of singular points, and a unique isolated finite singular point $(0, 0)$ (which is a saddle) if $c \neq 0$. The compactified vector field is $(z_2(a+bz_1+cz_2), 2z_1(a+bz_1+cz_2))$ and $(-z_2(az_1+b+cz_2), -2z_1(az_1+b+cz_2))$ in the local charts U_1 and U_2, respectively. So the infinite singular points are $(0, 0)$, $[-(a/b), 0] \in U_1$ and $(0, 0) \in U_2$ if $a \neq 0$ and $b \neq 0$ or $(0, 0) \in U_1$ and $(0, 0) \in U_2$ if $a = 0$ or $b = 0$. Moreover, all the orbits of the vector field $\mathbf{X}_2^2$ are contained in the hyperbolas: $xy = k$, $k \neq 0$, except for the two invariant straight lines: $x = 0$ and $y = 0$, and the straight line l.

Assume that $a = 0$ or $b = 0$. Then when $c = 0$, the straight line l coincides with one invariant straight line, we get the topological phase portrait given in Figure 2(3). When $c \neq 0$, the straight line l is parallel to one invariant straight line but does not coincide with it, we get the topological phase portrait given in Figure 2(4).

Assume that $a \neq 0$ and $b \neq 0$. Then, when $c = 0$, the straight line l

passes through the intersection point of the two invariant straight lines, the topological phase portrait is equivalent to Figure 2(5). When $c \neq 0$, the straight line l does not pass through the intersection point, i.e., the unique isolated finite saddle, of the two invariant straight lines, so the topological phase portrait is equivalent to Figure 2(6).

For the vector field $\mathbf{X}_3^2$, the straight line l is full of singular points, and there are no other finite singular points. The compactified vector field is $(-z_2^2(a+bz_1+cz_2),(2-z_1z_2)(a+bz_1+cz_2))$ and $(-2z_1z_2(az_1+b+cz_2),(z_2-2z_1^2)(az_1+b+cz_2))$ in the local charts U_1 and U_2, respectively. So the infinite singular points are $[0,-(a/b)] \in U_1$ and $(0,0) \in U_2$ if $b \neq 0$, or $(0,0) \in U_2$ if $b = 0$. Moreover, except the singular points on l, all other trajectories of $\mathbf{X}_3^2$ are contained in parabolas: $y = x^2 + c$, $c \in \mathbf{R}$. Therefore, the topological phase portrait is equivalent to Figure 2(7) if $b = 0$ or Figure 2(8) if $b \neq 0$.

This completes the proof of Theorem B. $\square$

7. Proof of Theorem C. By Lemma 4.1 of [**3**], a quadratic system having a polynomial first integral of degree 3 is topologically equivalent to one of the four vector fields $\mathbf{X}_i^3$ for $i = 1,2,3$ and 4 of Theorem C. We know from Theorem A that the vector field $\mathbf{X}_i^3$ for $i = 2,3$ and 4, has no polynomial first integrals of degree 1, and that the vector field $\mathbf{X}_1^3$ has a polynomial first integral of degree 1 if and only if $\alpha = b = c = 0$. Under the condition $\alpha^2 + b^2 + c^2 \neq 0$, by Theorems 5.1 and 5.10 of [**3**], the vector field $\mathbf{X}_1^3$ has no topological phase portraits 15 and 16 in Figure 1.1 of [**3**].

We now check the conditions in order that the vector field $\mathbf{X}_i^3$, $i \in \{1,2,3,4\}$ has no minimal polynomial first integrals of degree 2. By Theorem B we know that if a quadratic vector field has a minimal polynomial first integral of degree 2, then it has a unique straight line formed by singular points. The following proof depends on this fact.

For the vector field $\mathbf{X}_1^3$, it cannot be equivalent to any one of the vector fields $\mathbf{X}_i^2$ for $i = 1,2,3$ and 4. It is due to Neumann's lemma and the fact that the vector field $\mathbf{X}_1^3$ has no straight lines formed by singular points. So the vector field $\mathbf{X}_1^3$ has no polynomial first integrals of degree 2.

For the vector field $\mathbf{X}_2^3$, it has a straight line formed by singular points if and only if $\alpha = \beta = b = c = 0$. So the vector field $\mathbf{X}_2^3$ has a minimal polynomial first integral of degree 3 if and only if $\alpha^2+\beta^2+b^2+c^2\neq 0$. Furthermore, by Theorem 5.2 of [**3**], the vector field $\mathbf{X}_2^3$ has no phase portrait 14 in Figure 1.1 of [**3**] under this last condition.

The vector fields $\mathbf{X}_3^3$ and $\mathbf{X}_4^3$ have no straight lines formed by singular points for any selection of their parameters, so the degree of their minimal polynomial first integrals is three.

This completes the proof of Theorem C. □

8. Proof of Theorem D. Let $H(x,y)$ be a polynomial first integral of degree 4 for system (1). Assume that P,Q and H have the forms given in (11) with $n=4$. By Proposition 1, it follows that

$$\frac{\partial H}{\partial y}=P(ax+by+c), \qquad \frac{\partial H}{\partial x}=-\,Q(ax+by+c), \tag{16}$$

where a,b and c are real constants with $a^2+b^2\neq 0$.

By Theorem 2 and equations (16) we know that if $H(x,y)$ is a polynomial first integral, then (a,b,c) must satisfy the following linear algebraic equations

$$\begin{aligned}&(b_{11}+3a_{20})a+b_{20}b=0,\\&(a_{11}+b_{02})a+(a_{20}+b_{11})b=0,\\&a_{02}a+(a_{11}+3b_{02})b=0,\end{aligned} \tag{17}$$

and

$$\begin{aligned}&(b_{11}+2a_{20})c+(b_{01}+2a_{10})a+b_{10}b=0,\\&(a_{11}+2b_{02})c+a_{01}a+(a_{10}+2b_{01})b=0,\\&(a_{10}+b_{01})c+a_{00}a+b_{00}b=0.\end{aligned} \tag{18}$$

In what follows, we distinguish three different cases.

Case 1. $a=0$, $b\neq 0$. Without loss of generality, let $b=1$. It follows from equations (17) and (18) that $b_{20}=0$, $a_{20}=-b_{11}$, $a_{11}=-3b_{02}$, $b_{10}=b_{11}c$, $a_{10}=b_{02}c-2b_{01}$ and $b_{02}c^2-b_{01}c+b_{00}=0$. Hence, quadratic system (1) has the form (A). Furthermore, by using the expressions of c_{ij} in Theorem 2, the expression for first integral

$H(x,y)$ follows. Moreover, by straightforward calculations, we know that quadratic system (A) has no polynomial first integrals of degree less than 4. This means that the polynomial $H(x,y)$ is a minimal first integral of degree 4.

Case 2. $a \neq 0$ and $b = 0$. By changing the variables (x,y) to (y,x), we get that this class of systems is equivalent to the class $a = 0$ and $b \neq 0$.

Case 3. $a \neq 0$ and $b \neq 0$. Since system (1) is quadratic, we have that $a_{20}^2 + a_{11}^2 + a_{02}^2 + b_{20}^2 + b_{11}^2 + b_{02}^2 \neq 0$, so all the coefficients of the linear algebraic equations with respect to a and b in (17) cannot be zero simultaneously. The rest of the proof of the theorem is divided into six subcases.

Subcase 1. Assume that only the coefficients of the first and third equations in (17) are all equal to zero. Then

$$b_{20} = 0, \quad b_{11} = -3a_{20}, \quad a_{02} = 0,$$
$$a_{11} = -3b_{02}, \quad a_{20} \neq 0, \quad b_{20} \neq 0, \quad a = -\frac{a_{20}}{b_{02}}b,$$

and $b \neq 0$. Without loss of generality, let $b = 1$. From (18) we get that

$$c = \frac{b_{10}}{a_{20}} - \frac{b_{01} + 2a_{10}}{b_{02}},$$

and conditions (4) hold. By Theorem 2 we know that quadratic system (1) has the form (B), and the corresponding first integral $H(x,y)$. Straightforward calculations also show that the polynomial first integral $H(x,y)$ is minimal.

Subcase 2. Assume that only the coefficients of the first and second equations in (17) are all equal to zero. Working a similar way to the previous subcase, we obtain that qudratic system (1) has the form (C) with $a_{20} = b_{20} = 0$, and the corresponding minimal polynomial first integral $H(x,y)$ of degree 4.

Subcase 3. If only the coefficients of the second and third equations in (17) are all equal to zero, by interchanging the variables x and y, we get system (C) and the corresponding minimal polynomial first integral of degree 4 as in subcase 2.

Subcase 4. Assume that only the coefficients of the first equation in (17) are all equal to zero. Then we have that $b_{20}=0$ and $b_{11}=-3a_{20}$. Since $ab\neq 0$, it follows that $a_{20}a_{02}\neq 0$ and $a_{11}+3b_{02}\neq 0$. By (17) we get that

$$a=-\frac{a_{11}+3b_{02}}{a_{02}}b, \qquad a_{11}^2+4a_{11}b_{02}+3b_{02}^2+2a_{20}a_{02}=0.$$

By using equations (18), we get that $c=k_Db$ and the last two conditions in (6) hold. Theorem 2 guarantees that quadratic system (1) is of the form (D) and has the corresponding first integral $H(x,y)$. By straightforward calculations it is easy to see that the polynomial first integral is also minimal.

Subcase 5. If only the coefficients of the second, respectively third, equation in (17) are all equal to zero, then working in a similar way to the previous subcases, we can also obtain system (C), respectively system (D), interchanging the variables x and y, and the corresponding minimal polynomial first integrals $H(x,y)$ of degree 4.

Subcase 6. Finally, we assume that all the coefficients of system (17) are not equal to zero. Then equations (17) has nonzero solutions if and only if the second and third conditions in (7) hold. Moreover, we have

$$a=-\frac{a_{11}+3b_{02}}{a_{02}}b.$$

Substituting a into (18) gives

$$\begin{aligned}
&a_{02}(a_{10}+b_{01})c+[a_{02}b_{00}-a_{00}(a_{11}+3b_{02})]b=0,\\
&a_{02}(b_{11}+2a_{20})c+[a_{02}b_{10}-(b_{01}+2a_{10})(a_{11}+3b_{02})]b=0,\\
&a_{02}(a_{11}+2b_{02})c+[a_{02}(a_{10}+2b_{01})-a_{01}(a_{11}+3b_{02})]b=0.
\end{aligned} \tag{19}$$

If all the coefficients of c in (19) are zero, then we have (since $b\neq 0$)

$$\begin{aligned}
&a_{10}=-b_{01}, \quad b_{11}=-2a_{20}, \quad a_{11}=-2b_{02},\\
&b_{00}=a_{00}\frac{b_{02}}{a_{02}}, \quad b_{10}=a_{10}\frac{b_{02}}{a_{02}}, \quad b_{01}=a_{01}\frac{b_{02}}{a_{02}}.
\end{aligned}$$

Under these last conditions, the second and third conditions in (7) reduce to $a_{20}b_{02} = a_{02}b_{20}$ and $a_{20}a_{02} = b_{02}^2$. So the vector field associated to system (1) is of the form $[P, (b_{02}/a_{02})P]$. By Theorem 1 it has a minimal polynomial first integral $b_{02}x - a_{02}y$ of degree 1.

Assume that the coefficients of c in (19) are not all equal to zero. If $a_{10} + b_{01} \neq 0$, then $c = k_E b$. Substituting c into the last two equations in (19) yields the last two conditions in (7). Therefore, it follows from Theorem 2 that quadratic system (1) has the polynomial first integral (8). Moreover, straightforward calculation shows that the polynomial first integral (8) is also minimal.

Similarly, if $b_{11} + 2a_{20} \neq 0$, we can prove that when quadratic system (1) satisfies the conditions in (F), then it has a polynomial first integral of the form (8) with k satisfying (9). Furthermore, by straightforward calculations, we can obtain that the corresponding first integral is also minimal.

When $a_{11} + 2b_{02} \neq 0$, by the symmetry of a and b in (17) and (18) and interchanging x and y, we obtain that this class is equivalent to the class (F).

This completes the proof of Theorem D. $\square$

9. Proof of Theorem E. By the assumptions and Proposition 8, we know that quadratic system (1) has a polynomial first integral of degree 4 if and only if the linear algebraic equations (14) have a nonzero solution (l, m, n) and a, b and c satisfy equations (15). We separate the proof into three cases.

Case 1. $a_2 \neq 0$. Solving the first and second equations in (14) gives

$$m = -\frac{3a_1 + b_2}{a_2}l, \qquad n = \left[\frac{(a_1 + b_2)(3a_1 + b_2)}{a_2^2} - \frac{b_1}{a_2}\right]l.$$

Substituting m and n into the third equation in (14), we have

$$l(a_1 + b_2)[4a_2b_1 - (a_1 + 3b_2)(3a_1 + b_2)] = 0. \tag{20}$$

If $l = 0$, then $m = n = 0$. By (10) quadratic system (1) has no polynomial first integral of degree 4. So $l \neq 0$ and, without loss of generality, we can assume that $l = 1$.

If $a_1+b_2 \neq 0$, then $4a_2b_1-(a_1+3b_2)(3a_1+b_2)=0$. As the determinant of the coefficients with respect to b and c in the second and third equations of (15) is

$$\begin{vmatrix} 2a_1+b_2 & a_2 \\ b_1 & a_1+2b_2 \end{vmatrix} = \frac{5(a_1+b_2)^2}{4} \neq 0,$$

equations (15) have a solution (a,b,c) as shown in Theorem E(a). By Proposition 8, quadratic system (1) has a polynomial first integral of the form (10). By straightforward calculations, we know that under the conditions of Theorem E(a), quadratic system (1) has no polynomial first integrals of degree less than 4. So statement (a) in Theorem E follows.

If $a_1+b_2=0$, equations (15) can be written as

$$\begin{aligned} &c_1b+c_2c=0,\\ &a_1b+a_2c+2c_1l+c_2m=0,\\ &b_1b+b_2c+c_1m+2c_2n=0. \end{aligned} \tag{21}$$

When $a_1b_2-a_2b_1 \neq 0$, then we have

$$l=1, \quad m=-\frac{2a_1}{a_2}, \quad n=-\frac{b_1}{a_2}, \quad b=\frac{2c_2}{a_2}, \quad c=-\frac{2c_1}{a_2},$$

and a an arbitrary constant. Therefore, quadratic system (1) has the minimal polynomial first integral

$$H(x,y)=b_1y^2+2a_1xy-a_2x^2+2c_1y-2c_2x,$$

when $a_1b_2-a_2b_1=-b_2^2-a_2b_1=0$. If $b_2=0$, equations (21) reduce to $c=-(2c_1/a_2)$ and $c_1[b-(2c_2/a_2)]=0$. As $c_1 \neq 0$ (otherwise, $P(x,y)\equiv 0$), we have

$$l=1, \quad m=n=0, \quad b=\frac{2c_2}{a_2}, \quad c=-\frac{2c_1}{a_2}.$$

Moreover, under these last conditions, quadratic system (1) has the minimal polynomial first integral $H(x,y)=a_2x^2-2c_1y+2c_2x$.

If $b_2 \neq 0$, equations (21) reduce to

$$b = \frac{a_2}{b_2}c + \frac{2c_1}{b_2} + \frac{2c_2}{a_2}, \qquad (a_2c_1 + b_2c_2)\left(c + \frac{2c_1}{a_2}\right) = 0.$$

Since $a_2c_1 + b_2c_2 \neq 0$ (otherwise, P and Q have a common factor of degree 2), so we have

$$l = 1, \quad m = \frac{2b_2}{a_2}, \quad n = -\frac{b_1}{a_2}, \quad b = \frac{2c_2}{a_2}, \quad c = -\frac{2c_1}{a_2}.$$

Moreover, under these last conditions, quadratic system (1) has the minimal polynomial first integral

$$H(x, y) = b_2^2y^2 + 2a_2b_2xy + a_2x^2 - 2a2c_1y + 2a_2c_2x.$$

Case 2. $a_2 = 0$, $b_1 \neq 0$. Solving equations (14), we obtain

$$m = -\frac{a_1 + 3b_2}{b_1}n, \qquad l = \frac{(a_1 + b_2)(a_1 + 3b_2)}{b_1^2}n,$$

and

$$n(a_1 + b_2)(a_1 + 3b_2)(3a_1 + b_2) = 0. \tag{22}$$

In what follows, without loss of generality, we can assume that $n = 1$. By using conditions (15) and by distinguishing the two cases $a_1 = -b_2 = 0$ and $a_1 = -b_2 \neq 0$, we have respectively

$$l = m = 0, \quad n = 1, \quad b = -\frac{2c_2}{b_1}, \quad c = \frac{2c_1}{b_1},$$

$$l = 0, \quad m = -\frac{2b_2}{b_1}, \quad n = 1,$$

$$b = \frac{2b_2c_2}{a_1b_1}, \quad c = \frac{2b_2c_2}{a_1^2} - \frac{2b_2c_1}{a_1b_1} + \frac{2c_2}{a_1}.$$

Correspondingly, quadratic system (1) has the minimal polynomial first integrals $H(x, y) = b_1y^2 + 2c_1y - 2c_2x$ and $H(x, y) = b_1y^2 - 2b_2xy + 2c_1y - 2c_2x$, respectively.

Moreover, under the condition $a_1 = -3b_2 \neq 0$, respectively, $b_2 = -3a_1 \neq 0$, from the equations (14) and (15) we can get the values of l, m, n, a, b and c. It follows that the statement (b), respectively (c), holds.

Case 3. $a_2 = b_1 = 0$. Since $a_1 = b_2 = 0$ or $l = m = n = 0$, by (10) quadratic system (1) has no polynomial first integrals of degree 4. So we know from (14) and (15) that quadratic system (1) has a polynomial first integral of degree 4 if and only if one of the following three conditions holds: $b_2 = -3a_1 \neq 0$, $a_1 = -b_2 \neq 0$ or $a_1 = -3b_2 \neq 0$. Under the first condition, the statement (d) follows. Under the second condition, quadratic system (1) has the minimal polynomial first integral $H(x, y) = a_1xy - c_2x + c_1y$ of degree 2. The third condition is equivalent, by changing the variables $(x, y) \to (y, x)$ to the first one. This completes the proof of Theorem E. □

10. Proof of Theorem F. We will start from the classification given by Gasull, Liren and Llibre (see Lemma 3) of quadratic systems (1). In the following we assume that $P/Q \not\equiv$ constant, and that Q is of the form given in (11). System (X) is equivalent to the quadratic vector field $a(Q, Q)$ with $a \in \mathbf{R}$, and has a minimal polynomial first integral of degree 1. Hence, in the following we will prove our conclusions for systems (I)–(IX). Since the complete proof is very long, in the following we only give the main idea to prove our theorem. If some reader is interested in its complete proof, he can find it in the research report [**17**].

Lemma 9. *Quadratic system* (I): $\dot{x} = 1 + xy$, $\dot{y} = Q(x, y)$ *with a minimal polynomial first integral of degree* 4, *has one of the eight phase portraits given in Figures* 4(1)–4(8).

Proof. By some computations and Theorem D, we get that system (I) with a minimal polynomial first integral of degree 4 is equivalent to one of the following two systems:

$$(\mathrm{I}_1) \qquad \dot{x} = 1 + xy, \qquad \dot{y} = \frac{1}{6}c^2 - \frac{1}{6}cy - \frac{1}{3}y^2,$$

with the minimal polynomial first integral of degree 4

$$H(x,y) = -\frac{1}{6}c^3x + cy + \frac{1}{2}y^2 + \frac{1}{2}cxy^2 + \frac{1}{3}xy^3. \tag{23}$$

$$(\mathrm{I}_2) \quad \begin{aligned} \dot{x} &= 1 + xy, \\ \dot{y} &= \frac{3}{2}b_{11} + 6b_{01}^2 + \frac{15}{2}b_{01}b_{11}x + b_{01}y + \frac{3}{2}b_{11}^2x^2 + b_{11}xy - \frac{1}{3}y^2, \end{aligned}$$

where $b_{11} \neq 0$ and its minimal polynomial first integral of degree 4 is

$$\begin{aligned} H(x,y) = &-6b_{01}\left(1 + 4\frac{b_{01}^2}{b_{11}}\right)x + 4\frac{b_{01}}{b_{11}}y - 3\left(6b_{01}^2 + \frac{b_{11}}{4}\right)x^2 + xy \\ &- \frac{1}{3b_{11}}y^2 - \frac{9}{2}b_{01}b_{11}x^3 + \frac{2b_{-1}}{b_{11}}xy^2 - \frac{3}{8}b_{11}^2x^4 + \frac{1}{2}x^2y^2 - \frac{2}{9b_{11}}xy^3. \end{aligned}$$

For system (I_1), the compactified vector field is $(-z_2^3 - z_1z_2, (1/6)c^2z_2^2 - (1/6)cz_1z_2 - z_1z_2^2 - (4/3)z_1^2)$ and $(-(1/6)c^2z_2^3 + (1/6)cz_2^2 + (1/3)z_2, z_2^2 - (1/6)c^2z_1z_2^2 + (1/6)cz_1z_2 + (4/3)z_1)$ in the local charts U_1 and U_2, respectively. The polynomial first integral can be rewritten as

$$\begin{aligned} H(x,y) + \frac{1}{2}c^2 &= (y+c)^2\left[\frac{1}{6}x(2y-c) + \frac{1}{2}\right], \\ H(x,y) - \frac{5}{8}c^2 &= \left(y - \frac{c}{2}\right)\left[\frac{1}{3}x(y+c)^2 + \frac{1}{4}(2y+5c)\right]. \end{aligned}$$

From these expressions of the first integral, and by studying the singular points, we obtain that the phase portrait of system (I_1) is topologically equivalent to Figure 4(1) if $c \neq 0$ or to Figure 4(2) if $c = 0$.

For system (I_2), the local phase portrait of this system at all singular points in the Poincaré disc is topologically equivalent to

Figure 5(1)	if	$-\frac{3}{4}b_{01}^2 < b_{11} < 6b_{01}^2$ and $b_{11} > 0$;
Figure 5(2)	if	$-\frac{3}{4}b_{01}^2 < b_{11} < 6b_{01}^2$ and $b_{11} < 0$;
Figure 5(3)	if	$b_{11} = 6b_{01}^2 > 0$;
Figure 5(4)	if	$b_{11} > 6b_{01}^2$;
Figure 5(5)	if	$b_{11} = -\frac{3}{4}b_{01}^2 < 0$;
Figure 5(6)	if	$b_{11} < -\frac{3}{4}b_{01}^2$.

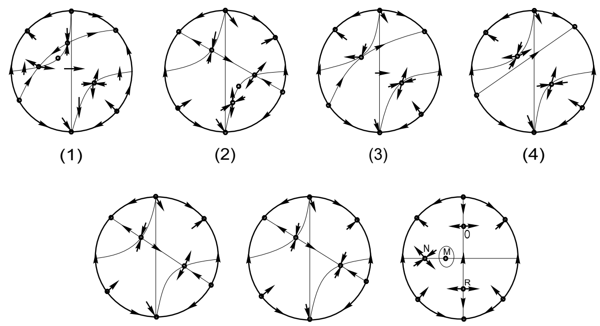

FIGURE 5. The local phase portraits at singular points in the Poincaré disc for some quadratic systems having minimal polynomial first integrals, where the hyperbolas are vertical isoclines in Figures 5(1)–5(6).

So, by Neumann's lemma, we can get the global phase portraits given in Figures 4(3)–4(8). Hence, the lemma follows. $\square$

Lemma 10. *Quadratic system* (II). $\dot{x} = xy$, $\dot{y} = Q(x,y)$, *with a minimal polynomial first integral of degree* 4, *has one of the* 22 *phase portraits given in Figures* 4(4)–4(25).

Proof. From Theorems D and E we get that system (II) having a minimal polynomial first integral of degree 4 is equivalent to one of the following seven systems:

$$(\mathrm{II}_1) \qquad \dot{x} = xy, \qquad \dot{y} = \frac{1}{6}c^2 - \frac{1}{6}cy - \frac{1}{3}y^2,$$

with the minimal polynomial first integral

$$H(x,y) = -\frac{1}{6}c^3x + \frac{1}{2}cxy^2 + \frac{1}{3}xy^3;$$

$$(\mathrm{II}_2) \qquad \dot{x} = xy, \qquad \dot{y} = b_{00} + b_{10}x + b_{20}x^2 - y^2,$$

with the minimal polynomial first integral

(II_3)

$$H(x,y) = -\frac{1}{2}b_{00}x^2 - \frac{1}{3}b_{10}x^3 - \frac{1}{4}b_{20}x^4 + \frac{1}{2}x^2y^2;$$

$$\dot{x} = xy, \qquad \dot{y} = 6b_{01}^2 + \frac{15}{2}b_{01}b_{11}x + b_{01}y + \frac{3}{2}b_{11}^2x^2 + b_{11}xy - \frac{1}{3}y^2,$$

where $b_{11} \neq 0$ and its minimal polynomial first integral is

$$H(x,y) = -24\frac{b_{01}^3}{b_{11}}x - 18b_{01}^2x^2 - \frac{9}{2}b_{01}b_{11}x^3 + \frac{2b_{01}}{b_{11}}xy^2 - \frac{3}{8}b_{11}^2x^4 + \frac{1}{2}x^2y^2 - \frac{2}{9b_{11}}xy^3.$$

$$(\mathrm{II}_4) \qquad \dot{x} = xy, \qquad \dot{y} = x\left(\frac{3}{4}b_2^2x + b_2y + c_2\right),$$

where $b_2 \neq 0$, its minimal polynomial first integral is

$$H(x,y) = -\frac{10}{9b_2^4}c_2^3x - \frac{13c_2^2}{12b_2^2}x^2 + \frac{5c_2^2}{9b_2^4}y^2 - \frac{2}{3}c_2x^3 + \frac{2c_2}{3b_2^2}xy^2 - \frac{10c_2}{27b_2^3}y^3 - \frac{3}{16}b_2^2x^4 + \frac{1}{2}x^2y^2 - \frac{4}{9b_2}xy^3 + \frac{1}{9b_2}y^4;$$

$$(\mathrm{II}_5) \qquad \dot{x} = xy, \qquad \dot{y} = y(a_2x - 3y + c_2),$$

with the minimal polynomial first integral

$$H(x,y) = \frac{1}{3}c_2x^3 - \frac{1}{4}a_2x^4 + x^3y;$$

$$(\mathrm{II}_6) \qquad \dot{x} = xy, \qquad \dot{y} = y\left(a_2x - \frac{1}{3}y + c_2\right),$$

with the minimal polynomial first integral

$$H(x,y) = -\frac{16}{a_2^2}c_2^3x - \frac{12c_2^2}{a_2}x^2 + \frac{16c_2^2}{a_2^2}xy - 3c_2x^3 + \frac{8c_2}{a_2}x^2y - \frac{16c_2}{3a_2^2}xy^2 - \frac{a_2}{4}x^4 + x^3y - \frac{4}{3a_2}x^2y^2 + \frac{16}{27a_2^2}xy^3;$$

$$(\mathrm{II}_7) \qquad \dot{x} = xy, \qquad \dot{y} = y\left(-\frac{1}{3}y + c_2\right),$$

with the minimal polynomial first integral

$$H(x,y) = -9c_2^3x + 9c_2^2xy - 3c_2xy^2 + \frac{1}{3}xy^3.$$

For system (II_1), it is easy to prove that the global phase portrait is topologically equivalent to Figure 4(9) if $c \neq 0$ or to Figure 4(10) if $c = 0$.

For system (II_2), without loss of generality, we can assume that $b_{10} \geq 0$. The compactified vector field is $(-z_1 z_2, b_{00} z_2^2 + b_{10} z_2 + b_{20} - 2z_1^2)$ and $(-b_{00} z_2^3 - b_{10} z_1 z_2^2 - b_{20} z_1^2 z_2 + z_2, -b_{00} z_1 z_2^2 - b_{10} z_1^2 z_2 - b_{20} z_1^3 + 2z_1)$ in the local charts U_1 and U_2, respectively.

Case 1. $b_{00} > 0$. System (II_2) has two finite singular points $O = (0, \sqrt{b_{00}})$ and $R = (0, -\sqrt{b_{00}})$ on the y-axis, which are both saddles.

Subcase 1. $b_{20} > 0$ and $b_{10}^2 - 4b_{20}b_{00} > 0$. The local phase portrait at all singular points is given in Figure 5(7). Moreover, by some computations we get that the global phase portrait is topologically equivalent to Figure 4(11) if $2b_{10}^2 = 9b_{20}b_{00}$ or to Figure 4(4) if $2b_{10}^2 \neq 9b_{20}b_{00}$.

Subcase 2. $b_{20} > 0$ and $b_{10} - 4b_{20}b_{00} = 0$. The unique finite singular point on the x-axis is a cusp. The global phase portrait is topologically equivalent to Figure 4(7).

Subcase 3. $b_{20} > 0$ and $b_{10} - 4b_{20}b_{00} < 0$. Then O and R are two unique finite singular points and the phase portrait is topologically equivalent to Figure 4(8).

Subcase 4. $b_{20} = 0$. Then in U_1 the unique infinite singular point $(0, 0)$ is nilpotent if $b_{10} \neq 0$ or linearly if $b_{10} = 0$. We get that the topological phase portrait is equivalent to Figure 4(12) if $b_{10} \neq 0$ or to Figure 4(9) if $b_{10} = 0$.

Subcase 5. $b_{20} < 0$. System (II_2) has a unique pair of infinite singular points. The phase portrait is topologically equivalent to Figure 4(13).

Case 2. $b_{00} = 0$. The unique finite singular point on the x-axis is degenerate.

Subcase 1. $b_{20} > 0$. Since the polynomial first integral can be written as

$$H(x,y) = \frac{x^2}{2}\left[y^2 - \frac{1}{2}b_{20}\left(x + \frac{2b_{10}}{3b_{20}}\right)^2 + \frac{2b_{10}^2}{9b_{20}}\right],$$

we can get that the phase portrait is topologically equivalent to Figure 4(5) if $b_{10} > 0$ or to Figure 4(14) if $b_{10} = 0$.

Subcase 2. $b_{20} = 0$. By the polynomial first integral $H(x,y) = x^2(y^2/2 - b_{10}x/3)$, we can prove that the phase portrait is topologically equivalent to Figure 4(15) if $b_{10} > 0$ or to Figure 4(10) if $b_{10} = 0$.

Subcase 3. $b_{20} < 0$. If $b_{10} > 0$, the invariant ellipse $y^2/2 - b_{20}x^2/4 - b_{20}x/3 = 0$ is formed by the coincidence of two separatrices of the nilpotent singular point (0,0) and that no other trajectories pass through this singular point. We can obtain that the phase portrait is topologically equivalent to Figure 4(16).

If $b_{10} = 0$, by it polynomial first integral it follows that the phase portrait is topologically equivalent to Figure 4(17).

Case 3. $b_{00} < 0$. Then the possible finite singular points are located on the x-axis.

Subcase 1. $b_{20} > 0$. We can obtain that the phase portrait is also topologically equivalent to Figure 4(6).

Subcase 2. If $b_{10} > 0$, the invariant parabola $y^2/2 - b_{10}x/3 - b_{00}/2 = 0$ passes through the finite saddle $(-b_{00}/b_{10}, 0)$ and the infinite singular point in U_1. We can get that the phase portrait is topologically equivalent to Figure 4(18). If $b_{10} = 0$, it is easy to prove that the phase portrait is topologically equivalent to Figure 4(19).

Subcase 3. $b_{20} < 0$. Let $D = b_{10}^2 - 4b_{20}b_{00}$. From the properties of the singular points, we can obtain that the topological phase portrait is equivalent to Figure 4(20) if $D > 0$, to Figure 4(21) if $D = 0$ or to Figure 4(22) if $D < 0$.

For system (II_3), we can prove that the phase portrait is topologically equivalent to Figure 4(11) if $b_{10} > 0$ or to Figure 4(14) if $b_{10} = 0$.

For system (II_4), by its polynomial first integral $H(x,y)$, we can obtain that the phase portrait is topologically equivalent to Figure 4(23) if $c_2 \neq 0$ or to Figure 4(24) if $c_2 = 0$.

System (II_5) for $a_2 \neq 0$ can be transformed into system (II_4) with $b_2 = -2$. When $a_2 = 0$, we can prove that the phase portrait is topologically equivalent to Figure 4(25), respectively Figure 4(10), if $c_2 \neq 0$, respectively $c_2 = 0$.

System (II_6) has the topological phase portrait given in Figure 4(23) if $c_2 \neq 0$ or in Figure 4(24) if $c_2 = 0$.

System (II_7) has the topological phase portrait given in Figure 4(25) if $c_2 \neq 0$ or in Figure 4(10) if $c_2 = 0$.

Summing up the above arguments, the proof of the lemma is completed. ◻

Lemma 11. *Quadratic system* (III): $\dot{x} = y + x^2$, $\dot{y} = Q(x,y)$ *with a minimal polynomial first integral of degree* 4, *has one of the three phase portraits given in Figures* 4(15), 4(18) *and* 4(26).

Proof. System (III) with a minimal polynomial first integral of degree 4 is equivalent to one of the following two forms

$$(\mathrm{III}_1) \qquad \dot{x} = y + x^2, \qquad \dot{y} = -cx - xy,$$

with the minimal first integral

$$H(x,y) = \frac{1}{2}c^2x^2 + \frac{1}{2}cy^2 + cx^2y + \frac{1}{3}y^3 + \frac{1}{2}x^2y^2;$$

$$(\mathrm{III}_2) \qquad \dot{x} = y + x^2, \qquad \dot{y} = b_{10}b_{01} - 2b_{01}^3 + b_{10}x + b_{01}y + 4b_{01}x^2 - xy,$$

where $b_{01} \neq 0$ and, having the minimal first integral,

$$\begin{aligned} H(x,y) = &- \frac{(b_{10} - 2b_{01}^2)^2}{2}x - \frac{b_{10}^2 - 4b_{01}^4}{4b_{01}}x^2 + \frac{b_{10} - 2b_{01}^2}{4b_{01}}y^2 \\ &- \frac{3b_{10} - 4b_{01}^2}{3}x^3 + \frac{b_{10} - 2b_{01}^2}{2b_{01}}x^2y + \frac{1}{2}xy^2 \\ &- \frac{1}{6b_{01}}y^3 - b_{01}x^4 + x^3y - \frac{1}{4b_{01}}x^2y^2. \end{aligned}$$

The phase portrait of system (III_1) is topologically equivalent to Figure 4(26) if $c > 0$, to Figure 4(15) if $c = 0$ or Figure 4(18) if $c < 0$.

For system (III_2), let $D = 3b_{01}^2 - b_{10} > 0$. We can prove that the topological phase portrait is equivalent to Figure 4(15), respectively 4(18) or 4(26), when $D = 0$, respectively $D < 0$ or $D > 0$. This completes the proof of the lemma. $\square$

Lemma 12. *Quadratic system* (IV): $\dot{x} = y$, $\dot{y} = Q(x, y)$ *has no minimal polynomial first integral of degree* 4.

Proof. By following the steps in the proof of Theorems D and E, it is easy to check the statement of the lemma. $\square$

Lemma 13. *Quadratic system* (V): $\dot{x} = -1 + x^2$, $\dot{y} = Q(x, y)$ *with a minimal polynomial first integral of degree* 4*, has one of the* 3 *topological phase portraits given in Figures* 4(1)*,* 4(9) *and* 4(25)*.*

Proof. From Theorems D and E, system (V) having a minimal polynomial first integral of degree 4 is topologically equivalent to

$$(\mathrm{V}_1) \qquad \dot{x} = -1 + x^2, \qquad \dot{y} = -cx - xy,$$

with the minimal first integral

$$(\mathrm{V}_2) \qquad \begin{gathered} H(x, y) = -cy + \frac{1}{2}c^2x^2 - \frac{1}{y}^2 + cx^2y + \frac{1}{2}x^2y^2; \\ \dot{x} = -1 + x^2, \qquad \dot{y} = b_{00} + b_{10}x + y + b_{20}x^2 - 3xy, \end{gathered}$$

with the minimal first integral

$$(\mathrm{V}_3) \qquad \begin{gathered} H(x, y) = -b_{00}x - y - \frac{1}{2}(b_{10} + b_{00})x^2 - xy \\ - \frac{1}{3}(b_{20} + b_{10})x^3 + x^2y - \frac{1}{4}b_{20}x^4 + x^3y; \\ \dot{x} = -1 + x^2, \qquad \dot{y} = b_{00} + b_{10}x - 2b_{00}x^2 - xy, \quad b_{00} \neq 0, \end{gathered}$$

with the minimal first integral

$$(\mathrm{V}_4)\qquad \begin{aligned} H(x,y) = {} & b_{10}x + \frac{b_{10}}{b_{00}}y - \frac{b_{00}^2 - b_{10}^2}{2b_{00}}x^2 - xy - \frac{1}{2b_{00}}y^2 - b_{10}x^3 \\ & - \frac{b_{10}}{b_{00}}x^2y + \frac{1}{2}b_{00}x^4 + x^3y + \frac{1}{2b_{00}}x^2y^2; \\ \dot{x} = {} & -1 + x^2, \qquad \dot{y} = (x+1)(a_2x - 3y + c_2), \end{aligned}$$

with the minimal first integral

$$(\mathrm{V}_5)\qquad \begin{aligned} H(x,y) = {} & -c_2x - y - \frac{1}{2}(a_2 - 2c_2)x^2 + 3xy + \frac{1}{3}(2a_2 - c_2)x^3 \\ & - 3x^2y - \frac{1}{4}a_2x^4 + x^3y; \\ \dot{x} = {} & -1 + X^2, \qquad \dot{y} = (x+1)\left(a_2x - \frac{1}{3}y + c_2\right), \end{aligned}$$

with $a_2 \neq 0$ and the minimal first integral

$$(\mathrm{V}_6)\qquad \begin{aligned} H(x,y) = {} & \frac{a_2}{4}(x-1)\left(\frac{4}{3a_2}y - x - 3 - \frac{4c_2}{a_2}\right)^3; \\ \dot{x} = {} & -1 + x^2, \qquad \dot{y} = (x+1)\left(-\frac{1}{3}y + c_2\right), \end{aligned}$$

with the minimal first integral

$$H(x,y) = \frac{1}{3}(x-1)(y - 3c_2)^3.$$

System (V_1) has the unique topological phase portrait given in Figure 4(9).

For system (V_2), let $D = 3b_{00} + b_{10} + b_{20}$. We can prove that the global phase portrait is topologically equivalent to Figure 4(9) or 4(1) if $D = 0$ or $D \neq 0$, respectively.

System (V_3) with the polynomial first integral rewritten as

$$H(x,y) = \frac{1}{2b_{00}}[(y + b_{00}x - b_{10})^2(x^2 - 1) + b_{10}^2],$$

has the unique topological phase portrait given in Figure 4(9).

For systems (V_4)–(V_6) we can obtain that the topological phase portrait is equivalent to Figure 4(25). This completes the proof of the lemma. □

Lemma 14. *Quadratic system* (VI): $\dot{x} = 1 + x^2$, $\dot{y} = Q(x, y)$ *with a minimal polynomial first integral of degree* 4*, has a unique topological phase portrait given in Figure* 4(19)*.*

Proof. System (VI) having a minimal polynomial first integral of degree 4 has one of the following two forms:

$$(\mathrm{VI}_1) \qquad \dot{x} = 1 + x^2, \qquad \dot{y} = -cx - xy,$$

with a minimal first integral

$$(\mathrm{VI}_2) \qquad \begin{gathered} H(x,y) = cy + \frac{1}{2}c^2x^2 + \frac{1}{2}y^2 + cx^2y + \frac{1}{2}x^2y^2; \\ \dot{x} = 1 + x^2, \qquad \dot{y} = b_{00} + b_{10}x + 2b_{00}x^2 - xy, \quad b_{00} \neq 0, \end{gathered}$$

with the minimal first integral

$$\begin{aligned} H(x,y) = &- b_{10}x + \frac{b_{10}}{b00}y - \frac{b_{10}^2 + b_{00}^2}{2b_{00}}x^2 + xy - \frac{1}{2b_{00}}y^2 \\ &- b_{10}x^3 + \frac{b_{10}}{b_{00}}x^2y - \frac{1}{2}b_{00}x^4 + x^3y - \frac{1}{2b_{00}}x^2y^2. \end{aligned}$$

System (VI_1) is equivalent to system (II_2) with $b_{00} = -1$ and $b_{10} = b_{20} = 0$; the phase portrait is topologically equivalent to Figure 4(19).

System (VI_2) with the polynomial first integral rewritten as

$$H(x,y) = -\frac{1}{2b_{00}}[(x^2+1)(b_{10} + b_{00}x - y)^2 - b_{10}^2],$$

also has the topological phase portrait given in Figure 4(19). □

Lemma 15. *Quadratic system* (VII): $\dot{x} = x^2$, $\dot{y} = Q(x, y)$ *with a minimal polynomial first integral of degree* 4*, has one of the two topological phase portraits given in Figures* 4(10) *and* 4(27)*.*

Proof. System (VII) having a minimal polynomial first integral of degree 4 is equivalent to

$$\text{(VII}_1\text{)} \qquad \dot{x} = x^2, \qquad \dot{y} = b_{00} + b_{10}x + b_{20}x^2 - 3xy,$$

with a minimal polynomial first integral

$$\text{(VII}_2\text{)} \qquad \begin{gathered} H(x,y) = -\frac{1}{2}b_{00}x^2 - \frac{1}{3}b_{10}x^3 - \frac{1}{4}b_{20}x^4 + x^3y; \\ \dot{x} = x^2, \qquad \dot{y} = x\left(a_2x - \frac{1}{3}y + c_2\right), \end{gathered}$$

with $a_2 \neq 0$, and the minimal first integral

$$\text{(VII}_3\text{)} \qquad \begin{gathered} H(x,y) = \frac{a_2}{4}x\left(\frac{4}{3a_2}y - x - \frac{4c_2}{a_2}\right)^3; \\ \dot{x} = x^2, \qquad \dot{y} = x\left(-\frac{1}{3}y + c_2\right), \end{gathered}$$

with the minimal first integral

$$H(x,y) = -9c_2^3x + 9c_2^2xy - 3c_2xy^2 + \frac{1}{3}xy^3.$$

System (VII$_1$) has the topological phase portrait given in Figure 4(27) if $b_{00} \neq 0$ or in Figure 4(10) if $b_{00} = 0$.

Systems (VII$_2$) and (VII$_3$) have the unique topological phase portrait given in Figure 4(10). □

Lemma 16. *Quadratic system* (VIII): $\dot{x} = x$, $\dot{y} = Q(x,y)$ *with a minimal polynomial first integral of degree* 4*, has a unique topological phase portrait given in Figure* 4(28).

Proof. System (VIII) having a minimal polynomial first integral of degree 4 is topologically equivalent to the following system:

$$\dot{x} = x, \qquad \dot{y} = b_{00} + b_{10}x - 2y + b_{20}x^2, \quad b_{20} \neq 0,$$

with the minimal polynomial first integral

$$H(xy) = -\frac{1}{2}b_{00}x^2 - \frac{1}{3}b_{10}x^3 + x^2y - \frac{1}{4}b_{20}x^4.$$

Its phase portrait is topologically equivalent to Figure 4(28). $\square$

Lemma 17. *Quadratic system* (IX): $\dot{x} = 1$, $\dot{y} = Q(x, y)$ *has no polynomial first integral of degree* 4.

Proof. Following the steps of the proof of Theorem D, it is easy to verify the argument of the lemma. $\square$

Summarizing the above nine lemmas, we complete the proof of Theorem F.

11. Proof of Theorem G. We claim that for a polynomial differential system (1), there cannot coexist infinitely many invariant straight lines and minimal polynomial first integrals of degree larger than 1.

Now we prove the claim. If the system has infinitely many invariant straight lines, then by Darboux theory of integrability (see, for instance, [**13**], [**9**]) the system has a rational first integral. Therefore, from the work of Poincaré on rational first integrals [**21**] we know that there exists a rational first integral $f/g = f(x, y)/g(x, y)$ of the system such that $f - cg$ with $c \in \mathbf{R}$ factorizes only for finitely many values of c. So if there are infinitely many invariant straight lines, the degrees of the polynomials f and g must be 1. Consequently, all the trajectories of the system are contained on invariant straight lines. But this is in contradiction with the fact that the system has a minimal polynomial first integral of degree larger than 1. In short, the claim is proved.

We now assume that system (1) has finitely many invariant straight lines. By Theorem A and the claim it follows that the quadratic functions P and Q of system (1) have no common factors of degree 2. Otherwise, the quadratic vector field (P, Q) has a minimal polynomial first integral of degree 1 and infinitely many invariant straight lines.

If P and Q have a common factor of degree 1, then the quadratic vector field (P, Q) has a singular straight line, denoted by $A = 0$.

Outside the line $A = 0$, the quadratic vector field (P, Q) is equivalent, scaling the variable t to a linear system. By Proposition 3(b) of [**4**], it has at most two invariant straight lines. So the result follows.

We now assume that P and Q are relatively prime. Then it is well known that a quadratic system (1) has at most five invariant straight lines (for instance, see Corollary 5(a) of [**4**] or [**26**]). From the proof of Proposition 1, we know that $\deg H \geq 3$. From the proofs of Theorems C and F, we know that quadratic systems having a minimal polynomial first integral of degree 3 and 4 have at most three invariant straight lines. In the following we will use Lemma 3 to prove that quadratic systems with more than three invariant straight lines have no polynomial first integrals of degree larger than 4. Since only systems (II), (VII) and (VIII) have a unique vertical invariant straight line, system (V) has two unique vertical invariant straight lines and system (X) has infinitely many invariant straight lines, without loss of generality, we need only to study the invariant straight lines of the form

$$y = kx + c, \tag{24}$$

for the first nine systems of Lemma 3. In what follows we prove our results for the nine systems one by one.

For system (I) the straight line (24) is invariant if and only if the following condition holds

$$Q(x, kx + c) \equiv k[1 + x(kx + c)].$$

This condition is verified if and only if the following three conditions hold:

$$\begin{aligned} k &= b_{00} + b_{01}c + b_{02}c^2, \\ b_{10} + b_{11}c + (b_{01} &- c + 2b_{02}c)k = 0, \\ b_{20} + b_{11}k &+ (b_{02} - 1)k^2 = 0. \end{aligned} \tag{25}$$

Substituting k from the first equation of (25) into the second and third equations of (25) yields

(26)

$$\begin{aligned} &b_{00}b_{01} + b_{10} + (-b_{00} + b_{01}^2 + 2b_{00}b_{02} + b_{11})c \\ &\qquad + (-b_{01} + 3b_{01}b_{02})c^2 + (-b_{02} + 2b_{02}^2)c^3 = 0, \\ &- b_{00}^2 + b_{00}^2 b_{02} + b_{00}b_{11} + b_{20} + (-2b_{00}b_{01} + 2b_{00}b_{01}b_{02} + b_{01}b_{11})c \\ &\qquad + (-b_{01}^2 - 2b_{00}b_{02} + b_{01}^2 b_{02} + 2b_{00}b_{02}^2 + b_{11}b_{02})c^2 \\ &\qquad + (-2b_{01}b_{02} + 2b_{01}b_{02}^2)c^3 + (-b_{02}^2 + b_{02}^3)c^4 = 0. \end{aligned}$$

Therefore, we know from the first equation of (26) that the necessary conditions for system (I) to have more than three invariant straight lines are

$$b_{02}(2b_{02}-1)=0, \quad b_{01}(3b_{02}-1)=0,$$
$$b_{01}^2-b_{00}+2b_{00}b_{02}+b_{11}=0, \quad b_{00}b_{01}+b_{10}=0,$$

which are equivalent to one of the following two conditions:

$$b_{02}=0, \quad b_{01}=0, \quad b_{00}=b_{11}, \quad b_{10}=0, \tag{27}$$

$$b_{02}=\frac{1}{2}, \quad b_{01}=0, \quad b_{11}=0, \quad b_{10}=0. \tag{28}$$

If the condition (27) holds, by the second equation of (26) system (I) has either infinitely many invariant straight lines if $b_{20}=0$ or none if $b_{20}\neq 0$.

If the condition (28) holds, the second equation of (26) can be written as

$$(c^2+2b_{00})^2=8b_{20}.$$

So the necessary and sufficient conditions in order that system (I) has more than three invariant straight lines are:

$$b_{11}=b_{10}=b_{01}=0, \quad b_{02}=\frac{1}{2}, \quad b_{20}>0, \quad b_{00}<-\sqrt{2b_{20}}.$$

Thus the matrix M_j in (12), $j=2,3,\dots,n-3$, is of the form

$$M_j=\begin{pmatrix}
0 & b_{20} & 0 & 0 & \cdots & 0 & 0\\
j & 0 & 2b_0 & 0 & \cdots & 0 & 0\\
0 & j-\frac{1}{2} & 0 & 3b_{20} & \ddots & \vdots & \vdots\\
0 & 0 & j-1 & 0 & \ddots & 0 & 0\\
0 & 0 & 0 & j-\frac{3}{2} & \ddots & (j-3)b_{20} & 0\\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 & (j-2)b_{20}\\
0 & 0 & 0 & 0 & \ddots & 2+\frac{j-1}{2} & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 1+\frac{j}{2}
\end{pmatrix}.$$

Since $j-k+1+(k/2) \neq 0$ for $k = 2, 3, \dots, j$, by omitting the first line of M_j we get a $(j-1)\times(j-1)$ matrix of rank $j-1$. By recurrence, it follows that the linear algebraic equations in D_j in (12) for $j \in \{2, 3, \dots, n-1\}$ only have the zero solution. From Proposition 1 and Theorem 2 we know that system (I) has no polynomial first integrals of degree larger than 4. Hence, for system (I), Theorem G follows.

For system (II) in a similar way to the previous proof, the straight line (24) is invariant if and only if the following three conditions hold:

$$\begin{aligned} &b_{00} + b_{01}c + b_{02}c^2 = 0, \\ &b_{10} + b_{11}c + (b_{01} - c + 2b_{02}c)k = 0, \\ &b_{20} + b_{11}k + (b_{02} - 1)k^2 = 0. \end{aligned} \tag{29}$$

If $b_{02} = 1$, we can easily prove that equation (29) has at most two solutions (k, c). It follows that system (II) has at most three invariant straight lines, including the vertical line $x = 0$.

If $b_{02} \neq 1$. We rearrange the second equation of (29) into

$$b_{10} + b_{01}k + (b_{11} - k + 2b_{02}k)c = 0.$$

This implies that the necessary conditions for equation (29) to have more than two solutions (k, c) are

$$b_{10} = b_{01} = b_{11} = 0, \quad b_{02} = \frac{1}{2}.$$

By a similar way to the proof of system (I), it follows that under these conditions system (II) has no polynomial first integrals of degree larger than 4.

For systems (III) and (IV) using similar arguments to the previous two systems, we know that they have at most three and two invariant straight lines, respectively.

System (V) has invariant straight lines of the form (24) if and only if the following three conditions hold:

$$\begin{gathered} k = -b_{00} - b_{01}c - b_{02}c^2, \\ b_{10} + b_{11}c + (b_{01} + 2b_{02}c)k = 0, \\ b_{20} + (b_{11} - 1)k + b_{02}k^2 = 0. \end{gathered} \tag{30}$$

Substituting k from the first equation of (30) into the other two equations gives

$$\begin{aligned}
&-b_{00}b_{01}+b_{10}+(-b_{01}^2-2b_{00}b_{02}+b_{11})c\\
&\qquad -3b_{01}b_{02}c^2-2b_{-2}^2c^3=0,\\
&-b_{00}+b_{00}^2b_{02}-b_{00}b_{11}+b_{20}+(b_{01}+2b_{00}b_{01}b_{02}-b_{01}b_{11})c\\
&\qquad +(b_{02}+b_{01}^2b_{02}+2b_{00}b_{02}^2-b_{11}b_{02})c^2\\
&\qquad +2b_{01}b_{02}^2c^3+b_{02}^3c^4=0.
\end{aligned}$$

For $b_{02}=0$, these two equations give that equation (30) has at most one solution (k,c). It follows that system (V) has at most three invariant straight lines, which include two vertical ones.

If $b_{02}\neq 0$, the matrix M_j in (12) can be written as

$$M_j=\begin{pmatrix}
* & * & 0 & \cdots & 0 & 0\\
2b_{02} & * & * & \cdots & 0 & 0\\
0 & 3b_{02} & * & \ddots & \vdots & \vdots\\
0 & 0 & 4b_{02} & \ddots & * & 0\\
\vdots & \vdots & \ddots & \ddots & * & *\\
0 & 0 & 0 & \ddots & (j-1)b_{02} & *\\
0 & 0 & 0 & \cdots & 0 & jb_{02}
\end{pmatrix},$$

where the stars represent the elements which are probably nonzero. In a similar way as the proof of system (I), we can get that system (V) has no polynomial first integrals of degree larger than 4.

For system (VI)–(IX), by making use of the same method as that for dealing with system (V), we can prove the following conclusions:

• Systems (VI) and (IX) have at most one invariant straight line if $b_{02}=0$ and no polynomial first integrals of degree larger than 4 if $b_{02}\neq 0$.

• Systems (VII) and (VIII) have at most two invariant straight lines if $b_{02}=0$ and no polynomial first integrals of degree larger than 4 if $b_{02}\neq 0$.

Summing up the above results, Theorem G follows. □

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REFERENCES

1. A.F. Andreev, *Investigation of the behavior of the integral curves of a system of two differential equations in the neighborhood of a singular point*, Trans. Amer. Math. Soc. **8** (1958), 183–207.

2. V.I. Arnold and Y.S. Ilyashenko, *Dynamical systems*–I, *Ordinary differential equations*, in *Encyclopaedia of Mathematical Sciences*, Vols. 1–2, Springer, Heidelberg, 1988.

3. J.C. Artés and J. Llibre, *Quadratic Hamiltonian vector fields*, J. Differential Equations **107** (1994), 80–95.

4. J.C. Artés, B. Grünbaum and J. Llibre, *On the number of invariant straight lines for polynomial differential systems*, Pacific J. Math. **184** (1998), 207–230.

5. A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.L. Maier, *Qualitative theory of second-order dynamic systems*, Wiley, New York, 1973.

6. L. Cairó, M.R. Feix and J. Llibre, *Integralbility and algebraic solutions for planar polynomial differential systems with emphasis on the quadratic systems*, Resenhas de Universidade de Sao Paulo **4** (2000), 127–161.

7. L. Cairó and J. Llibre, *Integrability of the* 2*-dimensional Lotka-Volterra system via polynomial inverse integrating factors*, J. Physics A: Math. Gen. **33** (2000), 2407–2417.

8. J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *On the integrability of two-dimensional flows*, J. Differential Equations **157** (1999), 163–182.

9. C. Christopher and J. Llibre, *Integrability via invariant algebraic curves for planar polynomial differential systems*, Annals Differential Equations **16** (2000), 5–19.

10. G. Darboux, *Mémoire sur les équations différentielles algébrique du premier ordre et du premier degré* (*Mélanges*), Bull. Sci. Math. 2éme Série **2** (1878), 60–96, 123–144, 151–200.

11. E.A.V. Gonzales, *Generic properties of polynomial vector fields at infinity*, Trans. Amer. Math. Soc. **143** (1969), 201–222.

12. A. Gasull, Sheng Liren and J. Llibre, *Chordal quadratic systems*, Rocky Mountain J. Math. **16** (1986), 751–782.

13. J.P. Jouanolou, *Equations de Pfaff algébriques*, Springer-Verlag, New York, 1979.

14. R.E. Kooij and C.J. Christopher, *Algebraic invariant curves and the integrability of polynomial systems*, Appl. Math. Lett. **6** (1993), 51–53.

15. S. Lang, *Fundamental of Diophantine Geometry*, Springer, New York, 1983.

16. S. Labrunie, *On the polynomial first integrals of the* (a, b, c) *Lotka-Volterra system*, J. Math. Phys. **37** (1996), 5539–5550.

17. J. Llibre and Xiang Zhang, *Polynomial first integrals of quadratic systems*, Centre de Recerca Matemàtica, Preprint núm. **428** (1999).

18. L. Markus, *Global structure of ordinary differential equations in the plane*, Trans. Amer. Math. Soc. **76** (1954), 127–148.

19. J. Moulin-Ollagnier, *Polynomial first integrals of the Lotka-Volterra system*, Bull. Sci. Math. **121** (1997), 463–476.

20. D. Neumann, *Classification of continuous flows on* 2*-manifolds*, Proc. Amer. Math. Soc. **48** (1975), 73–81.

21. H. Poincaré, *Sur l'intégration des équations différentielles du premier ordre et du premier degré* I *and* II, Rend. Mat. Palermo **5** (1891), 161–191; **11** (1897), 193–239.

22. M.J. Prelle and M.F. Singer, *Elementary first integrals of differential equations*, Trans. Amer. Math. Soc. **279** (1983), 215–229.

23. J.W. Reyn, *A bibliography of the qualitative theory of quadratic systems of differential equations in the plane*, 3rd ed., Delft University of Technology, Faculty of Technical Mathematics and Informatics, Report, 1994.

24. R. Roussarie, *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, Birkhauser Verlag, Boston, 1998.

25. M.F. Singer, *Liouvillian first integrals of differential equations*, Trans. Amer. Math. Soc. **333** (1992), 673–688.

26. Ye Yanqian, *Qualitative theory of polynomial differential systems*, Shanghai Scientific & Technical Publishers, Shanghai, 1995 (in Chinese).

27. Ye Yanqian et al., *Theory of limit cycles*, Amer. Math. Soc., Providence, R.I., 1984.

28. Zhang Zhifen, Ding Tongren, Huang Wenzao and Dong Zhenxi, *Qualitative theory of differential equations*, Amer. Math. Society, Providence, R.I., 1992.

29. H. Zholadek, *On algebraic solutions of algebraic Pfaff equations*, Stud. Math. **114** (1995), 117–126.

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