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JACOBI FORMS AND GENERALIZED RC-ALGEBRAS

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ABSTRACT. Using the recently found Rankin-Cohen type brackets on the spaces of Jacobi forms, we define generalized Rankin-Cohen algebras. We study their algebraic properties and give examples generalizing the elliptic cases.

1. Introduction. Classically, there are many interesting connections between differential operators and the theory of elliptic modular forms, and several interesting results have been obtained (see [**6**], [**8**], for instance). In 1956, Rankin gave a general description of the differential operators which sent modular forms to modular forms [**6**]. Later, Cohen constructed certain differential bilinear operators acting on the graded ring $M_*(\Gamma)$ of modular forms on the group $\Gamma \subset PSL(2, Z)$ and used them to construct modular forms with interesting Fourier coefficients [**4**]. In 1990, Zagier studied the algebraic properties of these bilinear operators and called them Rankin-Cohen brackets [**8**]. Moreover, the Rankin-Cohen brackets are shown to appear as the various terms in the (convergent) expansion of the composition of two symbols in a certain symbolic calculus associated with $SL(2, \mathbf{R})$ [**7**]. The existence of infinitely many identities among the Rankin-Cohen brackets motivated the definition of Rankin-Cohen algebras whose properties have been studied in detail in [**8**].

Recently, the theory of Jacobi forms has been studied extensively and systematically, first by Eichler and Zagier [**5**] and many others. It turns out that Jacobi forms are connected with modular forms of half-integral weight as well as integral weight, Siegel modular forms and elliptic curves. It was shown that the heat operator plays an important role connecting Jacobi forms and elliptic modular forms. In [**1**], [**2**], [**3**] the generalization of the Rankin-Cohen brackets, which involves the heat operator, to Jacobi forms has been found.

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Since there are also infinitely many relations among the Rankin-Cohen type brackets on the space of Jacobi forms, it is natural to define generalized Rankin-Cohen algebras which we will call GRC algebras. In this paper we study the properties of GRC-algebras on Jacobi forms. These results are the generalization of the results obtained in the elliptic case (see [**8**]).

2. Jacobi forms and differential operator. In this section we recall some basic notations of the theory of Jacobi forms and define the heat operator. In particular, we recall the Rankin-Cohen type brackets using the heat operator. Our conventions follow those used in [**5**].

Let $\Gamma(1)$ be the modular group, the set of 2×2 matrices with integer entries and determinant 1. The following slash operators on function $f:\mathcal{H}\times\mathbf{C}\to\mathbf{C}$ are given in [**5**]. For fixed integers k and m, let

$$(f|_{k,m}\begin{pmatrix} a & b \\ c & d\end{pmatrix})(\tau,z) := (c\tau+d)^{-k}e^{2\pi i m(-cz^2/(c\tau+d))}f\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right),$$

for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma(1)$, and

$$(f|_m[\lambda,\nu])(\tau,z) := e^{2\pi i m(\lambda^2\tau+2\lambda z)}f(\tau,z+\lambda\tau+\nu)\quad\text{for } [\lambda,\nu]\in\mathbf{Z}^2.$$

Using these slash operators we give the definition of Jacobi forms.

Definition 2.1. A Jacobi form of weight k and index m ($k,m\in\mathbf{Z}^+$) on $\Gamma(1)$ is a holomorphic function $f:\mathcal{H}\times\mathbf{C}\to\mathbf{C}$ satisfying

$$\begin{aligned}(f|_{k,m}M)(\tau,z) &= f(\tau,z)\quad\text{for } M\in\Gamma(1),\\ (f|_m X)(\tau,z) &= f(\tau,z)\quad\text{for } X\in\mathbf{Z}^2,\end{aligned}$$

and such that it has a Fourier expansion for the form

$$f(\tau,z)=\sum_{\substack{n=0\\ r\in Z,r^2\le 4nm}}^{\infty} c(n,r)q^n\zeta^r,$$

where $q=e^{2\pi i\tau}$ and $\zeta=e^{2\pi i z}$. If f has a Fourier expansion of the same form but with $r^2<4nm$, then f is called a *Jacobi cusp form* of weight k and index m.

We denote by $J_{k,m}$ the vector space of all Jacobi forms of weight k and index m and by $J_{k,m}^{\text{cups}}$ the vector space of all Jacobi cusp forms of weight k and index m. We identify the space $J_{k,0}$ as the space of elliptic modular forms of weight k which we also denote by M_k (similarly $J_{k,0}^{\text{cusp}} = M_k^{\text{cusp}}$ the space of elliptic cusp forms).

In [**5**] the action of the heat operator L on $J_{*,*}$ has been studied. It plays an important role connecting Jacobi forms and elliptic modular forms. It also turns out that the heat operator L_m can be used to construct the Rankin-Cohen type brackets on the space of Jacobi forms [**1**]. We recall the definition of the heat operator and the Rankin-Cohen type bilinear differential operators.

Definition 2.2. Heat operator. For any integer m the heat operator L_m is defined by

$$\begin{cases} L_m(f) = \Big(8\pi\, im\dfrac{\partial}{\partial\tau} - \dfrac{\partial^2}{\partial z^2}\Big)(f) & \text{if } f \in J_{*,m} \text{ with } m \in \mathbf{Z}^+ \\ L_0(f) = \dfrac{\partial}{\partial\tau}(f) & \text{if } f \in J_{*,0} \end{cases}.$$

Remark 2.3. Note that the normalization of the heat operator L is different from the one used in [**1**].

Definition 2.4. Rankin-Cohen type brackets. Let f_i with $i = 1, 2$ be complex-valued holomorphic functions on $\mathcal{H} \times \mathbf{C}$. Then, for all nonnegative integers n, and for some integers k_i, m_i, $i = 1, 2$, the nth Rankin-Cohen type bracket $[\ ,\sqcup_n$ is defined by

$$\begin{aligned}[f_1 f_2]_n = \sum_{r+s=n} &(-1)^r \binom{a_1+n-1}{s}\binom{a_2+n-1}{r} \\ &\cdot m_1^s m_2^r L_{m_1}^r(f_1) L_{m_2}^s(f_2)\end{aligned}$$

where $\alpha_i = k_i - (1/2)$, $i = 1, 2$.

We now recall the main result in [**1**].

Theorem 2.5. *Let $f_i \in J_{k_i,m_i}$ for $i = 1, 2$. Then, for any nonnegative integer n,*

1) $[f_1, f_2]_n \in J_{k_1+k_2+2n, m_1+m_2}$.
2) *If* f_1 *or* $f_2 \in J^{\mathrm{cusp}}_{*,*}$, *then* $[f_1 f_2]_n \in J^{\mathrm{cusp}}_{*,*}$.

Proof. See Theorem 3.1 in [**1**].

We also define a covariant differential operator for Jacobi forms, which we will need later.

Proposition 2.6. *Let* $\delta(k,m) = k - (1-\delta_{m,0})/2$. *Then the operator* $\mathcal{D}$ *defined by*

$$\mathcal{D}(f) := L_m(f) - \frac{1}{3}\,\delta(k,m) E_2(\tau) f, \quad f \in J_{k,m},$$

(*with* L_m *as in* (2.2)) *maps* $J_{*,*}$ *to* $J_{*+2,*}$. *Here* $E_2(\tau)$ *is the elliptic Eisenstein series of weight* 2, *i.e.,* $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n\tau}$.

Proof. It is well known that $E_2(\tau)$ satisfies, for any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$,

$$(c\tau + d)^{-2} E_2(M\tau) = E_2(\tau) + \frac{6c}{\pi i(c\tau + d)}.$$

A simple computation shows that

$$L_m(f)|_{k+2,m} M = L_m(f|_{k,m} M) + \frac{2\delta(k,m)}{\pi i(c\tau + d)}\, f|_{k,m} M,$$

for all $f \in J_{k,m}$. Together with equation (10) on page 33 of [**5**], this implies the proposition. □

3. Generalized Rankin-Cohen algebras. In [**8**], Zagier defined Rankin-Cohen algebras (RC algebras) over a field K as a graded K-vector space $R_* = \oplus_{k\geq 0} R_k$ together with bilinear operators $[\cdot,\cdot]_n : R_k \times R_l \to R_{k+l+2n}$ which hold all the algebraic identities satisfied by the Rankin-Cohen brackets. It was shown that under a rather general hypothesis, all RC-algebras arise as subalgebras of certain standard RC-algebras.

In this section we define generalized Rankin-Cohen algebras (GRC algebras) using the Rankin-Cohen type brackets considered in the last section and get the analogous properties of RC algebras. However, before we give a definition of generalized RC algebras, we want to mention some of the identities satisfied by the Rankin-Cohen type brackets on the spaces of Jacobi forms which can easily be proved from their definition.

Proposition 3.1. *Let* $f_i \in J_{k_i,m_i}$, $i = 1, 2, 3$, *and let* $[\ ,\ ]_n$ *be the* n*th Rankin-Cohen type bracket. Then one has the following identities*

$$[[f_1, f_2]_0, f_3]_0 = [f_1, [f_2, f_3]_0]_0, \tag{1}$$

$$[f_1, f_2]_n = (-1)^n [f_2, f_1]_n, \quad \textit{for all } n, \tag{2}$$

$$\begin{aligned}\delta(k_3, m_3)[[f_1, f_2]_1, f_3]_0 + \delta(k_1, m_1)[[f_2, f_3]_1, f_1]_0 \\ + \delta(k_2, m_2)[[f_3, f_1]_1, f_2]_0 = 0.\end{aligned} \tag{3}$$

For $m_1 = m_2 = m_3 = 0$, *it satisfies the Jacobi identity*

$$[[f_1, f_2]_1, f_3]_1 + [[f_2, f_3]_1, f_1]_1 + [[f_3, f_1]_1, f_2]_1 = 0. \tag{4}$$

It is also true that, for $m_1 = m_2 = 0$, *one has*[1]

$$\begin{aligned}&k_1^2(k_1+1)f_1^2[f_2, f_2]_2 - k_2^2(k_2+1)f_2^2[f_1, f_1]_2 \\ &\quad = -(k_1+1)(k_2+1)^2[f_1, f_2]_1^2 - k_2(k_2+1)(k_1+1)f_2[[f_1, f_2], f_1]_1.\end{aligned}$$

Recall from [**8**] that a simple counting argument shows that there exist many more universal identities, satisfied by the Rankin-Cohen type brackets (for instance, the permutations of the r-fold 2-brackets $[..[[f_1, f_2]_2 f_3] \dots]_2$ are linearly dependent for all sufficiently large r. More precisely, we see that there are at most $5 \times 2^r - 2r - 5$ unknowns with $r!/2$ equations coming from all possible permutations.)

Definition 3.2. A generalized RC algebra (GRC algebra) over a field K is a bigraded K-vector space $R_{*,*} = \oplus_{k,m} R_{k,m}$ (with $R_{0,0} = K \cdot 1$

and $\dim_K R_{k,m} < \infty$ for all k, m) together with bilinear operations $[\cdot,\cdot]_n : R_{k,m} \otimes R_{k',m'} \to R_{k+k'+2n,m+m'}$, $k, k', m, m', n \geq 0$, which satisfy (1)–(4) of Proposition 3.1 with the $f_i \in R_{k_i,m_i}$, $i = 1, 2, 3$.

Example 3.3. Theorem 2.5 implies that $(J_{*,*}, [.]_n)$ is a GRC algebra. Here $[.]_n$ is the nth Rankin-Cohen type bracket.

Next we define the notion of standard GRC-algebras.

Definition 3.4. Let $R*,*$ be a commutative bigraded algebra with unit over K with two derivations D_1 and D_2 of degree (2,0) and (1,0), respectively ($D_1(R_{*,*}) = R_{*+2,*}$ and $D_2(R_{*,*}) = R_{*+1,*}$) and assume that $D_2 R_{*,0} = 0$. Set $\mathcal{L}_m = D_1 - (1/m)D_2^2$ on $R_{*,m}$, $m \neq 0$, and $\mathcal{L}_0 = D_1$, and define a bilinear operator $[\ ,\ ]_{\mathcal{L},n}$ by

$$(5)\quad [f,g]_{\mathcal{L},n} = \sum_{r+s=n} (-1)^r \binom{\delta(k,m)+n-1}{s} \cdot \binom{\delta(k',m')+n-1}{r} \mathcal{L}_m^r(f)\mathcal{L}_{m'}^s(g)$$

for any $f \in R_{k,m}$ and $g \in R_{k',m'}$. Then $\mathcal{L}$ is called a generalized heat operator and the GRC algebra $(R_{*,*}, [\ ,\ ]_{\mathcal{L},*})$ is called a standard GRC algebra (note that $(R_{*,*}, [\ ,\ ]_{\mathcal{L},*})$ is the GRC algebra since $[f,g]_{\mathcal{L},n}$ is in $R_{k+k'+2n,m+m'}$).

Remark 3.5. Note that $J_{*,*}$ together with the heat operator given in Definition 2.2 does not define a standard GRC-algebra since $J_{*,*}$ is not closed under the action of the heat operator. However, by Proposition 2.6, $J_{*,*}$ is a standard GRC-algebra with $\mathcal{D}$ (here $D_1 = (4/2\pi i)(\partial/\partial\tau) - (\delta(k,m)E_2/3)$ and $D_2 = (1/2\pi i)(\partial/\partial z)$).

The following Proposition 3.6 shows the existence of a class of GRC-algebras in which the standard GRC-algebra is obtained as a special case of $\Phi = 0$. The proof of the proposition shows that such a wider class of GRC-algebras is realized as a subalgebra of the standard one. It is analogous to Proposition 1 given in [**8**].

Proposition 3.6. *Let $R_{*,*}$ be a bigraded commutative associative*

K-algebra with $R_{0,0} = K.1$ together with an element $\Phi \in R_{4,0}$ and a generalized heat operator $\mathcal{D}$. Define brackets $[\ ,\ ]_{\mathcal{D},\Phi,n}$, $n \geq 0$, on $R_{*,*}$, by

$$[f,g]_{\mathcal{D},\Phi,n} = \sum_{r+s=n} (-1)^r \binom{\delta(k,m)+n-1}{s}\binom{\delta(k',m')+n-1}{r} f_r g_s$$

where $f_r \in R_{k+2r,m}$, $g_s \in R_{k'+2s,m'}$, $r,s \geq 0$, satisfying

$$\begin{aligned} f_{r+1} &= \mathcal{D}f_r + r(\delta(k,m) - 1 + r)\Phi f_{r-1} \\ g_{s+1} &= \mathcal{D}g_s + s(\delta(k',m') - 1 + s)\Phi g_{r-1} \end{aligned}$$

with initial conditions $f_0 = f$ and $g_0 = g$. So $f_1 = \mathcal{D}(f)$, $g_1 = \mathcal{D}(g)$. Then $(R_{*,*}, [\ ,\ ]_{\mathcal{D},\Phi,*})$ is a GRC-algebra.

Definition 3.7. A GRC algebra will be called *canonical* if its brackets are given as in Proposition 3.6 for some generalized heat operator $\mathcal{D}$ and some element $\Phi \in R_{4,0}$.

Proof of Proposition 3.6. The proof of the proposition goes along the same lines as the proof of Proposition 1 in [**8**]. First we will embed $(R_{*,*}, [\ ,\ ,\ ]_{\mathcal{D},\Phi,*})$ into a standard GRC algebra $(\hat{R}_{*,*}, [\ ,\ ]_{\mathcal{L},*})$ for some larger graded ring $\hat{R}_{*,*}$ with a generalized heat operator $\mathcal{L}$. Define the operator $\mathcal{L}$ on $\hat{R}_{*,*} := R_{*,*} \otimes_K K[\phi]$ (ϕ has degree $(2,0)$) by

$$\mathcal{L}(f) = \mathcal{D}(f) + \delta(k,m)\phi f \in \hat{R}_{k+2,m}, \quad \mathcal{L}(\phi) = \Phi + \phi^2$$

where $f \in R_{k,m}$ and $\Phi \in \hat{R}_{4,0}$. This defines $\mathcal{L}$ on generators of $\hat{R}_{*,*}$, and we extend $\mathcal{L}$ uniquely such that it satisfies Leibnitz's rule if one of the elements is in $\hat{R}_{*,0}$. Note that $\mathcal{L}$ is a generalized heat operator since $\mathcal{D}$ is a generalized heat operator. We can now prove the proposition by showing that $[f,g]_{\mathcal{L},n} = [f,g]_{\mathcal{D},\Phi,n}$ for $f,g \in R_{*,*}$. First, observe that the brackets $[\ ,\ ]_{\mathcal{L},n}$ in equation (5) can be rewritten as

$$\sum_{n=0}^{\infty} \frac{[f,g]_{\mathcal{L},n}}{(n+\delta(k,m)-1)!(n+\delta(k',m')-1)!} X^n = \tilde{f}(-X)\tilde{g}(X)$$

where

$$\tilde{f}(X) = \sum_{n=0}^{\infty} \frac{\mathcal{L}^n(f)}{n!(\delta(k,m)-1+n)!}\, X^n,$$

$$\tilde{g}(X) = \sum_{n=0}^{\infty} \frac{\mathcal{L}^n(g)}{n!(\delta(k',m')-1+n)!}\, X^n.$$

Second, we claim that f_r satisfies

$$e^{-\phi X}\tilde{f}(X) = \sum_{r=0}^{\infty} \frac{f_r}{r!(\delta(k,m)-1+r)!}\, X^r$$

(and similarly for g).

Or, equivalently,

$$f_r = \sum_{n=0}^{r}(-1)^{r-n}\frac{r!(\delta(k,m)-1+r)!}{n!(\delta(k,m)-1+n)!(r-n)!}\,\phi^{r-n}\mathcal{L}^n(f) \in \hat{R}_{k+2r,m}.$$

This can be proved by showing that the above equation satisfies the recursion relation given in the proposition. Assume inductively that we have proved that $f_r \in R_{k+2r,m}$ for some r. Then we find

$$\begin{aligned}
\mathcal{D}(f_r) &= \mathcal{L}(f_r) - (\delta(k,m)+2r)\phi f_r \\
&= \sum_{n-0}^{r}(-1)^{r-n}\,\frac{r!(\delta(k,m)-1+r)!}{n!(\delta(k,m)-1+n)!(r-n)!} \\
&\quad \cdot [\phi^{r-n}\mathcal{L}_{n+1}(f) + (r-n)\phi^{r-n-1}(\Phi+\phi^2)\mathcal{L}^n(f) \\
&\quad - (\delta(k,m)+2r)\phi^{n-r+1}\mathcal{L}^n(f)] \\
&= \sum_{n=0}^{r+1}(-1)^{r+1-n}\,\frac{r!(\delta(k,m)-1+r)!}{n!(\delta(k,m)-1+n)!(r+1-n)!} \\
&\quad \cdot [n(n+\delta(k,m)-1) - (r-n)(r+1-n) \\
&\quad\quad + (\delta(k,m)+2r)(r+1-n)]\,\phi^{r-n+1}\mathcal{L}^n(f) \\
&\quad\quad + \Phi\sum_{n=0}^{r-1}(-1)^{r-n}\,\frac{r!(\delta(k,m)+r)!}{n!(\delta(k,m)+n)!(r-n-1)!}\,\phi^{r-n-1}\mathcal{L}^n(f) \\
&= f_{r+1} - r(\delta(k,m)-1+r)\Phi f_{r-1},
\end{aligned}$$

from the induction hypothesis. By multiplying $e^{-\phi X}\tilde{f}(X)$ by $e^{\phi X}\tilde{g}(-X)$ we find that $[f,g]_{\mathcal{L},n} = [f,g]_{\mathcal{D},\Phi,n}$. Hence the proposition becomes obvious. $\square$

The next theorem states a criterion for a GRC-algebra to be canonical which can be checked in a finite process.

Theorem 3.8. *Let $(R_{*,*},[\ ,\ ]_*)$ be a* GRC-*algebra which is finitely generated over a field of characteristic zero. Then the following statements are equivalent*:

(1) $(R_{*,*},[\ ,\ ]_*)$ *is a canonical* GRC-*algebra.*

(2) *For every homogeneous element $F \in R_{k',m'}$, there exists an element $G \in R_{*+2,0}$ such that*

(a) $[F,f]_1 \equiv \delta(k',m')fG \pmod F$ *for every $k,m \geq 0$ and all $f \in R_{k,m}$.*

(b) $[F,F]_2 \equiv (\delta(k',m')+1)G^2 - (\delta(k',m')+1)[F,G]_1 \pmod{F^2}$.

(3) *Property* 2 *holds for some homogeneous $F \in R_{*,0}$ which is not a divisor of zero.*

Specifically, if (F,G) are a pair of elements satisfying (a) *and* (b) *in* 2, *and with $F \in R_{*,0}$ which is not a divisor of zero, then the bracket on $R_{*,*}$ will agree with the canonical bracket associated with*

$$\begin{aligned} \mathcal{D}_{F,G}(f) &:= \frac{[F,f]_1 - \delta(k,m)fG}{\delta(k',m')F}, \quad f \in R_{k,m} \\ \Phi_{F,G} &:= \frac{[F,F]_2 + (\delta(k',m')+1)([F,G]_1 - G^2)}{\delta^2(k',m')(\delta(k',m')+1)F^2}. \end{aligned} \tag{6}$$

Proof. The proof is similar to that of Proposition 2 in [**8**]. First assume that $R_{*,*}$ is canonical with respect to some generalized heat operator $\mathcal{D} : \mathcal{R}_{*,*} \to R_{*+2,*}$ and $\Phi \in R_{4,0}$ and choose any homogeneous element $F \in R_{k',m'}$. Then properties (a) and (b) in 2 hold with $G = -\mathcal{D}(F)$ because of the identities

$$\begin{aligned} [F,f]_1 - \delta(k,m)fG &= [F,f]_{\mathcal{D},\Phi,1} + \delta(k,m)f\mathcal{D}(F) \\ &= \delta(k',m')\mathcal{D}(f)F, \quad f \in R_{k,m}, \end{aligned}$$

$$
\begin{aligned}
&[F,F]_2 + (\delta(k',m') + |)[F,G]_1 - (\delta(k',m')+1)G^2 \\
&\quad = (\delta(k',m')(\delta(k',m')+1)F\mathcal{D}^2(F) - (\delta(k',m')+1)^2\mathcal{D}(F)^2 \\
&\qquad + \delta(k',m')^2(\delta(k',m')+1)\Phi F^2) - (\delta(k',m'+1)(\delta(k',m')\mathcal{D}(F)F \\
&\qquad - (\delta(k',m')+1)F\mathcal{D}^2(F)) - (\delta(k',m')+1)(\mathcal{D}(F))^2 \\
&\quad = \delta(k',m')^2(\delta(k',m')+1)\Phi F^2.
\end{aligned}
$$

Conversely, suppose that $R_{*,*}$ contains elements $F \in R_{N,0}$, $G \in R_{N+2,0}$, for some N satisfying (a) and (b) in (2), and define $\mathcal{D}$ and Φ by (6). Then we claim that the brackets $[\ ,\ ]_{\mathcal{D},\Phi,*}$ induced by $\mathcal{D}$ and Φ agree with the given brackets. As in the proof of Proposition 3.6, we can assume that $(R_{*,*}, [\ ,\ ]_*)$ is a sub GRC algebra of a standard GRC algebra $(R_{*,*}, [\ ,\ ]_{\mathcal{L},*})$ since the assertion to be proved is equivalent to a collection of universal identities for the brackets of a GRC algebra and such identities are true by definition if they are true for standard algebras. Now the larger algebra $(R_{*,*}, [\ ,\ ]_{\mathcal{L},*})$ is canonical with a generalized heat operator $\mathcal{L}$ of degree (2.0) and Φ of degree (4,0). Therefore, we only have to show that in a ring with more than one choice of (F,G), as in (2) of the theorem, the induced brackets agree. Suppose that (F,G) satisfies (a) and (b) in (2), and let $\tilde{F} \in R_{\tilde{N},0}$ be an arbitrary homogeneous element of $R_{*,0}$. We now have to show that there is an element $\tilde{G} \in R_{\tilde{N}+2,0}$ such that $(\tilde{F},\tilde{G})$ also satisfy (a) and (b) in (2). We may start by choosing any $\tilde{G}$ satisfying (b). We set

$$
\tilde{G} = \frac{\delta(\tilde{N},0)G\tilde{F} - [F,\tilde{F}]_1}{\delta(N,0)F},
$$

which belongs to $R_{\tilde{N},0}$ by property (a) of (F,G). Then, for $f \in R_{k,m}$ we find

$$
\begin{aligned}
&\mathcal{D}_{F,G}(f) - \mathcal{D}_{\tilde{F},\tilde{G}}(f) \\
&\qquad = \frac{\delta(\tilde{N},0)\tilde{F}[f,F]_1 + \delta(N,0)F[\tilde{F},F]_1 + \delta(k,m)f[\tilde{F},F]_1}{\delta(N,0)\delta(\tilde{N},0)F\tilde{F}} = 0
\end{aligned}
$$

by the second identity of GRC brackets given in Proposition 3.1. From the third identity of GRC brackets given in Proposition 3.1, we find similarly that $\Phi_{F,G} - \Phi_{\tilde{F},\tilde{G}} = 0$. Therefore, the brackets constructed with $\mathcal{D}_{F,G}$ and $\Phi_{F,G}$ are the same as those constructed from $\tilde{F}$ and $\tilde{G}$, and thus the same as those constructed form any pair $(\tilde{F}, \tilde{G})$ satisfying (a) and (b) in (2). This proves the theorem.

ENDNOTE

1. This is the corrected version of equation (41) in [**8**] which contains several misprints.

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