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INDUCTION OF CHARACTERS, KERNELS AND LOCAL SUBGROUPS

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1. Introduction. Let G be a finite group, and let p be a prime number. Suppose that K is a subgroup of G having a character α such that $\alpha^G \in \mathrm{Irr}\,(G)$. If $\alpha(1) \leq p/2$, G.R. Robinson proved in [**5**] that

$$(\alpha_{\mathbf{N}_K(P)})^{\mathbf{N}_G(P)} \in \mathrm{Irr}\,(\mathbf{N}_G(P)),$$

where $P \in \mathrm{Syl}\,_p(K)$.

In [**2**], M. Isaacs proved this result when $\alpha(1) = 1$ by using elementary character theory. Robinson's general proof uses Green theory.

In the present note, we take a different approach and pay attention to the group $K/\ker(\alpha)$ instead of the degree of α.

Theorem A. *Let $K \subseteq G$ and suppose that $\alpha \in \mathrm{Irr}\,(K)$ induces $\alpha^G \in \mathrm{Irr}\,(G)$. Suppose that $P \in \mathrm{Syl}\,_p(K)$ is such that $P\ker(\alpha) \triangleleft K$. Then*

$$(\alpha_{\mathbf{N}_K(P)})^{\mathbf{N}_G(P)} \in \mathrm{Irr}\,(\mathbf{N}_G(P)).$$

Notice that if α is linear then $K/\ker(\alpha)$ is abelian, and we are in the hypothesis of Theorem A. Also, by using the Feit-Thompson theorem on linear groups, we will recover most of Robinson's theorem. The only case in the Robinson's situation which is not treated by our methods is when $\alpha(1) = (p-1)/2$. In this case, in view of the classification of finite simple groups, a description of the non-p-closed linear groups of degree $(p-1)/2$ is possible. As remarked by the referee, it might be possible to weaken the hypothesis of p-closure in Theorem A to still obtain the same conclusion.

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In our opinion, Theorem A has two extreme cases which are worth mentioning: when $K/\ker(\alpha)$ is a p-group and when $K/\ker(\alpha)$ is a p'-group. In the first case, a P-projective irreducible character of $\mathbf{N}_G(P)$ is obtained. In the second, Alperin celebrated p-weights appear.

Corollary B. *Suppose that $\alpha^G \in \mathrm{Irr}(G)$ where $\alpha \in \mathrm{Irr}(K)$ and $K/\ker(\alpha)$ is a p'-group. Let P be a Sylow p-subgroup of K. Then*

$$(\alpha_{\mathbf{N}_K(P)})^{\mathbf{N}_G(P)}$$

is a p-defect zero character of $\mathbf{N}_G(P)/P$.

Proof. By Theorem A, we have that the character $\eta = (\alpha_{\mathbf{N}_K(P)})^{\mathbf{N}_G(P)}$ is irreducible. Since $P \subseteq \ker(\alpha)$, it easily follows that $P \subseteq \ker(\eta)$. Also it is clear that η is a defect zero character of $\mathbf{N}_G(P)/P$ since it is induced from a p'-subgroup of $\mathbf{N}_G(P)/P$. ◻

In the hypothesis of Corollary B, we find it interesting to study to what extend χ uniquely determines (up to G-conjugacy) the p-subgroup P. This seems to be a difficult problem, however.

2. Proof of Theorem A. The main ingredient in the proof of Theorem A is the following lemma (which follows directly from Mackey's theorem).

2.1 Lemma. *Let $R \subseteq G$ and suppose that $\alpha \in \mathrm{Irr}(R)$. Then α^G is irreducible if and only if we have*

$$[\alpha^g_{R\cap R^g}, \alpha_{R\cap R^g}] = 0$$

for all $g \in G - R$.

Proof. See Lemma (2.1) of [**2**]. ◻

We will repeatedly use the following elementary result, which we state and prove for the reader's convenience.

2.2 Lemma. *Suppose that* $N \triangleleft G$ *and* $H \subseteq G$ *are such that* $NH = G$. *Write* $M = N \cap H$. *Then the restriction map* $\alpha \mapsto \alpha_H$ *is a bijection from* $\mathcal{A} = \{\alpha \in \mathrm{Char}\,(G) \mid N \subseteq \ker(\alpha)\}$ *onto* $\mathcal{B} = \{\beta \in \mathrm{Char}\,(H) \mid M \subseteq \ker(\beta)\}$. *In fact,*

$$[\alpha_H, \gamma_H] = [\alpha, \gamma]$$

for $\alpha, \gamma \in \mathcal{A}$.

Proof. Since the map $hM \mapsto hN$ is a group isomorphism, it follows that there is an additive bijection from $\mathcal{B}$ to $\mathcal{A}$ (sending $\mathrm{Irr}\,(H/M)$ onto $\mathrm{Irr}\,(G/N)$) whose inverse is the restriction map. □

2.3 Lemma. *Let* $N, M \triangleleft G$ *and let* $P \in \mathrm{Syl}_p(G)$. *Suppose that* $G = N\mathbf{N}_G(P) = M\mathbf{N}_G(P)$. *Let* $\alpha \in \mathrm{Irr}\,(G/N)$ *and* $\beta \in \mathrm{Irr}\,(G/M)$. *Then* $\alpha_{\mathbf{N}_G(P)} = \beta_{\mathbf{N}_G(P)}$ *if and only if* $\alpha = \beta$.

Proof. Assume that $\alpha_{\mathbf{N}_G(P)} = \beta_{\mathbf{N}_G(P)}$ and let $x \in G$. We want to show that $\alpha(x) = \beta(x)$. We have that the groups G/N and G/M have a normal Sylow p-subgroup. Therefore, so does $G/N \cap M$. Hence, $P(M \cap N)$ is also normal in G. Therefore, $G = (N \cap M)\mathbf{N}_G(P)$. Now we can write $x = yz$ where $y \in N \cap M$ and $z \in \mathbf{N}_G(P)$. Then

$$\alpha(x) = \alpha(yz) = \alpha(z) = \beta(z) = \beta(yz) = \beta(x),$$

as required. □

Next is Theorem A from the Introduction.

2.4 Theorem. *Let* $K \subseteq G$ *and suppose that* $\alpha \in \mathrm{Irr}\,(K)$ *induces* $\alpha^G \in \mathrm{Irr}\,(G)$. *Suppose that* $P \in \mathrm{Syl}_p(K)$ *is such that* $P\ker(\alpha) \triangleleft K$. *Then*

$$(\alpha_{\mathbf{N}_K(P)})^{\mathbf{N}_G(P)} \in \mathrm{Irr}\,(\mathbf{N}_G(P)).$$

Proof. Write $U = \mathbf{N}_K(P)$, $N = \mathbf{N}_G(P)$ and $V = \ker(\alpha) \cap U$. By the Frattini argument, we have that $\ker(\alpha)U = K$. Thus, $\beta = \alpha_U \in \mathrm{Irr}\,(U)$ by Lemma 2.2. We wish to prove that

$$\beta^N \in \mathrm{Irr}\,(N).$$

By Lemma 2.1, it suffices to check that, given $n \in N - U$, then

$$[\beta^n_{U\cap U^n}, \beta_{U\cap U^n}] = 0.$$

Otherwise, let $\gamma \in \mathrm{Irr}\,(U \cap U^n)$ be an irreducible constituent of $\beta^n_{U\cap U^n}$ and $\beta_{U\cap U^n}$.

Now $P \subseteq U \cap U^n$ and $U \cap U^n = \mathbf{N}_{K\cap K^n}(P)$. Since $P\ker(\alpha) \triangleleft K$, it follows that $P(\ker(\alpha)\cap K^n) = P\ker(\alpha)\cap K^n \triangleleft K\cap K^n$. By the Frattini argument, we have that

$$K \cap K^n = (\ker(\alpha) \cap K^n)\mathbf{N}_{K\cap K^n}(P) = (\ker(\alpha) \cap K^n)(U \cap U^n).$$

Also,

$$\mathbf{N}_{K\cap K^n}(P) \cap \ker(\alpha) \cap K^n = V \cap U^n.$$

Now γ is an irreducible constituent of $\beta_{U\cap U^n} = \alpha_{U\cap U^n}$. Hence, $V \cap U^n = \ker(\alpha) \cap (U \cap U^n)$ is contained in the kernel of γ. By Lemma 2.2, let

$$\hat{\gamma} \in \mathrm{Irr}\,(K \cap K^n/\ker(\alpha) \cap K^n)$$

be such that

$$\hat{\gamma}_{U\cap U^n} = \gamma.$$

By Lemma 2.2, we have that

$$[\alpha_{K\cap K^n}, \hat{\gamma}] = [\alpha_{U\cap U^n}, \gamma] \neq 0.$$

Now we repeat the argument above with K^n and α^n. Note first that $P\ker(\alpha^n) \triangleleft K^n$. Hence, $P(\ker(\alpha^n) \cap K) = K \cap P\ker(\alpha^n) \triangleleft K \cap K^n$, and by the Frattini argument we have that

$$K \cap K^n = (\ker(\alpha^n) \cap K)\mathbf{N}_{K\cap K^n}(P) = (\ker(\alpha^n) \cap K)(U \cap U^n).$$

Also,

$$\mathbf{N}_{K\cap K^n}(P) \cap \ker(\alpha^n) \cap K = U \cap V^n.$$

Now γ is an irreducible constituent of $\beta^n_{U\cap U^n} = \alpha^n_{U\cap U^n}$. Thus γ has $U \cap V^n$ in its kernel. By Lemma 2.2 there exists

$$\tilde{\gamma} \in \mathrm{Irr}\,(K \cap K^n/\ker(\alpha^n) \cap K)$$

such that

$$\tilde{\gamma}_{U \cap U^n} = \gamma.$$

Furthermore, by Lemma 2.2 we have that

$$[\alpha^n_{K \cap K^n}, \tilde{\gamma}] = [\alpha^n_{U \cap U^n}, \gamma] \neq 0.$$

Now by Lemma 2.3 applied to the normal subgroups $\ker(\alpha^n) \cap K$ and $\ker(\alpha) \cap K^n$ of $K \cap K^n$, we have that

$$\hat{\gamma} = \tilde{\gamma} = \tau.$$

Hence, τ is an irreducible constituent of both $\alpha^n_{K \cap K^n}$ and $\alpha_{K \cap K^n}$. This contradicts Lemma 2.1, since α^G is irreducible, $n \in G - K$ and $K \cap N = U$. □

As a consequence of Theorem A, we recover part of Robinson's theorem.

2.5 Theorem. *Let $K \subseteq G$ and suppose that $\alpha \in \mathrm{Irr}(K)$ induces $\alpha^G \in \mathrm{Irr}(G)$. Suppose that $P \in \mathrm{Syl}_p(K)$. If $\alpha(1) < (p-1)/2$, then*

$$(\alpha_{\mathbf{N}_K(P)})^{\mathbf{N}_G(P)} \in \mathrm{Irr}(\mathbf{N}_G(P)).$$

Proof. By the Feit-Thompson theorem on linear groups ([**1**]), we have that $K/\ker(\alpha)$ has a normal Sylow p-subgroup and Theorem A applies. □

Our Corollary B naturally leads us to study characters $\chi \in \mathrm{Irr}(G)$ which are of the form

$$\chi = \alpha^G$$

for some $\alpha \in \mathrm{Irr}(K)$ with $K/\ker(\alpha)$ a p'-group. To what extent are the Sylow p-subgroups of K determined by χ? In general, two different inductions of the same irreducible character have little in common and are difficult to compare. This is also the case here, and we are unable to prove or disprove whether or not the Sylow p-subgroups of K are uniquely determined by χ up to G-conjugacy. We succeed in proving this for groups with a normal p-complement, although our proof is (already) surprisingly hard. (Using Isaacs π-theory, the same result also holds for groups of odd order.)

2.6 Theorem. *Suppose that $\chi = \alpha^G$ is irreducible, where $K \subseteq G$, $\alpha \in \mathrm{Irr}\,(K)$ and $K/\ker(\alpha)$ is a p'-group. Suppose that G has a normal p-complement M. Then MK is the stabilizer in G of the character $(\alpha^{MK})_M \in \mathrm{Irr}\,(M)$.*

Proof. We argue by induction on $|G|$.

Notice that α^{MK} has p'-degree because α has p'-degree and $|MK : K|$ is a p'-number. Hence, by Corollary (11.29) of [**3**], we have that

$$\theta = (\alpha^{MK})_M \in \mathrm{Irr}\,(M).$$

Now we see that MK stabilizes θ, and by induction we certainly may assume that θ is G-invariant. So we are forced to prove that $MK = G$.

Now let P be a Sylow p-subgroup of K. Hence $MK = MP$ and it suffices to show that $P \in \mathrm{Syl}_p(G)$. By Theorem A we know that $(\alpha_{\mathbf{N}_K(P)})^{\mathbf{N}_G(P)}$ is irreducible. Write $\beta = \alpha_{\mathbf{N}_K(P)} \in \mathrm{Irr}\,(\mathbf{N}_K(P))$. Now $\mathbf{C}_M(P) = \mathbf{N}_G(P) \cap M$ is the normal p-complement of $\mathbf{N}_G(P)$. Since $\beta^{\mathbf{C}_M(P)\mathbf{N}_K(P)}$ has p'-degree, it follows, using the same argument as in the second paragraph, that

$$(\beta^{\mathbf{C}_M(P)\mathbf{N}_K(P)})_{\mathbf{C}_M(P)} = (\beta_{\mathbf{C}_{M\cap K}(P)})^{\mathbf{C}_M(P)}$$

is irreducible. Also, $\alpha_{\mathbf{C}_{M\cap K}(P)} = \beta_{\mathbf{C}_{M\cap K}(P)}$ is irreducible.

Now $M \cap K$ is P-invariant, $\alpha_{M\cap K}$ is P-invariant and irreducible, and since $\alpha_{\mathbf{C}_{M\cap K}(P)}$ is irreducible, it follows that $\alpha_{\mathbf{C}_{M\cap K}(P)}$ is the P-Glauberman correspondent of $\alpha_{M\cap K}$ (by Lemma 3.5 of [**4**], for instance). Since

$$\theta = (\alpha^{MK})_M = (\alpha_{M\cap K})^M$$

is irreducible, by Theorem A of [**4**], we have that θ^* (the P-Glauberman correspondent of θ) equals

$$(\alpha_{\mathbf{C}_{M\cap K}(P)})^{\mathbf{C}_M(P)} = (\beta^{\mathbf{C}_M(P)\mathbf{N}_K(P)})_{\mathbf{C}_M(P)}.$$

Assume now that $\mathbf{N}_G(P) < G$. By induction, we have in this case that the stabilizer of

$$(\beta^{\mathbf{C}_M(P)\mathbf{N}_K(P)})_{\mathbf{C}_M(P)} = \theta^*$$

in $\mathbf{N}_G(P)$ is $\mathbf{N}_K(P)\mathbf{C}_M(P) = P \times \mathbf{C}_M(P)$. We claim that, in this case, P is a Sylow p-subgroup of G. It suffices to show that P is a Sylow p-subgroup of $\mathbf{N}_G(P)$. Assume that $P \triangleleft Q$, where Q is some p-subgroup of G. Since θ is Q-invariant, by the uniqueness in the Glauberman correspondence (see Theorem 13.1.c of [**3**]), we deduce that θ^* is Q-invariant. Hence, $P = Q$ as claimed. Therefore, $G = MK$ as desired.

Assume now that $P \triangleleft G$. Note that $KM = M \times P \triangleleft G$. Also, we have that

$$\alpha^{KM} = \theta \times 1_P,$$

because $P \subseteq \ker(\alpha)$. Since $(\alpha^{KM})^G$ is irreducible, we have that $I_G(\alpha^{KM}) = KM$ by Problem 6.1 of [**3**]. This implies that $I_G(\theta) = KM$. Since θ is G-invariant, the proof of the theorem is complete. □

2.7 Corollary. *Suppose that G has a normal p-complement. Let $\chi \in \mathrm{Irr}(G)$. Suppose that $\alpha^G = \chi = \beta^G$, where $\alpha \in \mathrm{Irr}(U)$, $U/\ker(\alpha)$ is a p'-group, $\beta \in \mathrm{Irr}(V)$ and $V/\ker(\beta)$ is a p'-group. Then the Sylow p-subgroups of U and V are G-conjugate.*

Proof. Let M be the normal p-complement of G. The irreducible characters $(\alpha^{MU})_M$ and $(\beta^{MV})_M$ are G-conjugate since they are constituents of χ_M. Hence, the same happens to their stabilizers, which are MU and MV by Theorem 2.6. Since the Sylow p-subgroups of U and V are Sylow p-subgroups of MU and MV, respectively, the proof of the corollary easily follows. □

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