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AN IMPROVEMENT ON A THEOREM OF THE GOLDBACH-WARING TYPE

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ABSTRACT. Let p_i, $2 \leq i \leq 5$ be prime numbers. It is proved that all but $\ll x^{19193/19200+\varepsilon}$ positive even integers N smaller than x can be represented as

$$N = p_1^2 + p_2^3 + p_3^4 + p_4^5.$$

1. Introduction and statement of results. I.M. Vinogradov [**14**] proved the ternary Goldbach-conjecture in 1937. Its method was successfully applied to different problems in additive prime number theory by various mathematicians. Among them Prachar established in 1952, [**11**] the following result: *There exists a constant* $c > 0$ *such that all but* $\ll x(\log x)^{-c}$ *even integers* N *smaller than* x *are representable as*

$$N = p_1^2 + p_2^3 + p_3^4 + p_4^5 \tag{1.1}$$

for prime numbers p_i.

The author could improve upon this result in [**1**] by giving the following estimate: *There exists a positive number* δ *such that all but*

$$\ll x^{1-\delta}$$

positive even integers $N \leq x$ *are representable as in* (1.1).

Here the constant δ is very small and its value depends on the existence of the possible Siegel-zero (see [**3**]) of the Dirichlet series $L(s, \chi)$. Using a method first developed in [**2**] we will improve on this estimate by showing the following theorem:

Theorem. *All but* $\ll x^{19193/19200+\varepsilon}$ *positive even integers smaller than* x *can be represented as in* (1.1).

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Using the circle method the main difficulties arise on the major arcs, where we apply mean value estimates for Dirichlet polynomials and power moments of L-functions. Compared to [**1**] no special attention is paid to the possible Siegel zero and the Deuring-Heilbronn phenomena is not used.

2. Notation and structure of the proof. We will choose our notation similar to the one in [**8**]. By k we will always denote an integer $k \in \{2,3,4,5\}$, by p we denote a prime number and L denotes $\log x$. c is an effective positive constant and ε will denote an arbitrarily small positive number; both of them may take different values at different occasions. For example, we may write

$$L^c L^c \ll L^c, \qquad x^\varepsilon L^c \ll x^\varepsilon.$$

$d_2(n)$ denotes the number of divisors of n and $[a_1, \dots, a_n]$ denotes the least common multiple of the integers $a_1, \dots, a_n$. Be further

$$r \sim R \Longleftrightarrow R/2 < r \le R, \quad \sum_{\chi \bmod q}{}^{*} = \sum_{\substack{x \bmod q \\ x \text{primitive}}}, \quad \sum_{1 \le a \le q}{}^{*} = \sum_{\substack{1 \le a \le q \\ (a,q)=1}}^{q}.$$

$$P = N^{(7/150-\varepsilon)}, \quad Q = NP^{-1}L^{-E}, \quad (E > 0 \quad \text{will be defined later}),$$

and

$$\mu = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - 1.$$

We define for any characters $\chi, \chi_j \pmod q$, $q \le P$, and a fixed integer N:

$$C_k(a,\chi) = \sum_{l=1}^{q} \chi(l) e\left(\frac{al^k}{q}\right), \qquad C_k(a,\chi_0) = C_k(a,q)$$

$$Z(q,\chi_2,\chi_3,\chi_4,\chi_5) = \sum_{h=1}^{q}{}^{*} e\left(\frac{-hN}{q}\right) \prod_{k=2}^{5} C_k(h,\chi_k),$$

$$Y(q) = Z(q,\chi_0,\chi_0,\chi_0,\chi_0), \qquad A(q) = \frac{Y(q)}{\phi^4(q)},$$

$$S_k(\lambda,\chi) = \sum_{\sqrt[k]{x}/2^{k+1} \le n \le \sqrt[k]{x}} \Lambda(n)\chi(n)e(n^k\lambda),$$

$$T_k(\lambda) = \sum_{\sqrt[k]{x}/2^{k+1} \le n \le \sqrt[k]{x}} e(n^k \lambda),$$
$$W_k(\lambda, \chi) = S_k(\lambda, \chi) - E_0 T_k(\lambda),$$
$$E_0 = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

Using the circle method we define the major arcs M and minor arcs m as follows:

$$M = \sum_{q \le P} \sum_{a=1}^{q} {}^* I(a,q), I(a,q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq}\right],$$
$$m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus M.$$

Let

$$R(N) = \sum_{\substack{\sqrt[k]{x}/2^{k+1} \le n_k \le \sqrt[k]{x} \\ k \in \{2, \dots, 5\} \\ n_2^2 + \cdots + n_5^5 = N}} \Lambda(n_2) \cdots \Lambda(n_5).$$

Then we find

$$\begin{aligned} R(N) &= \int_{1/Q}^{1+1/Q} e(-N\alpha) \prod_{k=2}^{5} S_k(\alpha)\, d\alpha \\ &= \left(\int_M + \int_m\right) e(-N\alpha) \prod_{k=2}^{5} S_k(\alpha)\, d\alpha \\ &=: R_1(N) + R_2(N). \end{aligned} \tag{2.1}$$

Using Theorem 1 in [**5**] and Lemma 3 in [**11**], we obtain

$$\begin{aligned} \sum_{x/2 \le N < x} |I_2(N)|^2 &\le \max_{\alpha \in m} |S_5(\alpha)|^2 \int_m |S_2(\alpha) S_3(\alpha) S_4(\alpha)|^2 \\ &\ll x^{2\mu+1+\varepsilon} P^{-1/128}, \end{aligned}$$

from which we derive that

$$I_2(N) \ll N^{\mu} L^{-1000} \tag{2.2}$$

for all but $\ll x^{1+2\varepsilon}P^{-1/128} < x^{19193/19200+3\varepsilon}$ even integers $x/2 \le N < x$. In Sections 3–5 we will show that, for any given $G > 0$,

$$R_1(N) = \frac{1}{120} P_0 \sum_{q \le P} A(q) + O(x^{\mu} L^{-G}), \tag{2.3}$$

where

$$x^{\mu} \ll P_0 := \sum_{\substack{m_1+m_2+m_3+m_4=N \\ x/2^{k+1} < m_k \le x}} \frac{1}{m^{1-(1/k)}} \ll x^{\mu} \quad \text{for } N \in (x/2, x]. \tag{2.4}$$

In Section 6 we will derive from (2.3) that for all but $\ll x^{443/450+\varepsilon}$ positive even integers $x/2 < N \le x$, the following holds

$$R_1(N) = \frac{1}{120} P_0 \prod_{p \le P} s(p) + O(x^{\mu} L^{-G}). \tag{2.5}$$

Using that

$$\prod_{p \le P} s(p) \gg (\log P)^{-960}$$

(see [**1**, Lemma 4.5]) the theorem follows from (2.1), (2.2), (2.4) and (2.5).

3. The major arcs. We will make use of the following lemmas:

Lemma 3.1. *If* $(a, q) = 1$, *then*

$$C_k(a, \chi_q) \ll q^{1/2+\varepsilon}.$$

Proof. This is contained in Lemmas 5.1 and 5.2 in [**9**].

Lemma 3.2. *Let* $f(x), g(x)$ *and* $f'(x)$ *be three real differentiable and monotonic functions in the interval* $[a, b]$ *and* $|g(x)| \ll M$.

(i) *If* $|f'(x)| \gg m > 0$, *then*

$$\int_a^b g(x) e(f(x))\, dx \ll M/m.$$

(ii) *If* $|f''(x)| \gg r > 0$, *then*

$$\int_a^b g(x)e(f(x))\,dx \ll M/r^{1/2}.$$

(iii) *If* $|f'(x)| \le \theta < 1$, $g(x)$, $g'(x) \ll 1$, *then*

$$\sum_{a<n\le b} g(n)e(f(n)) = \int_a^b g(x)e(f(x))\,dx + O\left(\frac{1}{1-\theta}\right).$$

Proof. See Lemma 4.8 in [**13**].

Lemma 3.3. *For primitive characters* $\chi_1 \bmod r_i$, $i = 1, 2, 3, 4$, *and the principal character* $\chi_0 \bmod q$, *we have*

$$\sum_{\substack{q\le P\\ r|q}} \frac{|Z(q,\chi_0\chi_1,\chi_0\chi_2,\chi_0\chi_3,\chi_0\chi_4)|}{\phi^4(q)} \ll r^{-1+\varepsilon}(\log P)^c,$$

where $r = [r_1, r_2, r_3, r_4]$.

Proof. Let J denote the lefthand side in Lemma 3.3, and write $Z(q) = Z(q,\chi_0\chi_1,\chi_0\chi_2,\chi_0\chi_4,\chi_0\chi_4)$. Using Lemmas 4.1 and 4.3 a) in [**1**], we argue as in the proof of Lemma 6.7 in [**7**] and obtain

$$J \ll \sum_{u|a} \frac{|Z(ur)|}{\phi^4(ur)} \sum_{\substack{q\le P/ur\\ (q,r)=1}} |A(q)|,$$

where $a \ll 1$. From Lemma 3.1, we derive

$$\sum_{u|a} \frac{|Z(ur)|}{\phi^4(ur)} \ll r^{-1+\varepsilon}.$$

Lemma 3.3 follows therefore from

Lemma 3.4.
$$\sum_{q\le P}|A(q)|\ll(\log P)^c.$$

Proof. Using Lemmas 4.1, 4.4a) and (4.6) in [**1**], we find
$$\sum_{q\le P}|A(q)|\ll\prod_{p\le P}\left(1+\frac{c}{p}\right)\ll(\log P)^c.$$

Splitting the summation over n in residue classes modulo q we obtain
$$S_k\left(\frac{a}{q}+\lambda\right)=\frac{C_k(a,q)}{\phi(q)}\,T_k(\lambda)+\frac{1}{\phi(q)}\sum_{\chi \bmod q}C_k(a,\chi)W_k(\lambda,\chi)+O(L^2).$$

Thus we obtain from (2.1),
$$R_1(N)=R_1^m(N)+R_1^e(N)+O(x^\mu L^{-G})\quad\text{for any } G>0, \tag{3.1}$$

where
$$R_1^m(N)=\sum_{q\le P}\frac{1}{\phi^4(q)}\cdot\sideset{}{^*}\sum_{1\le a\le q}\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}C_k(a,q)e\left(-\frac{a}{q}\,N\right)T_k(\lambda)e(-\lambda N)\,d\lambda,$$
$$\begin{aligned}R_1^e(N)=&\sum_{\substack{k,l=2\\k<l}}^{5}\sum_{q\le P}\frac{1}{\phi^4(q)}\sideset{}{^*}\sum_{1\le a\le q}\int_{-1/Qq}^{1/Qq}\prod_{m\in\{k,l\}}C_m(a,q)T_m(\lambda)\\&\cdot\prod_{\substack{o=2\\o\ne k\\o\ne l}}^{5}\sum_{\chi \bmod q}C_0(a,\chi)W_0(\lambda,\chi)e\left(-\frac{a}{q}\,N-\lambda N\right)d\lambda\\&+\sum_{k=2}^{5}\sum_{q\le P}\frac{1}{\phi^4(q)}\sideset{}{^*}\sum_{1\le a\le q}\int_{-1/Qq}^{1/Qq}C_k(a,q)T_k(\lambda)\\&\cdot\prod_{\substack{l=2\\l\ne k}}^{5}\sum_{\chi \bmod q}C_l(a,q)W_l(\lambda,\chi)e\left(-\frac{a}{q}\,N-\lambda N\right)d\lambda\end{aligned}$$

$$
\begin{aligned}
&+\sum_{q\le P}\frac{1}{\phi^4(q)}\sideset{}{^*}\sum_{1\le a\le q}\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{t}\sum_{\chi \bmod q}\\
&\qquad\qquad\qquad\cdot C_k(a,\chi)W_k(\chi,\lambda)e\biggl(-\frac{a}{q}N-\lambda N\biggr)\,d\lambda,\\
&=: S_1+S_2+S_3+S_4.
\end{aligned}
$$

We first calculate $R_1^m(N)$. Applying Lemma 3.2 yields

$$
\begin{aligned}
T_k(\lambda)&=\int_{\sqrt[k]{x}/2^{k+1}}^{\sqrt[k]{x}} e(\lambda u^k)\,du+O(1)\\
&=\frac{1}{k}\int_{x/2^{k+1}}^{x} v^{1/k-1}e(\lambda v)\,dv+O(1)\\
&=\frac{1}{k}\sum_{x/2k+1<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}+O(1).
\end{aligned}
$$

Substituting this in $R_1^m(N)$, we see

$$
\begin{aligned}
R_1^m(N)&=\frac{1}{120}\sum_{q\le P}A(q)\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}\biggl(\sum_{x/2^{k+1}<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}\biggr)e(-N\lambda)\,d\lambda\\
&\quad+O\biggl(\biggl|\max_{2\le l\le 5}\sum_{q\le P}A(q)\int_{1/Qq}^{-1/Qq}\prod_{\substack{k=2\\k\ne l}}^{5}\sum_{2/2^{k+1}<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}\,d\lambda\biggr|\biggr).
\end{aligned}
$$

Using Lemma 3.3 and the trivial bound

$$
\sum_{x/2^{k+1}<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}\ll\min\biggl(\sqrt[k]{x},\,\frac{1}{x^{1-(1/k)}|\lambda|}\biggr) \tag{3.2}
$$

we derive

$$
\begin{aligned}
R_1^m(N) &= \frac{1}{120}\sum_{q\le P} A(q)\int_{-1/2}^{1/2}\prod_{k=2}^{5}\bigg(\sum_{x/2^{k+1}<m\le x}\frac{e(\lambda m)}{m^{1-(1/k)}}\bigg)e(-N\lambda)\,d\lambda \\
&\qquad + O\bigg(\sum_{q\le P}|A(q)|\int_{1/Qq}^{1/2}\frac{1}{x^{3-\mu}|\lambda|^4}\,d\lambda\bigg) + O(x^\mu L^{-G}) \\
&= \frac{1}{120}P_0\sum_{q\le P}A(q) + O((PQ)^3x^{\mu-3}L^c) + O(x^\mu L^{-G}) \\
&= \frac{1}{120}P_0\sum_{q\le P}A(q) + O(x^\mu L^{-G}),
\end{aligned} \tag{3.3}
$$

where P_0 is defined as in (2.4) and E is chosen sufficiently large in $Q = NP^{-1}L^{-E}$. In the sequel $E = E(G)$ is fixed. Now we estimate the terms S_i, $i = 1, 2, 3, 4$. Using Lemma 3.3 we can estimate S_4 in the following way:

$$
\begin{aligned}
|S_4| &\le \sum_{q\le P}\frac{1}{\phi^4(q)}\sum_{\chi_2 \bmod q}\sum_{\chi_3 \bmod q}\sum_{\chi_4 \bmod q}\sum_{\chi_5 \bmod q} \\
&\qquad \cdot |Z(q,\chi_2,\chi_3,\chi_4,\chi_5)|\int_{-1/Qq}^{1/Qq}\prod_{k=2}^{5}|W_k(\lambda,\chi_j)|\,d\lambda \\
&\le \sum_{r_2\le P}\sum_{r_3\le P}\sum_{r_4\le P}\sum_{\substack{r_5\le P\\ [r_2,r_3,r_4,r_5]\le P}}\sideset{}{^*}\sum_{\chi_2 \bmod r_3}\sideset{}{^*}\sum_{\chi_3 \bmod r_3}\sideset{}{^*}\sum_{\chi_4 \bmod r_4}\sideset{}{^*}\sum_{\chi_5 \bmod r_5} \\
&\qquad \cdot \int_{-1/Q[r_2,r_3,r_4,r_5]}^{1/Q[r_2,r_3,r_4,r_5]}\prod_{k=2}^{5}|W_k(\lambda,\chi_k)|\,d\lambda \\
&\qquad \cdot \sum_{\substack{q\le P\\ [r_2,r_3,r_4,r_5]|q}}\frac{|Z(q,\chi_2\chi_0,\chi_3\chi_0,\chi_4\chi_0,\chi_5\chi_0)|}{\phi^4(q)}, \\
&\ll L^c\sum_{r_2\le P}\sum_{r_3\le P}\sum_{r_4\le P}\sum_{r_5\le P}[r_2,r_3,r_4,r_5]^{-1+\varepsilon} \\
&\qquad \cdot \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\sideset{}{^*}\sum_{\chi_3 \bmod r_3}\sideset{}{^*}\sum_{\chi_4 \bmod r_4}\sideset{}{^*}\sum_{\chi_5 \bmod r_5}
\end{aligned}
$$

$$\cdot \int_{-1/Q[r_2,r_3,r_4,r_5]} \prod_{k=2}^{5} |W_k(\lambda, \chi_k)| \, d\lambda.$$

Using $[r_2, r_3, r_4, r_5] \geq (r_2 r_3)^{1/7} (r_4 r_5)^{5/14}$, we obtain

$$\begin{aligned}
S_4 \ll L^c & \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} \sum_{r_k\leq P} r_k^{-5/14+\varepsilon} \sideset{}{^*}\sum_{\chi_k \bmod r_k} \\
& \cdot |W_k(\lambda, \chi_k| \max_{|\lambda|\leq 1/Q} \sum_{r_l\leq P} r_l^{-5/14+\varepsilon} \\
& \cdot \sideset{}{^*}\sum_{\chi_l \bmod r_l} |W_l(\lambda, \chi_l| \sum_{r_m\leq P} r_m^{-1/7+\varepsilon} \sideset{}{^*}\sum_{\chi_m \bmod r_m} \\
& \cdot \left(\int_{-1/Qr_m}^{1/Qr_m} |W_m(\lambda, \chi_m|^2 \, d\lambda \right)^{1/2} \\
& \cdot \sum_{r_n\leq P} r_n^{-1/7+\varepsilon} \sideset{}{^*}\sum_{\chi_n \bmod r_n} \left(\int_{-1/Qr_n}^{1/Qr_n} |W_n(\lambda, \chi_n|^2 \, d\lambda \right)^{1/2} \\
\ll L^c & \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} I_k(\lambda) \max_{|\lambda|\leq 1/Q} I_l(\lambda) W_m W_n,
\end{aligned} \tag{3.4}$$

where

$$I_k(\lambda) = \sum_{r\leq P} r^{-5/14+\varepsilon} \sideset{}{^*}\sum_{\chi} |W_k(\lambda, \chi|,$$

$$W_k = \sum_{r\leq P} r^{-1/7+\varepsilon} \sideset{}{^*}\sum_{\chi} \left(\int_{-1/Qr}^{1/Qr} |W_k(\lambda, \chi|^2 \, d\lambda \right)^{1/2}.$$

Arguing similarly we obtain

$$\begin{aligned}
S_1 + S_2 + S_3 \ll L^c & \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} |T_k(\lambda)| \\
& \cdot \max_{|\lambda|\leq 1/Q} |T_l(\lambda)| \left(\int_{-1/Q}^{1/Q} |T_m(\lambda)|^2 \, d\lambda \right)^{1/2} W_n \\
& + L^c \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} |T_k(\lambda)| \max_{|\lambda|\leq 1/Q} |T_l(\lambda)| W_m W_n \\
& + L^c \max_{2\leq k<l<m<n\leq 5} \max_{|\lambda|\leq 1/Q} |T_k(\lambda)| \max_{|\lambda|\leq 1/Q} I_l(\lambda) W_m W_n.
\end{aligned} \tag{3.5}$$

We have trivially

$$\max_{|\lambda|\le 1/Q} |T_k(\lambda)| \ll x^{1/k}.$$

Using (3.2) we obtain

$$\left(\int_{-1/Q}^{1/Q} |T(\lambda)|^2 \, d\lambda\right)^{1/2} \ll x^{(1/k)-(1/2)}.$$

Thus we see from (3.1) and (3.3)–(3.5) that the proof of (2.3) reduces to the proof of the following two lemmas:

Lemma 3.5. *If* $P \le x^{(7/150)-\varepsilon}$ *and* $2 \le k \le 5$,

$$W_k \ll_B x^{1/k-1/2} L^{-B}$$

for any $B > 0$.

Lemma 3.6. *If* $P \le x^{(7/150)-\varepsilon}$ *and* $2 \le k \le 5$,

$$\max_{|\lambda|\le 1/Q} I(\lambda) \ll x^{1/k} L^{A}$$

for a certain $A > 0$.

For the proof of these lemmas we will appeal to the following lemmas:

Lemma 3.7. *For any* $P \ge 1$, $T \ge 1$ *and* $k = 0, 1$,

$$\sum_{q\le P} \sideset{}{^*}\sum_{\chi} \int_{-T}^{T} \left| L^{(k)}\left(\frac{1}{2} + it, \chi\right)\right|^4 dt \ll P^2 T (\log PT)^{4(k+1)}.$$

Lemma 3.8. *For any* $P \ge 1$, $T \ge 1$ *and any complex numbers* a_n

$$\sum_{q\le P} \sideset{}{^*}\sum_{\chi} \int_{-T}^{T} \left| \sum_{n=M+N}^{M} a_n \chi(n) n^{-it}\right|^2 dt \ll \sum_{n=M+N}^{M} (P^2 T + n)|a_n|^2.$$

Lemma 3.9. *Let $N^*(\alpha, T, q)$ denote the number of zeros $\sigma + it$ of all L-functions to primitive characters modulo q within the region $\sigma \geq \alpha$, $|t| \leq T$. Then*

$$\sum_{q \leq Q} N^*(\alpha, T, q) \ll T^{12(1-\alpha)/5} (\log QT)^c.$$

The lemmas 3.7–3.9 may be found in [**10**, Chapters 2, 3 and 5].

4. Proof of Lemma 3.5. In order to prove the lemma, it is enough to show that

$$W_{k,R} \ll x^{(1/k)-(1/2)} R^{1/7-\varepsilon} L^{-B}, \tag{4.1}$$

where

$$W_{k,R} = \sum_{r \sim R} \sum_{\chi}{}^{*} \left(\int_{-1/Qr}^{1/Qr} |W_k(\lambda, \chi|^2 \, d\lambda \right)^{1/2}$$

for $R \leq P/2$. Applying Lemma 1 [**4**], we see

(4.2)

$$\int_{-1/Qr}^{1/Qr} |W_k(\lambda, \chi)|^2 \, d\lambda$$
$$\ll (QR)^{-2} \int_{x/2^{k+2}}^{x} \Bigg| \sum_{\substack{t < m^k \leq t + Qr \\ x/2^{k+1} < m^k \leq x}} \Lambda(m)\chi(m) - E_0 \sum_{\substack{t < m^k \leq t + Qr \\ x/2^{k+1} < m^k \leq x}} 1 \Bigg|^2 dt.$$

We set $X = \max(x/2^{k+1}, t)$ and $X + Y = \min(x, t + Qr)$. In the sequel we will treat the case $R > L^D$ and $R \leq L^D$ for a sufficiently large constant $D > 0$ separately. In the first case we apply a slight modification of Heath-Brown's identity [**6**],

$$-\frac{\zeta^\iota}{\zeta}(s) = \sum_{j=1}^{K} \binom{K}{j} (-1)^{j-1} \zeta^\iota(s) \zeta^{j-1}(s) M^j(s)$$
$$- \frac{\zeta^\iota}{\zeta}(s)(1 - \zeta(s)M(s))^K,$$

with $K = 5$ and
$$M(s) = \sum_{n \le x^{1/5k}} \mu(n)$$
to the sum
$$\sum_{X < m^k \le X+Y}.$$
Arguing exactly as in part III, [**15**], we find by applying Heath Brown's identity and Perron's summation formula (see [**13**, Lemma 3.12]) that the inner sum of (4.2) – where always $E_0 = 0$ because of $R > L^D$ and the primitivity of the characters – is a linear combination of $O(L^c)$ terms of the form
$$S_k = \frac{1}{2\pi i} \int_{-T}^{T} F_k\left(\frac{1}{2} + iu, \chi\right) \frac{(X+Y)^{(1/2+iu)/k} - X^{(1/2+iu)/k}}{(1/2) + iu} \, du \\ + O(T^{-1} x^{(1/k)+\varepsilon}),$$
where $2 \le T \le x$,
$$F_k(x, \chi) = \prod_{j=1}^{10} f_{k,j}(s, \chi),$$
$$f_{k,j}(s, \chi) = \sum_{n \in I_{k,j}} a_{k,j}(n) \chi(n) n^{-s},$$
$$a_{k,j}(n) = \begin{cases} \log n \text{ or } 1 & j = 1, \\ 1 & 1 < j \le 5, , \\ \mu(n) & 6 \le 10 \end{cases} \tag{4.3}$$
$$I_j = (N_{k,j}, 2N_{k,j}], \quad 1 \le j \le 10,$$
$$\sqrt[k]{x} \ll \prod_{j=1}^{10} N_{k,j} \ll \sqrt[k]{x}, \quad N_{k,j} \le x^{1/5k}, \quad 6 \le j \le 10.$$
Since
$$\frac{(X+Y)^{(1/k)[(1/2)+iu]} - X^{(1/k)[(1/2)+iu]}}{(1/2) + iu} \\ \ll \min(QRx^{(1/2k)-1}, x^{1/2k}(|u|+1)^{-1})$$

by taking $T = x^{2\varepsilon}P^2(1+|\lambda|x)$ and $T_0 = x(QR)^{-1}$, we conclude that S_k is bounded by

$$\ll QRx^{(1/2k)-1}\int_{-T_0}^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|du \\ + x^{1/2k}\int_{T_0\le|u|\le T}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|\frac{du}{|u|}+x^{1/k}P^{-2},$$

Thus we derive from (4.2) that in order to prove (4.1) it is enough to show that

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)g\right|dt \ll x^{1/2k}R^{1/7-\varepsilon}L^{-B}, \tag{4.4}$$

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_{T_1}^{2T_1}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt \\ \ll x^{1/2k-1}QR^{8/7-\varepsilon}T_1L^{-B},\quad T_0<|T_1|\le T. \tag{4.5}$$

For the proof of (4.4) and (4.5) we will prove two propositions. We will need the estimate

$$\sum_{n\le x} d_2^k(n) \ll_k xL^{c(k)}. \tag{4.6}$$

We now establish

Proposition 1. *If there exist N_{k,j_1} and N_{k,j_2}, $1\le j_1,\ j_2\le 5$, such that $N_{k,j_1}N_{k,j_2}\ge P^{12/7+3\varepsilon}$, then (4.4) is true.*

Proof. We suppose without loss of generality that $j_1=1$, $a_1(n)=\log n$ and $j_2=2$, $a_2(n)=1$. Arguing exactly as in the proof of Proposition 1 in [**15**], we find

$$f_{k,1}\left(\frac{1}{2}+it,\chi\right) \ll L\left(\int_{-x^{1/k}}^{x^{1/k}}\left|L'\left(\frac{1}{2}+it+iv,\chi\right)\right|^4\frac{dv}{1+|v|}\right)^{1/4}+L,$$

and so we find by using Lemma 3.7,

$$\begin{aligned}
&\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|f_1\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\\
&\ll L^4\int_{-x^{1/k}}^{x^{1/k}}\frac{dv}{1+|v|}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_v^{T_0+v}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^4 dt+T_0R^2L^4\\
&\ll L^5\max_{|N|\le x^{1/k}}\int_{N/2}^{N}\frac{dv}{1+|v|}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_v^{T_0+v}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\\
&\quad+T_0R^2L^4\\
&\quad+L^5\max_{|N|\le x^{1/k}}N^{-1}\int_0^{T_0}dt\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{(N/2)+t}^{N+t}\left|L'\left(\frac{1}{2}+iv,\chi\right)\right|^4 dv\\
&\quad+T_0R^2L^4\\
&\ll R^2T_0L^c.
\end{aligned}$$

Using Lemma 3.8, (4.6) and Holder's inequality, we obtain

$$\begin{aligned}
&\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt\\
&\quad\ll\left(\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|f_{k,1}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/4}\\
&\qquad\cdot\left(\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|f_{k,2}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/4}\\
&\qquad\cdot\left(\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\left|\prod_{j=3}^{10}f_{k,j}\left(\frac{1}{2}+it,\chi\right)\right|dt\right)^{1/2}\\
&\quad\ll(R^2T_0)^{1/2}\left(R^2T_0+\frac{x^{1/k}}{N_{k,1}N_{k,2}}\right)^{1/2}L^c\\
&\quad\ll x^{1/2k}R^{1/7-\varepsilon}L^{-B},
\end{aligned}$$

by the definition of T_0 and the condition of the proposition.

Proposition 2. *Let $J=\{1,\dots,10\}$. If J can be divided into two nonoverlapping subsets J_1 and J_2 such that*

$$\max\Bigl(\prod_{j\in J_1}N_{k,j},\prod_{j\in J_2}N_{k,j}\Bigr)\ll x^{1/k}P^{-(12/7)-3\varepsilon}$$

then (4.4) *is true.*

Proof. let

$$\begin{aligned}F_{k,i}(s,\chi)&=\prod_{j\in J_i}f_{k,j}(s,\chi)\\&=\sum_{n\ll M_i}b_i(n)\chi(n)n^{-s},\\b_i(n)&\ll d_2^c(n),\quad i=1,2,\end{aligned}$$

where $M_i=\prod_{j\in J_i N_{k,j}}$, $i=1,2$. Applying Lemma 3.8, (4.3) and (4.6) we see

$$\begin{aligned}&\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\Bigl|F_k\Bigl(\frac12+it,\chi\Bigr)\Bigr|\,dt\\&\quad\ll\Bigl(\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\Bigl|F_{k,1}\Bigl(\frac12+it,\chi\Bigr)\Bigr|\,dt\Bigr)^{1/2}\\&\qquad\cdot\Bigl(\sum_{r\sim R}\sideset{}{^*}\sum_{\chi}\int_0^{T_0}\Bigl|F_{k,2}\Bigl(\frac12+it,\chi\Bigr)\Bigr|\,dt\Bigr)^{1/2}\\&\quad\ll(R^2T_0+M_1)^{1/2}(R^2T_0+M_2)^{1/2}\\&\quad\ll R^2T_0+x^{1/2k}RP^{-(6/7)-(3/2)\varepsilon}T_0^{1/2}+x^{1/2k}L^c.\end{aligned}$$

This proves the proposition because of $R>L^D$. Whereas for the proof of the proposition an estimate $P\ll x^{(7/130)-\varepsilon}$ would have been enough, we need the estimate $P\le x^{(7/150)-\varepsilon}$ in the following. Now we can prove (4.4). In view of Proposition 1 we assume

$$N_{k,i}N_{k,j}\le P^{12/7+3\varepsilon}\le x^{2/5k},\quad 1\le i,\quad j\le 5,\quad i\ne j.$$

Therefore, we see from (4.3) that there exists at most one $N_{k,j}$, $1 \le j \le 10$, with $N_{k,j} \ge x^{1/5k}$. Suppose such a $N_{k,j}$ is N_{k,j_0} if it exists (otherwise $N_{k,j_0} = 1$). Reorder the other $N_{k,j}$ as follows:

$$N_{k,j_1} \ge N_{k,j_2} \ge \dots \ge N_{k,j_K}, \quad K = 9 \text{ or } 10.$$

We find an integer $1 \le l \le K-1$ such that

$$N_{k,j_0} N_{k,j_1} \dots N_{k,j_{l-1}} \le x^{2/5k} \quad \text{and} \quad N_{k,j_0} N_{k,j_1} \dots N_{k,j_l} \ge x^{2/5k}.$$

Taking $M_1 = N_{k,j_0} N_{k,j_1} \dots N_{k,j_l}$ and $M_2 = N_{k,j_{l+1}} \dots N_{k,j_K}$, we have

$$M_1 \ll x^{2/5k} N_{k,j_l} \le x^{3/5k} \quad \text{and} \quad M_2 \ll x^{1/5k} M_1^{-1} \ll x^{3/5k}.$$

The sets M_1 and M_2 satisfy the conditions of Proposition 2 and therefore (4.4) is proved. The proof of (4.5) goes along the same lines. (4.1) is now proved in the case $R > L^D$. If $R \le L^D$ we can estimate the sum on the righthand side of (4.2) by using the zero expansion of the von Mangoldt function:

$$\begin{aligned}
&\sum_{\substack{t < m^k \le t+Qr \\ x/2^k < m^k \le x}} \Lambda(m)\chi(m) - E_0 \sum_{\substack{t < m^k \le t+Qr \\ x/2^k < m^k \le x}} 1 \\
&\qquad = \sum_{X < m^k \le X+Y} \Lambda(m)\chi(m) - E_0 \sum_{X < m^k \le X+Y} 1 \\
&\qquad \ll \sum_{|\mathrm{Im}\,\rho| \le x^{1/3k}} \left| \frac{(X+Y)^{\rho/k}}{\rho} - \frac{X^{\rho/k}}{\rho} \right| + O(x^{2/3k} L^2) \\
&\qquad \ll QRx^{(1/k)-1} \sum_{|\mathrm{Im}\,\rho| \le x^{1/3k}} x^{\beta - 1/k} + O(x^{2/3k} L^2),
\end{aligned}$$

where ρ runs over the nontrivial zeros of the L-function corresponding to $\chi \bmod r$ with $|\mathrm{Im}\,\rho| \le x^{1/3k}$ and $\beta = \mathrm{Re}\,\rho$. Applying Lemma 3.9 and the fact that $L(\sigma + it, \chi)$ with $\chi \bmod r \le L^D$ has no zeros in the region (see [**12**], VIII Satz 6.2)

$$\sigma \ge 1 - \delta(T) := 1 - \frac{c_0}{\log r + (\log(T+2))^{4/5}}, \quad |t| \le T,$$

where c_0 is an absolute constant and taking $T = x^{1/3k}$ we obtain from (4.2)

$$
\begin{aligned}
&\int_{-1/Qr}^{1/Qr} |W_k(\lambda,\chi)|^2 \, d\lambda \\
&\qquad \ll x^{(2/k)-1}\bigg(\sum_{|\operatorname{Im}\rho|\le x^{1/3k}} x^{(\beta-1)/k}\bigg)^2 + (QR)^{-2}x^{1+(4/3k)}L^4 \\
&\qquad \ll x^{(2/k)-1}L^c\Big(\max_{(1/2)\le\beta\le 1-\delta(T)} x^{(4/5k)(1-\beta)}x^{(1/k)(\beta-1)}\Big)^2 \\
&\qquad\quad + P^2x^{(4/3k)-1}L^{2E+4} \\
&\qquad \ll x^{(2/k)-1}\exp(-cL^{1/5}).
\end{aligned}
$$

This gives (4.1) for $R \le L^D$.

5. Proof of Lemma 3.6. To prove the lemma it is enough to show that

$$\max_{R\le P/2}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi} |W_k(\lambda,\chi_r)| \ll x^{1/k}R^{(5/14)-\varepsilon}L^A,$$

uniformly for $|\lambda| \le Q^{-1}$. Arguing as in the section before – we do not have to apply Gallagher's lemma here – we find

$$
\begin{aligned}
W_k(\lambda,\chi) \ll L^c \max_{I_{a_1},\dots,I_{a_{2k+1}}} &\bigg| \int_{-T}^{T} F\bigg(\frac{1}{2}+it,\chi\bigg)\,dt \\
&\cdot \int_{x/2^{k+1}}^{x} u^{(1/2k)-1} e\bigg(\frac{t}{2k\pi}\log u + \lambda u\bigg)\,du \bigg| + x^{1/k}P^{-1},
\end{aligned}
$$

for $T = P^3$. Estimating the inner integral by Lemma 3.2, we obtain

$$
\begin{aligned}
&\int_{x/2^{k+1}}^{x} u^{(1/2k)-1} e\bigg(\frac{t}{2k\pi}\log u + \lambda u\bigg)\,du \\
&\qquad \ll x^{(1/2k)-1}\min\bigg(\frac{x}{\sqrt{|t|+1}}, \frac{x}{\min_{x/2^{k+1}<u\le x}|t+2k\pi\lambda u|}\bigg).
\end{aligned}
$$

Taking $T_0 = 4k\pi x Q^{-1}$, we conclude that in order to prove this lemma it is enough to prove that for $P \le x^{(7/150)-\varepsilon}$ and $2 \le k \le 5$, the following holds

(5.1)
$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt \ll x^{1/2k}T_0^{1/2}R^{5/14-\varepsilon}L^c,$$

(5.2)
$$\sum_{r\sim R}\sum_{\chi}{}^{*}\int_{T_1}^{2T_1}\left|F_k\left(\frac{1}{2}+it,\chi\right)\right|dt \ll x^{1/2k}R^{5/14-\varepsilon}T_1L^c,$$
$$T_0<|T_1|\le T.$$

These estimates are shown in the same way as (4.4) and (4.5). here the condition $P\le x^{(7/150)-\varepsilon}$ is needed. Two propositions analogous to Propositions 1 and 2 are proved:

Proposition 3. *If there exist N_{k,j_1} and N_{k,j_2}, $1\le j_1,\ j_2\le 5$, such that $N_{k,j_1},N_{k,j_2}\ge P^{9/7+3\varepsilon}$, then* (5.1) *is true.*

Proposition 4. *Let $J=\{1,\dots,10\}$. If J can be divided into two nonoverlapping subsets J_1 and J_2 such that*

$$\max\left(\prod_{j\in J_1}N_{k,j},\prod_{j\in J_2}N_{k,j}\right)\ll x^{1/k}P^{-(9/7)-3\varepsilon},$$

then (5.1) *is true.*

Remark. Here we do not need to treat the case $R>L^D$ separately because we do not have to save a factor L^{-B}.

6. The singular series. We now derive (2.5) from (2.3). In the sequel we write $A(q,N)$ instead of $A(q)$ and $s(p,N)$ instead of $s(p)$ because we will argue for variable N.

Lemma 6.1. *For $P\le x^{(7/150)-\varepsilon}$, we have*

$$\sum_{N\le x}\left|\prod_{p\le P}s(p,N)-\sum_{q\le P}A(q,N)\right|\ll xP^{-(1/3)+\varepsilon}, \tag{6.1}$$

which implies that for all but $\ll x^{1+2\varepsilon}P^{-1/3}$ *even integers* N *with* $1 \le N \le x$*, the following holds*

$$\prod_{p\le P} s(p,N) = \sum_{q\le P} A(q,N) + O(x^{-\varepsilon}). \tag{6.2}$$

From here, (2.5) follows.

Proof. Equation (6.1) was proved in Lemma 5.1 in [**1**] for a sufficiently small ε for P as large as x^{ε}. We show that it also holds for $x^{\varepsilon} < P \le x^{(7/150)-\varepsilon}$. We argue exactly as in the proof of Lemma 5.1 in [**1**], but here we set: $V := \exp(\log x \log P/\log\log x)$ and $v = 3\log\log x/4\log P$. Denoting the lefthand side in (6.1) by J, we follow the proof of Lemma 5.1 in [**1**]:

$$\begin{aligned} J &\ll xV^{-v}L^{cL^{1/2}} + x^{1+\varepsilon}P^{-1/3} + x^{7/8+\varepsilon} + x^{(31/40)+\varepsilon} \\ &\quad \cdot \sum_{10\le m\le (2+\varepsilon)\log P/\log\log x} (m\log(xe))^m \\ &\ll x^{7/8+\varepsilon} + x^{1+\varepsilon}P^{-1/3} + x^{(31/40)+\varepsilon} \\ &\quad \cdot \sum_{10\le m\le (2+\varepsilon)\log P/\log\log x} (m(\log(xe))^m. \end{aligned} \tag{6.3}$$

For the calculation of the last sum, we have used $x^{(m-1)/2} \le V$ and therefore $m \le (2+\varepsilon)\log P(\log\log x)^{-1}$ for a sufficiently large x. We obtain as an upper bound:

$$\begin{aligned} &\ll P^{2+\varepsilon} \sum_{10\le m\le (2+\varepsilon)\log P/\log\log x} (\log(xe))^m \\ &\ll P^{2+\varepsilon}\exp((2+\varepsilon)\log P\log\log(xe)/\log\log x)\log P/\log\log x \\ &\ll P^{2+\varepsilon}\exp((2+2\varepsilon)\log P)\log P \ll P^{4+3\varepsilon}. \end{aligned} \tag{6.4}$$

We derive from (6.3) and (6.4)

$$\begin{aligned} J &\ll x^{7/8+\varepsilon} + x^{1+\varepsilon}P^{-1/3} + x^{(31/40)+\varepsilon}P^{4+3\varepsilon}L^c \\ &\ll x^{1+\varepsilon}P^{-1/3}. \end{aligned}$$

This completes the proof of Lemma 6.1.

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