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ON THE WEAK PROPERTY OF LEBESGUE OF $L^1(\Omega, \Sigma, \mu)$

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ABSTRACT. A Banach space X is said to have the *weak property of Lebesgue* if every Riemann integrable mapping from the closed interval $[a, b]$ to X is weakly continuous almost everywhere on $[a, b]$. In this paper we prove that if (Ω, Σ, μ) is a totally finite, complete and countably generated measure space, then $L^1(\Omega, \Sigma, \mu)$ has the weak property of Lebesgue.

In this paper we are concerned with the Riemann integration in Banach spaces which was first studied by Graves [**2**]. In [**3**], Gordon compiled many results of Graves and others, e.g., Alexiewicz and Orlicz [**1**] and studied the interesting problem of determining which Banach spaces X have the property of Lebesgue, that is, the property that every Riemann integrable mapping from $[a, b]$ to the space X is continuous almost everywhere. After encouragement in 1992 by Professor Joe Diestel of Kent State University, we studied some problems related to the Riemann integration in Banach spaces and in [**6**] established some new characterizations of the Schur property and the H property of Banach spaces using Riemann integration. Inspired mainly by [**3**], in [**5**] we introduced the concept of the weak property of Lebesgue of a Banach space X and did some preliminary study, in which we pointed out that the most familiar Banach spaces enjoy the weak property of Lebesgue.

In this paper we assume that (Ω, Σ, μ) is a totally finite, complete and countably generated measure space. Since (Ω, Σ, μ) is countably generated, there is a sequence $\{G_k\} \subset \Sigma$ such that for any $E \in \Sigma$, $\delta > 0$, there exists some G_k satisfying $\mu(E \Delta G_k) < \delta$, see [**4**, pp. 168–169]. Under the stated assumptions we prove that $L^1(\Omega, \Sigma, \mu)$ has the weak property of Lebesgue. This result, of course, implies that the well-known space $L^1[0, 1]$ has the weak property of Lebesgue.

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Throughout this paper X will denote a real Banach space and X^* its dual.

Definition. A Banach space X is said to have the weak property of Lebesgue, abbreviated WLP, if every Riemann integrable mapping f from $[a,b]$ to X is weakly continuous almost everywhere on $[a,b]$, i.e., there exists a subset $E \subset [a,b]$ with $m(E) = b-a$ where $m(E)$ is a Lebesgue measure of E, such that for all $x^* \in X^*$, $\langle f(t), x^* \rangle$ is continuous at every $t \in E$.

For convenience, if the above is true for all x^* in a subset M of X^*, we say X has M-WLP.

Lemma 1. *Let X and M be as above. If X^* is the closure of M and X has the M-*WLP*, then X has the* WLP.

Proof. Let $f : [a,b] \to X$ be Riemann integrable on $[a,b]$, and let K satisfy the inequality $\|f(t)\| \le K$ for all $t \in [a,b]$. Since X has the M-WLP, there exists a subset $E \subset [a,b]$ with $m(E) = b-a$ such that for all $x^* \in M$, $\langle f(t), x^* \rangle$ is continuous at every $t \in E$.

For any $x^* \in X^*$, $\varepsilon > 0$, there exists $y^* \in M$ such that $\|x^* - y^*\| \le \varepsilon/(4K)$. For any fixed $t \in E$, we have

$$\begin{aligned} |\langle f(s) - f(t), x^* \rangle| &\le |\langle f(s) - f(t), x^* - y^* \rangle| + |\langle f(s) - f(t), y^* \rangle| \\ &\le \frac{\varepsilon}{2} + |\langle f(s) - f(t), y^* \rangle|. \end{aligned}$$

Since $\langle f(t), y^* \rangle$ is continuous at t, there exists $\delta > 0$ such that $|\langle f(s) - f(t), y^* \rangle| < \varepsilon/2$ whenever $|s-t| < \delta$, $s \in [a,b]$. Therefore, we have $|\langle f(s) - f(t), x^* \rangle| < \varepsilon$ whenever $|s-t| < \delta$, $s \in [a,b]$.

Thus X has the WLP.

Remark [**5**, Corollary 7]. If X is a real Banach space with separable dual, then we can prove that X has the WLP easily by using Lemma 1.

Now we are going to prove that $L^1(\Omega, \Sigma, \mu)$ has the WLP.

Lemma 2. *Let* $x(\cdot),\ y(\cdot) \in L^1(\Omega,\Sigma,\mu),\ A, B \in \Sigma,\ A \cap B = \Phi$. *If*

$$\int_A |x(r)|\,d\mu > c_1, \quad \int_B |y(r)|\,d\mu > c_2,$$

then

$$\|x+y\| > c_1 + c_2 \quad \text{or} \quad \|x-y\| > c_1 + c_2.$$

Proof. Let $x^* = \chi_A(r)\mathrm{sgn}\, x(r)$, $y^* = \chi_B(r)\mathrm{sgn}\, y(r)$. Clearly, $x^*, y^* \in (L^1(\Omega,\Sigma,\mu))^*$ and $\|x^* \pm y^*\| \le 1$. So we have

$$\begin{aligned}\|x+y\| + \|x-y\| &\ge \langle x+y, x^*+y^*\rangle + \langle x-y, x^*-y^*\rangle \\ &= 2\langle x, x^*\rangle + 2\langle y, y^*\rangle \\ &= 2\int_A |x(s)|\,d\mu + 2\int_B |y(s)|\,d\mu \\ &> 2(c_1+c_2).\end{aligned}$$

Then $\|x+y\| > c_1 + c_2$ or $\|x-y\| > c_1 + c_2$.

Lemma 3. *Assume that* $f : [a,b] \to L^1(\Omega,\Sigma,\mu)$ *is Riemann integrable. Then there exists* $E \subset [a,b]$, $m(E) = b-a$ *such that, for any* $t \in E$ *and any* $A \in \Sigma$,

$$\int_A |f(s)-f(t)|\,d\mu \to 0 \quad \text{whenever} \quad \mu(A) \to 0,$$
$$s \to t \quad \text{for } s \in [a,b].$$

Proof. Suppose that Lemma 3 is not true. Then there exist $\beta > 0$, $E_\beta \subset [a,b]$ such that Lebesgue outer measure of E_β, $m^*(E_\beta) > 0$; and for any $t \in E_\beta$, any $\delta_1 > 0$ and any $\delta_2 > 0$, there exist $s \in [a,b]$ and $A \in \Sigma$ such that $|s-t| < \delta_1$ and $\mu(A) < \delta_2$, but $\int_A |f(s)-f(t)|\,d\mu > \beta$.

Let P be a partition of $[a,b]$, and let $I_1, I_2, \dots, I_n$ be all those intervals of the partition satisfying $m^*(I_i \cap E_\beta) > 0$, $i = 1, 2, \dots, n$. Clearly,

$$\sum_{i=1}^n m(I_i) \ge m^*(E_\beta).$$

Since $m^*(I_1 \cap E_\beta) > 0$, we can take ξ_1, $\eta_1 \in I_1$ such that $\int_\Omega |f(\xi_1) - f(\eta_1)|\, d\mu > \beta$. Thus, $\|(f(\xi_1) - f(\eta_1))m(I_1)\| > \beta m(I_1)$.

Suppose that we have taken ξ_i, $\eta_i \in I_i$, $i = 1, \dots, k$, $1 \le k < n$ such that

$$\left\| \sum_{i=1}^{k} (f(\xi_i) - f(\eta_i)) m(I_i) \right\| > \beta \sum_{i=1}^{k} m(I_i).$$

By using absolute continuity of the integral, we can choose $\delta > 0$ such that if $\mu(A) < \delta$ and $A \in \Sigma$, then

$$\int_A \left| \sum_{i=1}^{k} (f(\xi_i) - f(\eta_i)) m(I_i) \right| d\mu < \left\| \sum_{i=1}^{k} (f(\xi_i) - f(\eta_i)) m(I_i) \right\| - \beta \sum_{i=1}^{k} m(I_i). \tag{1}$$

Since $m^*(I_{k+1} \cap E_\beta) > 0$, we can take ξ_{k+1}, $\eta_{k+1} \in I_{k+1}$, $A_{k+1} \in \Sigma$; $\mu(A_{k+1}) < \delta$ such that

$$\int_{A_{k+1}} |(f(\xi_{k+1}) - f(\eta_{k+1})) m(I_{k+1})|\, d\mu > \beta m(I_{k+1}). \tag{2}$$

Using (1) with A_{k+1} in place of A, we have

$$\int_{A_{k+1}} \left| \sum_{i=1}^{k} (f(\xi_i) - f(\eta_i)) m(I_i) \right| d\mu < \left\| \sum_{i=1}^{k} (f(\xi_i) - f(\eta_i)) m(I_i) \right\| - \beta \sum_{i=1}^{k} m(I_i),$$

from which we obtain

$$\int_{\Omega - A_{k+1}} \left| \sum_{i=1}^{k} (f(\xi_i) - f(\eta_i)) m(I_i) \right| d\mu > \beta \sum_{i=1}^{k} m(I_i). \tag{3}$$

By Lemma 2, from (2) and (3) we have

$$\left\| \sum_{i=1}^{k+1} (f(\xi_i) - f(\eta_i)) m(I_i) \right\| > \beta \sum_{i=1}^{k+1} m(I_i).$$

By induction we have proved that there exist ξ_i, $\eta_i \in I_i$, $i = 1, 2, \dots, n$, such that

$$\left\| \sum_{i=1}^{n} (f(\xi_i) - f(\eta_i)) m(I_i) \right\| > \beta \sum_{i=1}^{n} m(I_i) > \beta m^*(E_\beta).$$

Note that P is arbitrarily chosen, so we have a contradiction of (2) of Theorem 5 in [**3**]. The proof of Lemma 3 is complete.

Lemma 4. *Let M be the set of all simple functions on Ω. Then $M \subset (L^1(\Omega,\Sigma,\mu))^*$ and $L^1(\Omega,\Sigma,\mu)$ has the M-*WLP.

Proof. Assume that $f : [a,b] \to L^1(\Omega,\Sigma,\mu)$ is Riemann integrable. Note that, for any $A \in \Sigma$,

$$\int_A f(t)\, d\mu = \langle f(t), \chi_A \rangle \quad t \in [a,b],$$

where χ_A satisfies

$$\chi_A(r) = \begin{cases} 1 & r \in A \\ 0 & r \in \Omega - A. \end{cases}$$

Since f is Riemann integrable, f is scalarly Riemann integrable. Thus, $\int_A f(t)\, d\mu$ is continuous almost everywhere on $[a,b]$.

If

$$E_k = \left\{ t \in [a,b] : \int_{G_k} f(t)\, d\mu \text{ is continuous at } t \right\},$$

where G_k, $k = 1, 2, \dots$ are as at the beginning, then $m(E_k) = b - a$. Take E as in Lemma 3, and put

$$E_0 = E \cap \left(\bigcap_{k=1}^{\infty} E_k \right).$$

Clearly, $m(E_0) = b - a$.

Now we will prove that, for any $y^* \in M$, $\langle f(t), y^* \rangle$ is continuous at every $t \in E_0$. Clearly we may write

$$y^* = \sum_{l=1}^{m} c_l \chi_{A_l}.$$

Then

$$\langle f(t), y^* \rangle = \Big\langle f(t), \sum_{l=1}^{m} c_l \chi_{A_l} \Big\rangle.$$

It suffices to prove that $\langle f(t), \chi_{A_l} \rangle$, $1 \le l \le m$, is continuous at every $t \in E_0$.

Thus, let t be an arbitrary element of E_0. By Lemma 3 we have that, for any $\varepsilon > 0$, there exist $\delta_1 > 0$, $\delta_2 > 0$ such that if $s \in [a, b]$ and $A \in \Sigma$ such that $|s - t| < \delta_1$ and $\mu(A) < \delta_2$, then

$$\int_A |f(s) - f(t)| \, d\mu < \varepsilon.$$

Since (Ω, Σ, μ) is countably generated by $\{G_k\}$, for A_l there exists some G_k such that $\mu(A_l \Delta G_k) < \delta_2$. Therefore, we have

$$\begin{aligned} |\langle f(s) - f(t), \chi_{A_l} \rangle| &\le |\langle f(s) - f(t), \chi_{A_l} - \chi_{G_k} \rangle| + |\langle f(s) - f(t), \chi_{G_k} \rangle| \\ &\le \int_{A_l \Delta G_k} |f(s) - f(t)| \, d\mu + \left| \int_{G_k} (f(s) - f(t)) \, d\mu \right| \\ &< \varepsilon + \left| \int_{G_k} (f(s) - f(t)) \, d\mu \right|. \end{aligned}$$

Since $\int_{G_k} f(t) \, d\mu$ is continuous at $t \in E_0$, there exists $\delta_3 > 0$ such that

$$\left| \int_{G_k} (f(s) - f(t)) \, d\mu \right| < \varepsilon$$

whenever $s \in [a, b]$ and $|s - t| < \delta_3$.

So, for any $\varepsilon > 0$, let $\delta = \min(\delta_1, \delta_3)$ and note that if $|s - t| < \delta$, then

$$|\langle f(s) - f(t), \chi_{A_l}(s) \rangle| < 2\varepsilon.$$

Now we have proved that $\langle f(t), \chi_{A_l} \rangle$, $1 \le l \le m$, is continuous at every $t \in E_0$. Hence, for any $y^* \in M$, $\langle f(t), y^* \rangle$ is continuous at every $t \in E_0$, and $L^1(\Omega, \Sigma, \mu)$ has the M-WLP.

Theorem 5. *If (Ω, Σ, μ) is a totally finite, complete, and countably generated measure space, then $L^1(\Omega, \Sigma, \mu)$ has the* WLP.

Proof. Let M be as in Lemma 4. It is easy to see that the closure of $M = (L^1(\Omega, \Sigma, \mu))^*$. By using Lemmas 1 and 4, we complete the proof.

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