ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 2001

C^*-ALGEBRAS OF DYNAMICAL SYSTEMS OF QUASI ROTATIONS ON TORI

CARLA FARSI AND NEIL WATLING

ABSTRACT. In this note we determine the isomorphism classes of the crossed product C^*-algebras of affine (n, λ) quasi rotations of $\mathbf{T}^n$.

1. Introduction. There have been considerable contributions to the computation of K-theoretical and isomorphism invariants of C^*-algebras of dynamical systems on the n-torus $\mathbf{T}^n$, which include certain noncommutative tori [**5**], [**3**], [**7**]. Riedel [**5**] classified the crossed products of $C(\mathbf{T}^n)$ by minimal rotations of $\mathbf{T}^n$, i.e., minimal transformations of $\mathbf{T}^n$ with degree matrix $D(\phi) = I_n$. He showed that the set of eigenvalues of ϕ is a complete isomorphism invariant. When ϕ is a minimal homeomorphism of $\mathbf{T}^n$ with quasi discrete spectrum, Packer [**3**] computed the tracial range of $K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z})$. For $n = 2$, Rouhani [**7**] classified, by using K-theoretical invariants, the isomorphism classes of the crossed product C^*-algebras $C(\mathbf{T}^2) \rtimes_{\alpha_\phi} \mathbf{Z}$, where ϕ is an (affine) irrational quasi rotation of $\mathbf{T}^2$. That is an (affine) transformation that has a unitary eigenvalue $\lambda = e^{2\pi i\theta}$ (θ irrational) with a unitary eigenfunction f having degree matrix $D(f) = [n, m] \neq 0$, where n, m are relatively prime and the degree matrix $D(\phi)$ satisfies $\operatorname{rank}_{\mathbf{Q}}(D(\phi) - I_2) = 1$. The concept of quasi rotation admits a natural generalization to an n quasi rotation for transformations $\phi : \mathbf{T}^n \to \mathbf{T}^n$. Roughly speaking, ϕ is now required to have $n-1$ eigenvalues while the degree matrix $D(\phi)$ still satisfies $\operatorname{rank}_{\mathbf{Q}}(D(\phi) - I_n) = 1$. (See Definition 2 and Lemma 3.)

Our main result, which generalizes the main theorem in [**7**] to $\mathbf{T}^n$, $n \geq 3$, is the characterization, using K-theoretical invariants, of the isomorphism classes of crossed products $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$ of $\mathbf{T}^n$, where ϕ is an affine n quasi rotation, provided some additional conditions are

Received by the editors on September 4, 1997.
1991 AMS *Mathematics Subject Classification.* Primary 46L80, 46L40.
Key words and phrases. Dynamical systems, tori, C^*-algebras, classification.
First author partially supported by a University of Colorado Junior Faculty Award.

also satisfied (cf. Theorem 12). More precisely, the K-theory groups $K_*(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z})$, $* = 0, 1$, and the tracial range $\tau_*^\phi(K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}))$ are shown to be complete isomorphism invariants.

Our results also generalize some of [**3**] to the broader contest of n quasi rotations (in fact every n quasi rotation has topologically quasi discrete spectrum [**8**]).

In detail, the contents of this note are as follows. In Section 2 we consider affine transformations $\phi = aA$, $a \in \mathbf{T}^n$, $A \in GL(n, \mathbf{Z})$, of $\mathbf{T}^n$ satisfying $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$. By detailing the conjugacy classes in $GL(n, \mathbf{Z})$ of matrices A satisfying $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$, we are able to compute the K-theory of the crossed products $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$, $\phi = aA$, $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$. The proof of the conjugacy classes lemma is rather technical and is given in an Appendix at the end of this note. In Section 3 we compute the tracial range of $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$, $\phi = aA$, $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$, under the additional hypothesis that ϕ is an n quasi rotation. Section 4 details further properties of n quasi rotations, which are used in Section 5 where we state and prove our main result, Theorem 12. The main step in its proof establishes that K-theory and tracial range determine uniquely, up to isomorphism, a standard C^*-algebra isomorphic to $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$.

2. K-theory of $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$. In this section we compute the K-theory of the crossed products $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$, where $\alpha_\phi(f) = f \circ \phi^{-1}$. Here $\phi(z) = aA(z)$, $a \in \mathbf{T}^n$, $A \in GL(n, \mathbf{Z})$, $z \in \mathbf{T}^n$, is an affine transformation of $\mathbf{T}^n$ satisfying $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$. Note that A is acting on $\mathbf{T}^n$ by a group automorphism and that A has a topological interpretation as the degree matrix, $D(\phi)$, of ϕ.

The K-theory of $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$ only depends on the conjugacy class of A in $GL(n, \mathbf{Z})$. The structure of the conjugacy classes of elements A in $GL(n, \mathbf{Z})$ having $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$ is given in the following lemma, the proof of which is given in the Appendix. See also [**7**] for a proof when $n = 2$.

Lemma 1 (Conjugacy classes lemma). *Let $A \in GL(n, \mathbf{Z})$ with $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$. Then,*

(1) *If* $\det(A) = 1$, *then* A *is conjugate in* $GL(n, \mathbf{Z})$ *to*

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \\ & & M & 1 \end{bmatrix},$$

where $M \in \mathbf{Z} \setminus \{0\}$.

(2) *If* $\det(A) = -1$, *then* A *is conjugate in* $GL(n, \mathbf{Z})$ *to*

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \\ & & 0 & -1 \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}.$$

We will refer to

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \\ & & M & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \\ & & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix}$$

as the standard form for A and write S_1^M, S_2 and S_3, respectively.

Now we can compute the K-theory of $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$, $\phi = aA$, $a \in \mathbf{T}^n$, $A \in GL(n, \mathbf{Z})$, $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$.

By applying the Pimsner-Voiculescu sequence [**1**], we get

$$0 \longrightarrow \mathbf{Z}^{2n-1} \big/ \operatorname{Im}(1-\phi_0) \longrightarrow K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}) \longrightarrow \operatorname{Ker}(1-\phi_1) \longrightarrow 0,$$
$$0 \longrightarrow \mathbf{Z}^{2^{n-1}} \big/ \operatorname{Im}(1-\phi_1) \longrightarrow K_1(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}) \longrightarrow \operatorname{Ker}(1-\phi_0) \longrightarrow 0,$$

where $\phi_* : K^*(\mathbf{T}^n) \to K^*(\mathbf{T}^n)$, $* = 0, 1$, is induced by α_ϕ.

When $\det(A) = 1$, $A = S_1^M$ and thus ϕ_*, $* = 0, 1$, can be written as

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ -M & 1 \end{bmatrix} \bigotimes I_{2^{n-3}} & 0 \\ 0 & I_{2^{n-2}} \end{bmatrix},$$

for $n \geq 3$, so that

$$K_*(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}) \cong \mathbf{Z}^{3.2^{n-2}} \oplus \mathbf{Z}_M^{2^{n-3}}, \quad * = 0, 1, \quad n \geq 3.$$

When $\det(A) = -1$, if $A = S_2$, then $\phi_* = \begin{bmatrix} I_{2^{n-2}} & 0 \\ 0 & -I_{2^{n-2}} \end{bmatrix}$ and therefore,

$$K_*(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}) \cong \mathbf{Z}^{2^{n-1}} \oplus \mathbf{Z}_2^{2^{n-2}}, \quad * = 0, 1, \quad n \geq 2.$$

If $A = S_3$, then

$$\phi_* = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigotimes I_{2^{n-3}} & 0 & 0 \\ 0 & -I_{2^{n-3}} & 0 \\ 0 & 0 & I_{2^{n-3}} \end{bmatrix}$$

and so,

$$K_*(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z} \cong \mathbf{Z}^{2^{n-1}} \oplus \mathbf{Z}_2^{2^{n-3}}, \quad * = 0, 1, \quad n \geq 3.$$

3. The tracial range of $K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z})$. As shown in [**5**] and [**7**], K-theory groups isomorphism does not necessarily imply crossed product C^*-algebra isomorphism. Indeed, Riedel showed in [**5**] that the set of eigenvalues is a complete isomorphism invariant for crossed products by minimal rotations. Moreover, for affine transformations of $\mathbf{T}^2$, Rouhani [**7**] required the existence of a unitary eigenvalue $\lambda = e^{2\pi i\theta}$ (θ irrational) associated to a unitary eigenfunction f with degree matrix $D(f) = [n, m] \neq 0$, n, m relatively prime. (In this case, $\operatorname{rank}_{\mathbf{Q}}(A - I_2) = 1$). He was thus able to compute the tracial range of $K_0(C(\mathbf{T}^2) \rtimes_{\alpha_\phi} \mathbf{Z})$ and show that the tracial range together with the K-theory groups are complete isomorphism invariants.

Generalizing Rouhani's work to higher dimensions we will assume the existence of $n-1$ eigenvalues and thus complete the tracial range of $K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z})$.

Definition 2. Let $\phi = (\phi_1, \dots, \phi_n) : \mathbf{T}^n \to \mathbf{T}^n$ be a homeomorphism of $\mathbf{T}^n$. Then $\phi_i(z_1, \dots, z_n) = z_1^{a_{i,1}} \cdots z_n^{a_{i,n}} e^{2\pi\sqrt{-1}F_i(z_1,\dots,z_n)}$, where $a_{i,j} \in \mathbf{Z}$ and $F_i(z_1, \dots, z_n)$ is continuous and real valued. We say that ϕ is an n quasi rotation if there exist unitaries $f_1, \dots, f_{n-1} \in C(\mathbf{T}^n)$, $\lambda_1, \dots, \lambda_{n-1} \in \mathbf{T} \setminus \{1\}$ such that

(1) $D(\phi) = [a_{i,j}]_{i,j=1,\dots,n} \neq I_n$,

(2) $f_i \circ \phi = \lambda_i f_i$,

(3) gcd {all minors of size $(n-1) \times (n-1)$ of $[D(f_i)]_{i=1,\dots,n-1}\} = 1$.

Note that condition (3) above is equivalent to requiring that the matrix $[D(f_i)]$ be completable (by adding another row) to a matrix in $SL(n, \mathbf{Z})$, thus generalizing Rouhani's condition for $n = 2$ that $D(f_1) = [n, m] \neq 0$, n, m relatively prime.

Affine transformations $\phi = aA : \mathbf{T}^n \to \mathbf{T}^n$, $a \in \mathbf{T}^n$ and $A \in GL(n, \mathbf{Z})$, with $A = S_1^M, S_2, S_3$ are n quasi rotations. In fact, the ordered sets $z_1, \dots, z_{n-1}$, respectively $z_1, \dots, z_{n-2}, z_{n-1}z_n$, and $a_1, \dots, a_{n-1}$, respectively $a_1, \dots, a_{n-2}, a_{n-1}a_n$, are a set of eigenfunctions and eigenvalues for ϕ.

The following lemmas are an easy consequence of Definition 2.

Lemma 3. *Let ϕ be an n quasi rotation with associated degree matrices $D(\phi)$ and $[D(f_i)]$. Then,*

(1) $D(f_i) \neq [0, \dots, 0]$ *for all* $i = 1, \dots, n-1$,

(2) $D(f_i)(D(\phi) - I_n) = 0$ *for all* $i = 1, \dots, n-1$,

(3) $\operatorname{rank}_{\mathbf{Q}}(D(\phi) - I_n) = 1$.

Lemma 4. *Let ϕ be an n quasi rotation with associated degree matrices $D(\phi)$ and $[D(f_i)]$. Then, for any matrix $Y \in M((n-1) \times n, \mathbf{Z})$ such that $Y(D(\phi) - I_n) = 0$, there exists a matrix $\Lambda \in M(n-1, \mathbf{Z})$ such that $Y = \Lambda[D(f_i)]$.*

Proof. First notice that for all $K \in GL(n, \mathbf{Z})$, $[D(f_i)]K$ satisfies (3) of Definition 2. This follows since, by (3), we can choose a matrix $R \in M(1 \times n, \mathbf{Z})$ such that

$$\det \begin{bmatrix} [D(f_i)] \\ R \end{bmatrix} = 1.$$

So

$$\det \begin{bmatrix} [D(f_i)] \\ R \end{bmatrix} K = \det \begin{bmatrix} [D(f_i)]K \\ RK \end{bmatrix} = \pm 1.$$

If we write $D(\phi) = KSK^{-1}$, with S the standard form for $D(\phi)$, then both YK and $[D(f_i)]K$ are solutions of $X(S - I_n) = 0$. Since $[D(f_i)]K$ also satisfies (3) of Definition 2 its rows span the left null space of $S - I_n$, which implies $YK = \Lambda[D(f_i)]K$ for some $\Lambda \in M(n-1, \mathbf{Z})$, and hence $Y = \Lambda[D(f_i)]$. □

Now, to compute the tracial range of $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$, where ϕ is an n quasi rotation, let $[D(f_i)]$ be the degree matrix relative to the eigenfunctions f_i associated to the eigenvalues $\lambda_i = e^{2\pi\sqrt{-1}\theta_i}$, $0 < \theta_i < 1$, of ϕ. For a fixed i, define $\rho_i : C(\mathbf{T}) \to C(\mathbf{T}^n)$ by $\rho_i(g) = g \circ f_i$. ρ_i induces a homomorphism between the rotation algebra A_{λ_i} and $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$. By the naturality of the Pimsner-Voiculescu sequence, it follows that the image of the exponential of the Rieffel projection [**6**] in $K_1(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z})$ is f_i, [**7**]. By Lemma 4, $[D(f_i)]$ generates the kernel of $(\iota - \alpha_\phi^*)$ in $H^1(\mathbf{T}^n, \mathbf{Z})$. Therefore, for any trace τ^ϕ on $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$ ([**1**, pp. 99–100], [**4**]),

$$\tau_*^\phi(K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z})) = \mathbf{Z} + \theta_1\mathbf{Z} + \cdots + \theta_{n-1}\mathbf{Z}.$$

4. Some properties of n quasi rotations. In this section we will derive some additional properties of n quasi rotations, which we will use in Section 5 in the proof of our main result.

Proposition 5. *Let ϕ be an n quasi rotation with associated degree matrices $D(\phi)$ and $[D(f_i)]$ (relative to the eigenvalues λ_i and to the eigenfunctions f_i). Let μ_i and g_i (with degree matrix $[D(g_i)]$) be another system of eigenvalues and eigenfunctions for ϕ satisfying* (1), (2) *and* (3) *of Definition* 2. *Then $\mu_i = \prod_{j=1}^{n-1} \lambda_j^{\alpha_{i,j}}$ for some $\Lambda = [\alpha_{i,j}] \in GL(n-1, \mathbf{Z})$.*

Proof. Since $[D(g_i)] = \Lambda[D(f_i)]$, $\Lambda = [\alpha_{i,j}] \in M(n-1, \mathbf{Z})$ with $[D(f_i)]$ and $[D(g_i)]$ both satisfying (3) of Definition 2, it follows that $\det(\Lambda) = \pm 1$ since $\det(\Lambda)$ is a factor of all the $(n-1) \times (n-1)$ minors of

$[D(g_i)]$. By using the eigenvalue equations $f_i \circ \phi = \lambda_i f_i$ and $g_i \circ \phi = \mu_i g_i$, we get $h_i \circ \phi = \nu_i h_i$, with $h_i = \prod_{j=1}^{n-1} f_j^{\alpha_{i,j}} \overline{g_i}$ and $\nu_i = \prod_{j=1}^{n-1} \lambda_j^{\alpha_{i,j}} \overline{\mu_i}$ for $i = 1, \dots, n-1$. Note that $D(h_i) = 0$ and therefore $h_i(z) = e^{2\pi\sqrt{-1}H_i(z)}$. So $h_i \circ \phi = \nu_i h_i$ becomes $e^{2\pi\sqrt{-1}[H_i(\phi(z)) - H_i(z)]} = \nu_i$ or $H_i(\phi(z)) - H_i(z) = c_i$ for some constant c_i and for all $z \in \mathbf{T}^n$. Thus, $H_i(\phi^k(z)) - H_i(z) = kc_i$ for all $k \in \mathbf{Z}$ and for all $z \in \mathbf{T}^n$. Hence, as the lefthand side is bounded, $c_i = 0$, that is, $\nu_i = 1$ for $i = 1, \dots, n-1$. So we must have $\mu_i = \prod_{j=1}^{n-1} \lambda_j^{\alpha_{i,j}}$. $\square$

Proposition 6. *Let ϕ be an n quasi rotation with associated degree matrices $D(\phi)$ and $[D(f_i)]$.*

Let

$$\hat{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} \quad \text{and} \quad \hat{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}.$$

Then, for any $\Lambda \in GL(n-1, \mathbf{Z})$ with

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \vdots \\ \Lambda_{n-1} \end{bmatrix},$$

$$\Lambda_i \hat{f} \circ \phi = \Lambda_i \hat{\lambda}\, \Lambda_i \hat{f}, \quad i = 1, \dots, n-1.$$

Proof. Straightforward using $f_i \circ \phi = \lambda_i f_i$. $\square$

Lemma 7. *If $\phi = aA$ is an affine n quasi rotation with associated degree matrices $D(\phi) = A$ and $[D(f_i)]$, then $\lambda_i = D(f_i)(a)$.*

Proof. We can write $f_i(z) = D(f_i)(z)e^{2\pi\sqrt{-1}F_i(z)}$, and using $f_i \circ \phi = \lambda_i f_i$ obtain

$$D(f_i)(a)D(f_i)A(z)e^{2\pi\sqrt{-1}F_i(\phi(z))} = \lambda_i D(f_i)(z)e^{2\pi\sqrt{-1}F_i(z)}.$$

Now we observe that $D(f_i) = D(f_i)A$ so $e^{2\pi\sqrt{-1}[F_i(\phi(z)) - F_i(z)]} = \lambda_i \overline{D(f_i)(a)}$. Repeating the same argument as that in the proof of Proposition 5, we get $\lambda_i = D(f_i)(a)$. $\square$

5. Complete isomorphism invariants of $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$. Now we consider affine n quasi rotations, i.e., affine transformations $\phi = aA : \mathbf{T}^n \to \mathbf{T}^n$, $a \in \mathbf{T}^n$, and $A \in GL(n, \mathbf{Z})$, which are also n quasi rotations.

As mentioned in Section 3, in the particular case $A = S_1^M, S_2$, respectively S_3, the ordered sets $z_1, \dots, z_{n-1}$, respectively $z_1, \dots, z_{n-2}, z_{n-1} z_n$, and $a_1, \dots, a_{n-1}$, respectively $a_1, \dots, a_{n-2}, a_{n-1}a_n$, are a set of eigenfunctions and eigenvalues for ϕ. The tracial range of $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$ is completely determined by the eigenvalues λ_i of ϕ. To recover information on crossed product C^*-algebra isomorphism classes from information on the tracial range, we now restrict to quasi rotations with IRRI eigenvalues, that is,

Definition 8. An affine n quasi rotation with IRRI eigenvalues is an affine n quasi rotation ϕ with eigenvalues $\lambda_i = e^{2\pi\sqrt{-1}\theta_i}$, $i = 1, \dots, n-1$, such that $\theta_1, \dots, \theta_{n-1}$, are irrational and rationally independent $(\bmod\,(1))$.

In the particular case $A = S_1^M, S_2$, respectively S_3, ϕ has IRRI eigenvalues if $q_1, \dots, q_{n-1}$, respectively $q_1, \dots, q_{n-2}, q_{n-1} + q_n$, are irrational and rationally independent. (Where $a_j = e^{2\pi\sqrt{-1}q_j}$, $q_j \in \mathbf{R}$, $j = 1, \dots, n$.)

Our main result, Theorem 12, characterizes crossed products of affine n quasi rotations with IRRI eigenvalues, provided λ_{n-1} is fixed. We will now state and prove some additional results needed in the proof of Theorem 12.

Proposition 9. *Let $\phi = aA$ be an affine n quasi rotation with* IRRI *eigenvalues and $KAK^{-1} = S$ be the standard form for A.*

(i) *ϕ is topologically conjugate to the affine n quasi rotation with* IRRI *eigenvalues $\psi = sS$, where*

$$s = K(a) = \begin{bmatrix} K(a)_1 \\ \vdots \\ K(a)_n \end{bmatrix}.$$

(ii) $\psi = sS$, $s = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$, *is topologically conjugate to* $\omega = rS$, *with*

$$r = \begin{bmatrix} s_1 \\ \vdots \\ s_{n-1} \\ 1 \end{bmatrix} \quad \text{if } S = S_1^M, S_2$$

or

$$r = \begin{bmatrix} s_1 \\ \vdots \\ s_{n-2} \\ s_{n-1}s_n \\ 1 \end{bmatrix} \quad \text{if } S = S_3.$$

(iii) $\omega = rS$ *is topologically conjugate to* $\eta = tS$ *where* $t = L(r)$ *for any* $L \in GL(n, \mathbf{Z})$ *commuting with* S.

Proof. (i) If we define $\delta(z) = K(z)$, ϕ and ψ are topologically conjugate via δ. It remains to show that ψ is an n quasi rotation with IRRI eigenvalues. Put

$$X = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \quad \text{or} \quad X = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 1 \end{bmatrix}$$

according to $S = S_1^M, S_2$ or S_3. Now $XS = X$ and hence $XKA = XK$ so by Lemma 4, there exists $\Lambda \in GL(n-1, \mathbf{Z})$ such that $\Lambda[D(f_i)] = XK$. Moreover, by Proposition 6 and Lemma 7, $XK(a) = \Lambda[D(f_i)](a)$ is an IRRI system of eigenvalues for ϕ. It is now straightforward to show that $XK(a)$ is an IRRI system of eigenvalues for ψ with eigenfunctions $z_1, \dots, z_{n-1}$ or $z_1, \dots, z_{n-2}, z_{n-1}z_n$.

(ii) $\delta(z) = dI_n(z)$ with

$$d = \begin{bmatrix} 1 \\ \vdots \\ d_{n-1} \\ 1 \end{bmatrix},$$

$d_{n-1} = s_n^{-1/M}, s_n^{-1}$ if $S = S_1^M$, S_3 respectively, or

$$d = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ s_n^{1/2} \end{bmatrix}$$

if $S = S_2$ intertwines ψ and ω.

(iii) $\delta(z) = L(z)$ intertwines ω and η. □

The previous proposition's proof motivates the following definition (cf. also [**3**]).

Definition 10. Let $\phi = aA$ and $KAK^{-1} = S$ be the standard form of A. If $S = S_1^M, S_2$, respectively S_3, we will call the ordered sets $z_1, \dots, z_{n-1}$, respectively $z_1, \dots, z_{n-2}, z_{n-1}z_n$, and $K(a)_1, \dots, K(a)_{n-1}$, respectively $K(a)_1, \dots, K(a)_{n-2}, K(a)_{n-1}K(a)_n$, a standard set of eigenfunctions and eigenvalues for ϕ.

Definition 11. Let $\lambda \in \mathbf{T} \setminus \{1\}$. We say ϕ is an affine (n, λ) quasi rotation if ϕ is an affine n quasi rotation and there exists a standard set of eigenvalues $\lambda_1, \dots, \lambda_{n-1}$ for ϕ such that $\lambda_{n-1} = \lambda^{\pm 1}$.

Theorem 12. *Let $\phi = aA$ and $\psi = bB$ be affine (n, λ) quasi rotations with* IRRI *eigenvalues. Then the following are equivalent*:

(i) $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z} \cong C(\mathbf{T}^n) \rtimes_{\alpha_\psi} \mathbf{Z}$.

(ii) $K_*(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}) \cong K_*(C(\mathbf{T}^n) \rtimes_{\alpha_\psi} \mathbf{Z})$, $* = 0, 1$, *and for any tracial state τ^ϕ on $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$, respectively τ^ψ on $C(\mathbf{T}^n) \rtimes_{\alpha_\psi} \mathbf{Z}$, we have*

$$\tau_*^\phi(K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z})) \cong \tau_*^\psi(K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\psi} \mathbf{Z})).$$

(iii) *ϕ and ψ are topologically conjugate via an affine transformation.*

Proof. As (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) are trivial, we only need to show (2) $\Rightarrow$ (3). Since $C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z}$ and $C(\mathbf{T}^n) \rtimes_{\alpha_\psi} \mathbf{Z}$ have the same K-theory, A

and B are both conjugate to the same standard form S (see Section 2). That is, there exist $K_1, K_2 \in GL(n, \mathbf{Z})$ such that $K_1 A K_1^{-1} = S$ and $K_2 B K_2^{-1} = S$. Thus, by Proposition 9, ϕ is topologically conjugate to $\omega_1 = r_1 S$, where

$$r_1 = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \\ 1 \end{bmatrix},$$

and ψ is topologically conjugate to $\omega_2 = r_2 S$, where

$$r_2 = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{n-1} \\ 1 \end{bmatrix},$$

with $\lambda_1, \dots, \lambda_{n-1}$, respectively $\mu_1, \dots, \mu_{n-1}$, a standard set of eigenvalues for ϕ, respectively ψ. Note that ϕ and ψ are affine (n, λ) quasi rotations so we can assume $\lambda_{n-1} = \lambda^{\pm 1} = \mu_{n-1}^{\pm 1}$. As

$$\tau_*^{\phi}(K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\phi} \mathbf{Z})) \cong \tau_*^{\psi}(K_0(C(\mathbf{T}^n) \rtimes_{\alpha_\psi} \mathbf{Z})),$$

there exists $\Lambda = [\alpha_{i,j}] \in GL(n-1, \mathbf{Z})$ such that

$$\Lambda \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{n-1} \end{bmatrix}.$$

Moreover, $\alpha_{n-1,j} = 0$ for $j = 1, \dots, n-2$, $\alpha_{n-1,n-1} = \pm 1$ because $\lambda_{n-1} = \mu_{n-1}^{\pm 1}$ and the λ_i's are IRRI. Thus,

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{n-1} \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{1,1} & \cdots & \cdots & \alpha_{1,n-1} & \beta_1 \\ \vdots & & & \vdots & \vdots \\ \alpha_{n-2,1} & \cdots & \cdots & \alpha_{n-2,n-1} & \beta_{n-2} \\ 0 & \cdots & \cdots & \pm 1 & 0 \\ 0 & \cdots & \cdots & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \\ 1 \end{bmatrix},$$

$$\beta_1, \dots, \beta_{n-2} \in \mathbf{Z}.$$

Finally, by Proposition 9(iii), ω_1 is topologically conjugate to ω_2 with L the matrix above where we choose $\beta_i = 0$, respectively $\beta_i = \alpha_{i,n-1}$ for $i = 1, \dots, n-2$ if $S = S_1^M$ or S_2, respectively S_3. □

Remark 13. If $\phi = aA$ is an affine n quasi rotation with A having standard form S_2, the three conditions of Theorem 12 are equivalent since $\begin{bmatrix} \Lambda & 0 \\ 0 & \pm 1 \end{bmatrix}$ commutes with S_2 for any $\Lambda \in GL(n-1, \mathbf{Z})$.

Appendix

Proof of Lemma 1. We will prove Lemma 1 using induction on the size of A. For $n = 2$, the result was proved by Rouhani:

Lemma 14 [**7**]. *Let* $A \in GL(2, \mathbf{Z})$ *with* $\operatorname{rank}_{\mathbf{Q}}(A - I_2) = 1$.

(i) *If* $\det(A) = 1$, *then* A *is conjugate in* $GL(2, \mathbf{Z})$ *to* $\begin{bmatrix} 1 & 0 \\ M & 1 \end{bmatrix}$, $M \in \mathbf{Z} \setminus \{0\}$.

(ii) *If* $\det(A) = -1$, *then* A *is conjugate in* $GL(2, \mathbf{Z})$ *to* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Now suppose that for every matrix $\tilde{A} \in GL(k, \mathbf{Z})$, $k < n$, Lemma 1 holds, and consider $A = [a_{i,j}] \in GL(n, \mathbf{Z})$. Firstly, since $\det(A) = \pm 1$, $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$, there is at least one nonzero off diagonal element. By conjugating A with elementary matrices, we can assume that $a_{n,1} \neq 0$. Let $E_1 = \gcd(a_{n,1}, a_{n,2}) \neq 0$, choose t_1, r_1 such that $t_1(a_{n,1}/E_1) - r_1(a_{n,2}/E_1) = 1$, and put $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, with $\alpha = (\alpha_{n,1}/E_1)$, $\beta = (\alpha_{n,2}/E_1)$, $\gamma = r_1$ and $\delta = t_1$. Define the matrix $K \in GL(n, \mathbf{Z})$ by $K = \begin{bmatrix} B & 0 \\ 0 & I_{n-2} \end{bmatrix}$. Now KAK^{-1} is a matrix whose last row is $[E_1, 0, a_{n,3}, a_{n,4}, \dots, a_{n,n}]$. Conjugating again by the elementary matrix $e_{2,3}$ (which switches rows 2 and 3), we get that A is similar to a matrix having as last row $[E_1, a_{n,3}, 0, a_{n,4}, \dots, a_{n,n}]$ so that by proceeding as before and then conjugating by $e_{2,4}$ etc., A is similar to a matrix having as last row $[E, 0, \dots, 0, a_{n,n}]$, where $E = \gcd(a_{n,1}, \dots, a_{n,n-1}) \neq 0$.

Now, by using the fact that $\operatorname{rank}_{\mathbf{Q}}(A - I_n) = 1$, we see that

$$A \sim \begin{bmatrix} a'_{1,1} & 0 & \cdots & 0 & a'_{1,n} \\ a'_{2,1} & 1 & \ddots & \vdots & a'_{2,n} \\ \vdots & & \ddots & 0 & \vdots \\ a'_{n-1,1} & 0 & \cdots & 1 & a'_{n-1,n} \\ E & 0 & \cdots & 0 & a_{n,n} \end{bmatrix} \sim \begin{bmatrix} 1 & a'_{2,1} & 0 & \cdots & 0 & a'_{2,n} \\ 0 & a'_{1,1} & 0 & \cdots & 0 & a'_{1,n} \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & a'_{n-1,1} & 0 & \cdots & 1 & a'_{n-1,n} \\ 0 & E & 0 & \cdots & 0 & a_{n,n} \end{bmatrix}.$$

Since $\det(A) = \pm 1$, it follows that $\det(\tilde{A}) = \pm 1$, where

$$\tilde{A} = \begin{bmatrix} a'_{1,1} & 0 & \cdots & 0 & a'_{1,n} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ a'_{n-1,1} & 0 & \cdots & 1 & a'_{n-1,n} \\ E & 0 & \cdots & 0 & a_{n,n} \end{bmatrix}.$$

Suppose now that $\det(A) = \det(\tilde{A}) = 1$. Either $\operatorname{rank}_{\mathbf{Q}}(\tilde{A} - I_{n-1}) = 1$, so by the induction hypothesis there exists $K \in GL(n-1, \mathbf{Z})$ such that $K\tilde{A}K^{-1}$ is in standard form or $\operatorname{rank}_{\mathbf{Q}}(\tilde{A} - I_{n-1}) = 0$ and $\tilde{A} = I_{n-1}$. Therefore,

$$\begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a''_{1,2} & \cdots & a''_{1,n} \\ 0 & & & \\ \vdots & & K\tilde{A}K^{-1} & \\ 0 & & & \end{bmatrix}.$$

Conjugating by

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & & \ddots & 1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

we have

$$A \sim \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ M & \cdots & 0 & 1 & 0 \\ a''_{1,n-1} & \cdots & \cdots & a''_{1,n} & 1 \end{bmatrix}, \quad M \in \mathbf{Z}.$$

If $M \neq 0$, then $a''_{1,j} = 0$ for $j = 2, \dots, n-2, n$. Now choose t and r such that $t(M/F) - r(a''_{1,n-1}/F) = 1$, where $F = \gcd(M, a''_{1,n-1})$ and put $B = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ with $\alpha = (a''_{1,n-1})/F$, $\beta = -M/F$, $\gamma = t$ and $\delta = -r$. Conjugating by

$$\begin{bmatrix} I_{n-2} & 0 \\ 0 & B \end{bmatrix},$$

we have

$$A \sim \begin{bmatrix} 1 & & \\ & \ddots & \\ F & & 1 \end{bmatrix} \sim S_1^F.$$

If $M = 0$, without loss of generality, we can assume that $a''_{1,n-1} \neq 0$. We can then proceed as in the first part of the proof to obtain

$$A \sim \begin{bmatrix} 1 & & \\ & \ddots & \\ F & & 1 \end{bmatrix} \sim S_1^F,$$

with $F = \gcd(a''_{1,j})$, $j = 2, \dots, n$.

Finally, suppose that $\det(A) = \det(\tilde{A}) = -1$. By the induction hypothesis, there exists $K \in GL(n-1, \mathbf{Z})$ such that $K\tilde{A}K^{-1}$ is in standard form. Therefore,

$$A \sim \begin{bmatrix} 1 & a''_{1,2} & \cdots & a''_{1,n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X \end{bmatrix},$$

where

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\text{rank}_{\mathbf{Q}}(A - I_n) = 1$, we have $a''_{1,j} = 0$ for $j = 2, \dots, n-2$ and $a''_{1,n-1} = 0$ or $-a''_{1,n}$, respectively. In the first case, conjugation by $e_{1,n-1}$ gives

$$A \sim \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & a''_{1,n} \\ & & 0 & -1 \end{bmatrix}.$$

But, by Lemma 14,

$$\begin{bmatrix} 1 & a''_{1,n} \\ 0 & -1 \end{bmatrix}$$

is conjugate to

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus A is conjugate to S_2 or S_3. In the second case, conjugation by

$$\begin{bmatrix} 1 & & a''_{1,n} \\ & \ddots & \\ & & 1 \end{bmatrix}$$

gives $A \sim S_3$.

REFERENCES

1. B. Blackadar, *K-theory for operator algebras*, Math. Sci. Res. Inst. Publ., vol. 5, Springer, New York, 1986.

2. R. Ji, *On the crossed product C^*-algebra associated with Furstenberg transformations on tori*, Ph.D. Dissertation, SUNY at Stony Brook, New York, 1986.

3. J. Packer, *K-theoretical invariants for C^*-algebras associated to transformations and induced flows*, J. Funct. Anal. **67** (1986), 25–59.

4. M. Pimsner, *Ranges of traces on K_0 of reduced crossed products by free groups*, Lecture Notes in Math., vol. 1132, Springer-Verlag, New York, 1985.

5. N. Riedel, *Classification of the C^*-algebras associated with minimal rotations*, Pacific J. Math. **101** (1982), 153–161.

6. M. Rieffel, *C^*-algebras associated with irrational rotations*, Pacific J. Math. **93** (1981), 415–429.

7. H. Rouhani, *Quasi rotation C^*-algebras*, Pacific J. Math. **148** (1991), 131–151.

8. ———, *A Furstenberg transformation of the 2-torus without quasi-discrete spectrum*, Canad. Math. Bull. **33** (1990), 316–322.

Department of Mathematics, University of Colorado, Campus Box 395, Boulder, CO 80309
E-mail address: farsi@euclid.colorado.edu

Department of Mathematics, Widener University, Chester, PA 19013
E-mail address: watling@kuratowski.math.widener.edu