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SIMPLE GEOMETRIC CHARACTERIZATION OF SUPERSOLVABLE ARRANGEMENTS

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1. Introduction. An arrangement of hyperplanes is a finite collection of $\mathbf{C}$-linear subspaces of dimension $(l-1)$ in $\mathbf{C}^l$. For such an arrangement $\mathcal{A}$, there is a natural projective arrangement $\mathcal{A}^*$ of hyperplanes in $\mathbf{CP}^{l-1}$ associated to it. Let $M(\mathcal{A}) = \mathbf{C}^l - \cup\{H : H \in \mathcal{A}\}$ and $M(\mathcal{A}^*) = \mathbf{CP}^{l-1} - \cup\{H^* : H^* \in \mathcal{A}^*\}$. Then it is clear that $M(\mathcal{A}) = M(\mathcal{A}^*) \times \mathbf{C}^*$. The central problem in the theory of arrangements is to find a connection between the topology or differentiable structure of $M(\mathcal{A})$, respectively $M(\mathcal{A}^*)$, and the combinatorial geometry of $\mathcal{A}$, respectively $\mathcal{A}^*$.

More specifically, we would like to know the homotopy properties of $M(\mathcal{A})$ and how these properties relate to various other well-known properties of arrangements. Many people have asked the following questions. Precisely when is $M(\mathcal{A})$ a $K(\pi,1)$ space?

In [**2**], Brieskorn considers the Coxeter group W acting on $\mathbf{R}^l$. W also acts as a reflection group in $\mathbf{C}^l$. Let $\mathcal{A} = \mathcal{A}(W)$ be its reflection arrangement. Brieskorn conjectured that $\mathcal{A}(W)$ is a $K(\pi,1)$ arrangement for all Coxeter groups W. He proved this for some of the groups by representing M as the total space of a sequence of fibrations. Deligne [**3**] settled the question by proving that the complement of complexification of a real simplicial arrangement is $K(\pi,1)$. This result proves Brieskorn's conjecture because the arrangement of a Coxeter group is simplicial. Recently, Jambu and Terao [**4**] introduced the property of supersolvability of an arrangement. This property is combinatorial in nature, that is, it depends only on the pattern of intersection of the hyperplanes or equivalently on the lattice associated to the arrangement. It turns out that complement $M(\mathcal{A})$ of a supersolvable arrangement is the total space of a fiber bundle in which the base and fiber are $K(\pi,1)$ spaces. The long exact homotopy sequence of the bundle shows that

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$M(\mathcal{A})$ is $K(\pi, 1)$ also. As far as we know, supersolvable arrangements are the most general kind of complex arrangements having the $K(\pi, 1)$ property. Therefore it is desirable to have a simple geometric characterization for a supersolvable arrangement which one can check easily whether a given arrangement is supersolvable. The following is our main theorem.

Consider an arrangement $\mathcal{A}^*$ in $\mathbf{CP}^2$ which corresponds to $\mathcal{A}$ in $\mathbf{C}^3$. Then a point x of intersection of lines $\mathcal{A}^*$ is called the *center* of $\mathcal{A}^*$ if for each intersection point y of $\mathcal{A}^*$ there is a line l in $\mathcal{A}^*$ connecting x and y.

Main theorem. *$\mathcal{A}$ is a supersolvable arrangement if and only if $\mathcal{A}^*$ has a center.*

2. Preliminaries of arrangements and their lattices. We begin by recalling some terminology in lattice theory.

Definition 2.1. A poset is a set in which a binary relation $x \leq y$ is defined which satisfies for all x, y, z the following conditions

P1 (Reflexive). For all x, $x \leq x$.

P2 (Antisymmetry). If $x \leq y$ and $y \leq x$, then $x = y$.

P3 (Transitivity). If $x \leq y$ and $y \leq z$, then $x \leq z$.

If $x \leq y$, we shall say that x is less than or equal to y. If $x \leq y$ and $x \neq y$, one writes $x < y$.

An *upper bound* of a subset X of a poset P is an element $a \in P$ such that $x \leq a$ for every $x \in X$. The least upper bound is an upper bound less than or equal to every other upper bound; it is denoted by $\sup X$. By P2, $\sup X$ is unique if it exists. The notion of lower bound of X and greatest lower bound ($\inf X$) of X are defined dually. Again, by P2, $\inf X$ is unique if it exists.

Definition 2.2. A *lattice* is a poset P in which any two of whose elements have a greatest lower bound or "meet" denoted by $x \wedge y$, and a least upper bound or "join" denoted by $x \vee y$.

Definition 2.3. An element y covers an element x in a lattice L if and only if $x < y$, but $x < z < y$ for no element z in L.

Definition 2.4. A chain in a lattice L is any linearly ordered subset of L.

Definition 2.5. A lattice having no infinite chains is said to be semimodular whenever it has the covering property: for all lattice elements x, y, if x and y cover $x \wedge y$, then $x \vee y$ covers x and y.

Definition 2.6. Let L be a lattice with finite length. The *length* of a chain C of L is defined as $|C| - 1$. The rank of $a \in L$, denoted by $r(a)$, is the length of the longest chain in L below a. Let $\hat{0} = \inf L$ and $\hat{1} = \sup L$. Then $r(\hat{0}) = 0$. The rank of L (rank L) is defined to be $r(\hat{1})$. If a in L has rank 1, then a is called a point or an atom of the lattice.

Definition 2.7. A *point lattice* (or atomic lattice) is a lattice in which every element is a joint of points. A *geometric lattice* is a semimodular point lattice with no infinite chains.

In this paper an arrangement $\mathcal{A}$ is a finite collection of hyperplanes $\{H_1, \ldots, H_n\}$ through the origin in $\mathbf{C}^l$. Arrangements in our sense are sometimes called central. If we view $\mathbf{C}^l$ as affine space and allow $\mathcal{A}$ to contain affine hyperplanes, we call $\mathcal{A}$ an affine arrangement. A projective arrangement is a finite set of projective hyperplanes in projective space $\mathbf{CP}^{l-1}$. Recall the canonical bundle $p : \mathbf{C}^l - \{0\} \mapsto \mathbf{CP}^{l-1}$ with fiber $\mathbf{C}^*$, which identifies z with λz for $\lambda \in \mathbf{C}^*$.

Proposition 2.8. *Let $\mathcal{A}$ be a nonempty arrangement with complement $M = M(\mathcal{A})$, and let $M^* = p(M)$. The restriction $p : M \mapsto M^*$ is a trivial fibration so that $M = M^* \times \mathbf{C}^*$. There is a projective $(l-1)$ arrangement $\mathcal{A}^*$ such that $M^* = M(\mathcal{A}^*)$ where $M(\mathcal{A}^*)$ is the complement of the projective arrangement in $\mathbf{CP}^{l-1}$.*

Proof. Let $H \in \mathcal{A}$. The restriction of p to $\mathbf{C}^l - H$ has base space

$\mathbf{CP}^{l-1} - \mathbf{CP}^{l-2} \cong \mathbf{C}^{l-1}$. Thus $p : \mathbf{C}^l - H \mapsto \mathbf{C}^{l-1}$ is a trivial bundle and $p : M \mapsto M^*$ is a subbundle. The rest of the proposition follows easily. □

Since the complement of a hyperplane in a projective space is an affine space, a nonempty projective arrangement may be viewed as an affine arrangement and vice versa. Proposition 2.8 above gives a close connection between central l-arrangements and affine $(l-1)$-arrangements.

Following Orlik-Solomon [**5**], we define the lattice $L(\mathcal{A})$ of an arrangement. The set $L(\mathcal{A})$ is the set of all intersections of subsets of $\mathcal{A}$, partially ordered by reverse inclusion, i.e., $X \leq Y \Leftrightarrow Y \subseteq X$. Thus $\mathbf{C}^l$ is the minimal element.

Define a rank function r on $L(\mathcal{A})$ by $r(X) = \operatorname{codim} X = l - \dim_{\mathbf{C}} X$ for $X \in L(\mathcal{A})$. Call H_i an *atom* of $L(\mathcal{A})$. Define the *join* by $X \vee Y = X \cap Y$ and the *meet* by $X \wedge Y = \cap\{Z : Z \in L(\mathcal{A}), X \cup Y \subset Z\}$.

Lemma 2.9. *Let $\mathcal{A}$ be an arrangement. Then*

(i) *for every $X \in L(\mathcal{A})$ all maximal linear ordered subsets*

$$X_0 = C^l < X_1 < \cdots < X_p = X$$

have the same cardinality;

(ii) *every element of $L(\mathcal{A}) - \{\mathbf{C}^l\}$ is a join of atoms*;

(iii) *for all X, Y in $L(\mathcal{A})$ the rank function satisfies*

$$r(X \wedge Y) + r(X \vee Y) \leq r(X) + r(Y).$$

Thus $L(\mathcal{A})$ is a geometric lattice.

Definition 2.10. Let $L_p = L_p(\mathcal{A}) := \{X \in L(\mathcal{A}) : r(X) = p\}$. The Hasse diagram of $L(\mathcal{A})$ has vertices labeled by the elements of $L(\mathcal{A})$ and arranged on levels L_p, $p \geq 0$. Suppose $X \in L_p$ and $Y \in L_{p+1}$. An edge connects X with Y if $X < Y$.

Example 2.1. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbf{C}^3$ consisting of the elements $\{(x, y, z) \in \mathbf{C}^3 : x = y\}$, $\{(x, y, z) \in \mathbf{C}^3 :$

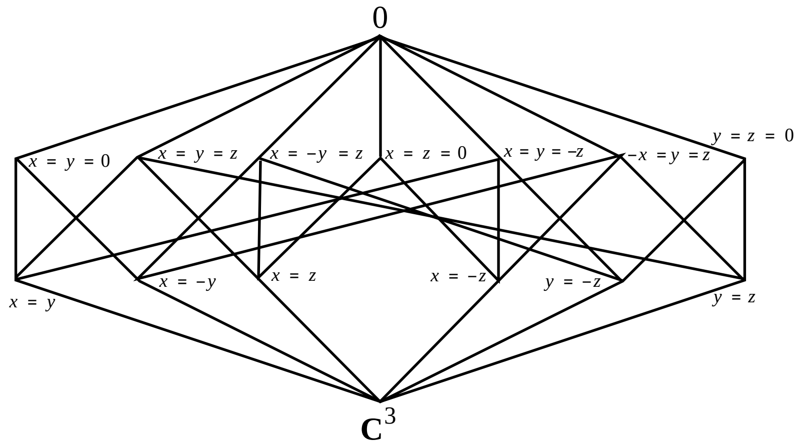

FIGURE 1. Lattice of Example 2.1.

$x = -y\}$, $\{(x, y, z) \in \mathbf{C}^3 : x = z\}$, $\{(x, y, z) \in \mathbf{C}^3 : x = -z\}$, $\{(x, y, z) \in \mathbf{C}^3 : y = -z\}$, $\{(x, y, z) \in \mathbf{C}^3 : y = z\}$. (See Figure 1.)

Definition 2.11. A hyperplane arrangement $\mathcal{A}$ in $\mathbf{C}^{l+1}$ is strictly linearly fibered if there is a choice of coordinates $(x_1, \ldots, x_l, y)$ on $\mathbf{C}^{l+1}$ so that the restriction $\mathbf{P}$, of the projection $\mathbf{C}^{l+1} \mapsto \mathbf{C}^l$, $(x_1, \ldots, x_l, y) \mapsto (x_1, \ldots, x_l)$, to the complement $M(\mathcal{A})$ is a fiber bundle projection, with base $\mathbf{P}(M(\mathcal{A})) = M(\mathcal{B})$, the complement of an arrangement $\mathcal{B}$ in $\mathbf{C}^l$, and fiber the complement of finitely many points in $\mathbf{C}$. We say that $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$.

Definition 2.12. An arrangement $\mathcal{A} = \mathcal{A}_1$ of finitely many points in $\mathbf{C}^1$ is fiber-type. An arrangement $\mathcal{A} = \mathcal{A}_l$ of hyperplanes in $\mathbf{C}^l$ is fiber-type if $\mathcal{A}$ is strictly linearly fibered over a finite-type arrangement $\mathcal{A}_{l-1}$ in $\mathbf{C}^{l-1}$.

3. Arrangements with supersolvable lattice. Let L be a finite geometric lattice with minimal element $\hat{0}$, the maximal element $\hat{1}$, and rank function r. We shall briefly review the notions of modular elements of L and supersolvable lattices, both of which are defined by Stanley [**6**] and [**7**].

Definition 3.1. A pair $(x, y) \in L \times L$ is said to be a *modular pair* when

$$z = (z \vee x) \wedge y \quad \text{for all } z \quad \text{with } x \wedge y \leq z \leq y.$$

Remark 3.2. In a geometric lattice $(x, y) \in L \times L$ is a modular pair if and only if $r(x) + r(y) = r(x \vee y) + r(x \wedge y)$, cf. [**1**, p. 83].

Definition 3.3. An element $x \in L$ is called a *modular element* if it forms a modular pair with every $y \in L$, i.e., if $a \leq y$, then $a \vee (x \wedge y) = (a \vee x) \wedge y$.

The following theorem is due to Theorem 1 of Stanley [**6**].

Theorem 3.4. *An element $x \in L$ is modular if and only if no two complements of x are comparable, i.e., if $x \wedge y = x \wedge z = \hat{0}$, $x \vee y = x \vee z = \hat{1}$ and $y \geq z$, then $y = z$.*

The following lemma due to Terao [**8**] provides a characterization of modular elements.

Lemma 3.5. *An element x of L is a modular element if and only if x forms a modular pair with any y satisfying $x \wedge y = \hat{0}$.*

Stanley [**7**] defined a lattice to be supersolvable.

Definition 3.6. A geometric lattice L is supersolvable if there exists a maximal modular chain

$$\hat{0} = x_0 < x_1 < \cdots < x_l = \hat{1},$$

i.e., $l = \operatorname{rank} L$ and each x_i is a modular element, $i = 0, \ldots, l$.

In [**4**], Jambu and Terao first studied those arrangements $\mathcal{A}$ whose lattice $L(\mathcal{A})$ is supersolvable. In [**8**] Terao has shown the following theorem.

Theorem 3.7. *An arrangement $\mathcal{A}$ is fiber-type if and only if $L(\mathcal{A})$ is supersolvable.*

Corollary 3.8. *For any arrangement $\mathcal{A}$, if $L(\mathcal{A})$ is supersolvable, then $M(\mathcal{A})$ is a $K(\pi, 1)$ space.*

From now on, we consider only the lattice L of a central arrangement $\mathcal{A}$ in $\mathbf{C}^3$. We can easily see the following.

Lemma 3.9. *For $x, y, z \in L$, $y \vee (x \wedge z) = (y \vee x) \wedge z$ if $y = \hat{0}$ or $z = \hat{1}$.*

Lemma 3.10. *Each atom is modular.*

Proof of Lemma 3.10. Let x be an atom, i.e., $r(x) = 1$. In view of Lemma 3.5, we only need to prove that x forms a modular pair with any $y \in L$ satisfying $x \wedge y = \hat{0}$. Furthermore, by Remark 3.2, for such a y we only need to show that

$$r(x) + r(y) = r(x \vee y) + r(z \wedge y). \tag{3.1}$$

Case 1. $r(y) = 0$, i.e., $y = \hat{0}$. In this case $r(x) + r(y) = 1 + 0 = 1$, while $r(x \vee y) + r(x \wedge y) = r(x) + r(\hat{0}) = 1 + 0 = 1$. So (3.1) holds.

Case 2. $r(y) = 1$. In this case both x and y are planes and $x \vee y$ is a line in $L(\mathcal{A})$. Therefore, $r(x \wedge y) + r(x \vee y) = r(\hat{0}) + r(l) = 0 + 2 = 2$, while $r(x) + r(y) = 1 + 1 = 2$. So (3.1) holds.

Case 3. $r(y) = 2$. In this case y is a line not containing in the plane x because of $x \wedge y = \hat{0}$. It follows that $x \vee y = \hat{1}$. Therefore, $r(x) + r(y) = 1 + 2 = 3$, while $r(x \vee y) + r(x \wedge y) = r(\hat{0}) + r(\hat{1}) = 0 + 3 = 3$. So (3.1) holds.

Case 4. $r(y) = 3$, i.e., $y = \hat{1}$. In this case $x \wedge \hat{1} = x$. So there is nothing to prove in this case. □

Lemma 3.11. *Let* $x \in L$ *and* $\operatorname{codim}(x) = 2$. x *is modular if and only if* $\operatorname{codim}(x \wedge z) = 1$ *for each* $z \in L$ *different from* x *with* $\operatorname{codim}(z) = 2$. (*In other words, the line* x *is modular if and only if for each line* $z \in L$, *there is a plane* $y \in L$ *containing both* x *and* z.)

Proof. $\Leftarrow$. Suppose $x \in L$ and $\operatorname{codim}(x) = 2$. Suppose further that $\operatorname{codim}(x \wedge z) = 1$ for each $z \in L$ with $\operatorname{codim}(z) = 2$. We want to prove that x is modular. As in the proof of Lemma 3.10, we only need to show that (3.1) holds for all $y \in L$ with $x \wedge y = \hat{0}$. Obviously we only need to consider those $y \in L$ with $r(y) \leq 2$.

Case 1. $r(y) = 0$, i.e., $y = \hat{0}$. In this case $r(x \vee y) = r(x)$ and $r(x \wedge y) = r(y)$. So (3.1) holds.

Case 2. $r(y) = 1$. In this case x is a line and y is a plane. The line x is not contained in the plane y because $x \wedge y = \hat{0}$. It follows that $x \vee y = \hat{1}$. Therefore, $r(x) + r(y) = 2 + 1 = 3$ while $r(x \vee y) + r(x \wedge y) = r(\hat{1}) + r(\hat{0}) = 3 + 0 = 3$. So (3.1) holds.

Case 3. $r(y) = 2$. In this case y is a line different from x. $r(x \vee y) = r(\hat{1}) = 3$, while $r(x \wedge y) = 1$ by hypothesis. So $r(x) + r(y) = 2 + 2 = 3 + 1 = r(x \vee y) + r(x \wedge y)$ and (3.1) holds.

$\Rightarrow$. Conversely, suppose $x \in L$, $\operatorname{codim}(x) = 2$ and x is modular. We want to prove that $\operatorname{codim}(x \wedge z) = 1$ for each $z \in L$ different from x with $\operatorname{codim}(z) = 2$. Since x is modular, (x, z) is a modular pair. By Definition 3.1, we have

$$r(x) + r(z) \; = \; r(x \, \vee \, y) + r(x \, \wedge \, z).$$

Observe that $x \vee z = \hat{1}$. The above equality implies

$$2 + 2 \; = \; 3 + r(x \, \wedge \, z)$$

which implies $r(x \wedge z) = 1$. □

Consider the arrangement $\mathcal{A}^*$ in $\mathbf{CP}^2$ which corresponds to $\mathcal{A}$ in $\mathbf{C}^3$; Lemma 3.11 shows that a point x_0 of intersection of lines in $\mathcal{A}^*$ is modular in $L(\mathcal{A})$ if and only if for each intersection point y of $\mathcal{A}^*$ there is a line l in $\mathcal{A}^*$ connecting x_0 and z. We call such x_0 the *center* of L (or $\mathcal{A}^*$). Thus we have the following theorem.

Theorem 3.12. *$L(\mathcal{A})$ is a supersolvable lattice if and only if $\mathcal{A}^*$ has a center.*

Remark 3.14. From Theorem 3.12 we can give an easy proof that an arrangement in $\mathbf{CP}^2$ with supersolvable lattice is a fiber-type. For we let the center c and $\mathcal{A}^*$ be $c = (0 : 1 : 0)$ and a line l_∞ passing through c be an infinity. If we look at the complement $M(\mathcal{A}^*)$ as a subset of $\mathbf{C}^2$, we can assume that the other lines passing c are presented by the equations $x = k_1, \ldots, x = k_m$ and the rest of the lines in $\mathcal{A}^*$ are $y = a_1x + b_1, \ldots, y = a_nx + b_n$. Thus, $M(\mathcal{A}^*)$ is such a bundle with the base $B = \mathbf{C} - \{k_1, \ldots, k_m\}$ and the fiber $F_x = \mathbf{C} - (z_1x + b_1, \ldots, a_nx + b_n\}$ for each $x \in B$. So $\mathcal{A}^*$ is a fiber-type.

Example 3.1. We construct an arrangement $\mathcal{A}^*$ in $\mathbf{CP}^2$ with a supersolvable lattice. Let l_1, l_2 and l_3 be three non-collinear lines in $\mathbf{CP}^2$ and let A, B and C be their three intersection points. Let X_A, X_B and X_C be the sets of lines other than l_1, l_2 and l_3 passing through A, B and C, respectively. Let C be the center of $\mathcal{A}^*$. In order to make C a center of $\mathcal{A}^*$ we can show that it must have $|X_C| \geq \max\{|X_A|, |X_S|\}$. In case the equality holds, its lattice is unique up to an isomorphism and its diffeomorphic structure is determined by this lattice.

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