ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 2001

MULTIPLICITY OF POSITIVE SOLUTIONS FOR HIGHER ORDER STURM-LIOUVILLE PROBLEMS

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ABSTRACT. We establish the existence of an arbitrary number of positive solutions to the $2m$th order Sturm-Liouville type problem

$$\begin{aligned}(-1)^m y^{(2m)}(t) &= f(t, y(t)), \quad 0 \le t \le 1,\\ \alpha y^{(2i)}(0) - \beta y^{(2i+1)}(0) &= 0, \quad 0 \le i \le m-1,\\ \gamma y^{(2i)}(1) + \delta y^{(2i+1)}(1) &= 0, \quad 0 \le i \le m-1,\end{aligned}$$

where $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous. We accomplish this by making growth assumptions on f which we state in terms which generalize assumptions in recent works regarding superlinear and/or sublinear growth in f.

1. Introduction. In [**5**], Erbe and Tang consider the boundary value problem (BVP)

$$-\Delta u = F(r, u) \quad \text{in } R < r < \widehat{R} \tag{1.1}$$

where $r = |x|$, $x \in \mathbf{R}^n$, subject to one of the boundary conditions

$$\begin{aligned} u = 0 \text{ on } |x| = R, &\qquad u = 0 \text{ on } |x| = \widehat{R}\\ u = 0 \text{ on } |x| = R, &\qquad \frac{\partial u}{\partial r} = 0 \text{ on } |x| = \widehat{R}\\ \frac{\partial u}{\partial r} = 0 \text{ on } |x| = R, &\qquad u = 0 \text{ on } |x| = \widehat{R}.\end{aligned} \tag{1.2}$$

(Here, $(\partial u/\partial r)$ denotes differentiation in the radial direction.) In a radially symmetric setting, after a change of variable (1.1), (1.2) become

$$-u'' = f(t, u), \quad 0 < t < 1, \tag{1.3}$$

$$\begin{aligned}\alpha u(0) - \beta u'(0) &= 0\\ \gamma u(1) + \delta u'(1) &= 0.\end{aligned} \tag{1.4}$$

1991 AMS *Mathematics Subject Classification.* 34B10, 34B15, 35J25, 35P30.
Key words and phrases. Multiple solutions, positive solutions, Sturm-Liouville problem, cone theory.
Received by the editors on April 20, 1999, and in revised form on November 10, 1999.

Erbe and Tang were able to establish criteria for the existence of multiple positive solutions of (1.3), (1.4) by making certain assumptions on the growth of f. It is the aim of this work to generalize these results for second order BVP's to arbitrary even order problems.

More specifically, we consider the $2m$th order differential equation

$$(-1)^m y^{(2m)}(t) = f(t, y(t)), \quad 0 \le t \le 1, \tag{1.5}$$

satisfying the Sturm-Liouville type boundary conditions

$$\begin{aligned} \alpha y^{(2i)}(0) - \beta y^{(2i+1)}(0) = 0, \quad 0 \le i \le m-1, \\ \gamma y^{(2i)}(1) + \delta y^{(2i+1)}(1) = 0, \quad 0 \le i \le m-1, \end{aligned} \tag{1.6}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $\alpha, \beta, \gamma, \delta \ge 0$, and

$$\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0.$$

These last two assumptions are simply so that we have a nonnegative Green's function for the homogeneous problem

$$(-1)^m y^{(2m)}(t) = 0 \tag{1.7}$$

satisfying the boundary conditions (1.6).

The higher order Sturm-Liouville problem given by (1.5), (1.6) is not only a generalization of (1.3) and (1.4) but it also encompasses the Lidstone BVP,s which have been of recent interest; see Davis and Henderson [**1**]. In [**1**], the existence of at least three positive, symmetric solutions is established for the Lidstone BVP's (i.e., when $\beta = \delta = 0$) via the Leggett-Williams fixed point theorem [**9**]. The techniques presented here are very different from those in [**1**] because we do not rely on the Leggett-Williams fixed point theorem, and we are also able to establish the existence of *any* number of positive solutions in appropriate annular regions.

Our primary goal is to establish growth conditions on $h(t,y) := (f(t,y)/y)$, $0 < y < \infty$, $0 \le t \le 1$, which yield existence and multiplicity criteria for positive solutions of (1.5) and (1.6). Accomplishing this requires us to define the extended real-valued functions $h_0(t)$ and $h_\infty(t)$ by

$$h_0(t) := \lim_{y \to 0^+} h(t,y)$$

$$h_\infty(t) := \lim_{y \to \infty} h(t,y).$$

The cases in which $h_0(t) \equiv 0$ and $h_\infty(t) \equiv \infty$ are referred to as superlinearities of $f(t,y)$ with respect to y at $y=0$ and $y=\infty$. On the other hand, $h_0(t) \equiv \infty$ and $h_\infty(t) \equiv 0$ are referred to as sublinearities of $f(t,y)$ with respect to y at $y=0$ and $y=\infty$. Each of the superlinear and sublinear cases has been discussed in [**4**], [**6**], [**7**], [**8**], [**11**]. Most recently, Lian, Wong, and Yeh [**10**] relaxed the superlinear and sublinear conditions above and instead assumed only certain smallness or largeness conditions for $h(t,y)$ at $y=0$ and as $y \to \infty$. We will extend these results in the same spirit as Erbe and Tang [**5**]. In doing so, we will assume that $h(t,y) \not\equiv 0$ on any subinterval of $[0,1]$ for all $0 < y < \infty$.

2. Existence of a positive solution. Our main tool will be the following fixed point theorem of cone expansion/compression type due to Krasnosels'kii. See Deimling's text [**2**] for the proof.

Theorem 2.1 (Krasnosels'kii). *Let E be a Banach space, let $K \subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that either*

(i) $||Au|| \leq ||u||$, $u \in K \cap \partial\Omega_1$ *and* $||Au|| \geq ||u||$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $||Au|| \geq ||u||$, $u \in K \cap \partial\Omega_1$ *and* $||Au|| \leq ||u||$, $u \in K \cap \partial\Omega_2$ *holds.*

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Our aim is to apply Theorem 2.1 to a completely continuous operator whose kernel is the Green's function for (1.7) and (1.6). For the case $m=1$, this Green's function is

$$G(t,s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s) & 0 \leq s \leq t \leq 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s) & 0 \leq t \leq s \leq 1. \end{cases}$$

If we let $G_1(t,s) := G(t,s)$, then for $2 \leq j \leq m$ we can recursively define

$$G_j(t,s) = \int_0^1 G(t,r) G_{j-1}(r,s)\, dr. \tag{2.1}$$

As a result, $G_m(t,s)$ is the Green's function for (1.7) and (1.6). Note that $G(t,r) \le G(r,r)$ for $0 \le t, r \le 1$. This quickly leads to

$$G_m(t,s) \le \left[\int_0^1 G(r,r)\,dr\right]^{m-1} G(s,s) = I^{m-1}G(s,s), \quad 0 \le t \le 1,\ 0 \le s \le 1, \tag{2.2}$$

where

$$I = \int_0^1 G(r,r)\,dr.$$

Since $G_1(t,s) > 0$ and therefore $G_2(t,s) = \int_0^1 G(t,r)G_1(r,s)\,dr > 0$, it follows that $G_m(t,s) > 0$. Moreover, $(G(t,s)/G(s,s)) \ge \sigma$ for $1/4 \le t \le 3/4$ and $0 \le s \le 1$ where

$$\sigma := \min\left\{\frac{\gamma + 4\delta}{4(\gamma+\delta)}, \frac{\alpha + 4\beta}{4(\alpha+\beta)}\right\} < 1.$$

Hence

$$\begin{aligned} G_2(t,s) &= \int_0^1 G(t,r)G_1(r,s)\,dr \\ &\ge \sigma \int_0^1 G(r,r)G_1(r,s)\,dr \\ &\ge \sigma \int_{1/4}^{3/4} G(r,r)G_1(r,s)\,dr \\ &\ge \sigma^2 G(s,s) \int_{1/4}^{3/4} G(r,r)\,dr. \end{aligned}$$

Continuing inductively, we see that

$$G_m(t,s) \ge \sigma^m J^{m-1} G(s,s), \\ 1/4 \le t \le 3/4,\ 0 \le s \le 1,$$

where

$$J = \int_{1/4}^{3/4} G(r,r)\,dr.$$

The inequalities (2.2) and (2.3) will be important in the proof of our main result.

It is well known that the BVP (1.5), (1.6) has a solution y if and only if y solves the operator equation

$$(Ay)(t) := \int_0^1 G_m(t,s) f(s, y(s))\, ds = y(t).$$

Establishing the existence of a solution of (1.5), (1.6) is equivalent then to proving the existence of a fixed point of A which we will do by using Theorem 2.1. First, the operator $A : C[0,1] \to C[0,1]$ is completely continuous. Define the cone $K \subset C[0,1]$ by

$$K := \{\varphi \in C[0,1] : \varphi(t) \geq 0 \quad \text{and} \quad \min_{1/4 \leq t \leq 3/4} \varphi(t) \geq \Sigma ||\varphi||\}$$

where

$$\Sigma := \frac{\sigma^m J^{m-1}}{I^{m-1}} < 1 \quad \text{and} \quad ||\varphi|| := \sup_{0 \leq t \leq 1} |\varphi(t)|.$$

For convenience and cleaner notation, define the constants

$$\eta := \left[\left(\int_0^1 G(s,s)\, ds \right)^m \right]^{-1} = (I^m)^{-1}$$

$$\mu := \left[\sigma^m J^{m-1} \int_{1/4}^{3/4} G(s,s)\, ds \right]^{-1} = (\sigma^m J^m)^{-1}.$$

In terms of η and μ, we state the following conditions which govern the behavior of $f(t,y)$.

(C_1) There is a $p > 0$ such that $f(t,y) \leq \eta p$ for $0 \leq t \leq 1$ and $0 \leq y \leq p$.

(C_2) There is a $q > 0$ such that $f(t,y) \geq \mu q$ for $(1/4) \leq t \leq (3/4)$ and $\Sigma q \leq y \leq q$.

We are now ready to state our first theorem which establishes the existence of a positive solution of (1.5), (1.6) based on the aforementioned conditions.

Theorem 2.2. *Suppose there exist distinct $p, q > 0$ such that condition (C_1) holds for p and condition (C_2) holds for q. Then (1.5), (1.6) has a positive solution y such that $||y||$ is between p and q.*

Proof. Without loss of generality we assume $0 < p < q$. If $y \in K$ and $||y|| = p$, then using the estimate in (2.2) we have

$$\begin{aligned} Ay(t) &= \int_0^1 G_m(t,s) f(s, y(s))\, ds \\ &\le \int_0^1 I^{m-1} G(s,s) f(s,y(s))\, ds \\ &\le I^{m-1} \eta p \int_0^1 G(s,s)\, ds \\ &= p \end{aligned}$$

which implies $||Ay|| \le ||y||$ for $||y|| = p$. Likewise, if $y \in K$ and $||y|| = q$, then using (2.3) we have, for $1/4 \le t \le 3/4$,

$$\begin{aligned} Ay(t) &= \int_0^1 G_m(t,s) f(s,y(s))\, ds \\ &\ge \int_{1/4}^{3/4} G_m(t,s) f(s,y(s))\, ds \\ &\ge \mu q \int_{1/4}^{3/4} G_m(t,s)\, ds \\ &\ge \mu q \sigma^m J^{m-1} \int_{1/4}^{3/4} G(s,s)\, ds \\ &= q \end{aligned}$$

which implies $||Ay|| \ge ||y||$ for $||y|| = q$. By Theorem 2.1, the operator A has a fixed point in $K \cap (\overline{\Omega}_q \setminus \Omega_p)$ where

$$\Omega_p := \{y \in C[0,1] : ||y|| < p\} \quad \text{and} \quad \Omega_q := \{y \in C[0,1] : ||y|| < q\}.$$

As a result, the BVP (1.5), (1.6) has a solution y such that $p \le ||y|| \le q$. □

Corollary 2.1. *The* BVP (1.5), (1.6) *has a positive solution provided*

(C_3) $h_0(t) < \eta$ *for* $0 \le t \le 1$ *and* $h_\infty(t) > (\mu/\Sigma)$ *for* $(1/4) \le t \le (3/4)$, *or*

(C_4) *$h_0(t) > (\mu/\Sigma)$ for $(1/4) \le t \le (3/4)$ and $h_\infty(t) < \eta$ for $0 \le t \le 1$.*

Proof. Suppose (C_3) holds. Then

$$\lim_{y\to 0^+} \frac{f(t,y)}{y} < \eta, \quad 0 \le t \le 1,$$

and

$$\lim_{y\to\infty} \frac{f(t,y)}{y} > \frac{\mu}{\Sigma}, \quad 1/4 \le t \le 3/4.$$

So there is a sufficiently small $p > 0$ and sufficiently large $q > 0$ such that

$$\frac{f(t,y)}{y} \le \eta, \quad 0 \le t \le 1,\ 0 < y \le p,$$

and

$$\frac{f(t,y)}{y} \ge \frac{\mu}{\Sigma}, \quad \frac{1}{4} \le t \le \frac{3}{4},\ y \ge \Sigma q.$$

Hence

$$f(t,y) \le \eta y \le \eta p, \quad 0 \le t \le 1,\ 0 \le y \le p,$$

and

$$f(t,y) \ge \frac{\mu}{\Sigma} y \ge \mu q, \quad \frac{1}{4} \le t \le \frac{3}{4},\ \Sigma q \le y \le q.$$

So we have shown that (C_3) implies (C_1) and (C_2).

For the rest of the proof, assume (C_4) holds. Then

$$\lim_{y\to 0^+} \frac{f(t,y)}{y} > \frac{\mu}{\Sigma}, \quad \frac{1}{4} \le t \le \frac{3}{4},$$

and

$$\lim_{y\to\infty} \frac{f(t,y)}{y} < \eta, \quad 0 \le t \le 1.$$

This implies that there are $0 < p < q$ such that

$$\frac{f(t,y)}{y} \ge \frac{\mu}{\Sigma}, \quad \frac{1}{4} \le t \le \frac{3}{4},\ 0 < y \le p,$$

and

$$\frac{f(t,y)}{y} \le \eta, \quad 0 \le t \le 1, \; y \ge q. \tag{2.4}$$

So

$$f(t,y) \ge \frac{\mu}{\Sigma} y \ge \mu p, \quad \frac{1}{4} \le t \le \frac{3}{4}, \; \Sigma p \le y \le p.$$

Therefore condition (C_2) holds at p.

In order to show that (C_1) holds, we consider two cases.

Case 1. Suppose $f(t,y)$ is bounded for $0 \le t \le 1$ and $0 \le y < \infty$, i.e., $f(t,y) \le M$ for $0 \le t \le 1$ and $0 \le y < \infty$. By (2.4) above, there is a $p^* \ge q$ such that $f(t,y) \le M \le \eta p^*$ for $0 \le t \le 1$ and $0 \le y \le p^*$ provided $p^* \ge (M/\eta)$. Hence (C_1) holds for $p = p^*$.

Case 2. Suppose $f(t,y)$ is unbounded. Then there is a $t_0 \in [0,1]$ and $p^* \ge q$ such that $f(t,y) \le f(t_0,p^*)$ for $0 \le t \le 1$ and $0 \le y \le p^*$. Hence $f(t,y) \le f(t_0,p^*) \le \eta p^*$ and (C_1) holds for $p = p^*$.

An application of Theorem 2.2 yields the result. □

3. An arbitrary number of positive solutions. We are also able to establish the existence of multiple solutions by utilizing our previous conditions.

Theorem 3.1. *The* BVP (1.5), (1.6) *has at least two positive solutions if* (C_1) *holds for some* $p > 0$, *and in addition, we have*

$$h_0(t) > \frac{\mu}{\Sigma}, \quad \frac{1}{4} \le t \le \frac{3}{4}$$

and

$$h_\infty(t) > \frac{\mu}{\Sigma}, \quad \frac{1}{4} \le t \le \frac{3}{4}.$$

Moreover, $0 < ||y_1|| < p < ||y_2||$.

Proof. As in the proof of the last corollary, there exist p_1, p_2 with $0 < p_1 < p < p_2$ satisfying

$$f(t,y) \geq \mu p_1, \quad \text{for } \frac{1}{4} \leq t \leq \frac{3}{4},\ \Sigma p_1 \leq y \leq p_1,$$

and

$$f(t,y) \geq \mu p_2, \quad \text{for } \frac{1}{4} \leq t \leq \frac{3}{4},\ \Sigma p_2 \leq y \leq p_2.$$

By Theorem 2.2, we obtain the existence of the solutions y_1, y_2 of the BVP (1.5), (1.6) with $0 < p_1 < ||y_1|| < p < ||y_2|| < p_2$. □

Similarly, we have the following result.

Theorem 3.2. *The* BVP (1.5), (1.6) *has at least two positive solutions if* (C_2) *holds for some* $p > 0$, *and in addition, we have*

$$h_0(t) < \eta, \quad 0 \leq t \leq 1 \quad \textit{and} \quad h_\infty(t) < \eta, \quad 0 \leq t \leq 1. \tag{3.2}$$

Moreover, $0 < ||y_1|| < p < ||y_2||$.

Criteria for the existence of three (or more) positive solutions may be stated in a similar manner. As examples, we give the following two corollaries.

Corollary 3.1. *Suppose* (C_3) *in Corollary* 2.1 *holds, and suppose there exist* $0 < p_1 < p_2$ *such that* (C_1) *holds at* $p = p_2$ *and* (C_2) *holds at* $p = p_1$. *Then the* BVP (1.5), (1.6) *has at least three positive solutions* y_1, y_2, y_3 *satisfying* $0 < ||y_1|| < p_1 < ||y_2|| < p_2 < ||y_3||$.

Corollary 3.2. *Suppose* (C_4) *in Corollary* 2.1 *holds, and suppose there exist* $0 < p_1 < p_2$ *such that* (C_1) *holds at* $p = p_1$ *and* (C_2) *holds at* $p = p_2$. *Then the* BVP (1.5), (1.6) *has at least three positive solutions* y_1, y_2, y_3 *satisfying* $0 < ||y_1|| < p_1 < ||y_2|| < p_2 < ||y_3||$.

For our final results, we utilize the previous cone expansion/compression techniques iteratively to obtain various sufficient conditions for the existence of n solutions for any $n \in \mathbf{N}$. We state two different sufficient conditions for odd n and two different sufficient conditions for even n.

Theorem 3.3 (Any odd number of solutions). *The* BVP (1.5), (1.6) *has at least* n *positive solutions where* $n = 2k+1$, $k \in \mathbf{N}$ *provided* (C_3) *in Corollary* 2.1 *holds and there are* $0 < p_1 < p_2 < \cdots < p_{n-1}$ *such that* (C_2) *holds at* p_{2i-1}, $i = 1, \dots, k$, *while at the same time* (C_1) *holds at* p_{2i}, $i = 1, \dots, k$. *Moreover,* $0 < ||y_1|| < p_1 < ||y_2|| < p_2 < \cdots < ||y_{n-1}|| < p_{n-1} < ||y_n||$.

Theorem 3.4 (Any odd number of solutions). *The* BVP (1.5), (1.6) *has at least* n *positive solutions where* $n = 2k+1$, $k \in \mathbf{N}$ *provided* (C_4) *in Corollary* 2.1 *holds and there are* $0 < p_1 < p_2 < \cdots < p_{n-1}$ *such that* (C_1) *holds at* p_{2i-1}, $i = 1, \dots, k$, *while at the same time* (C_2) *holds at* p_{2i}, $i = 1, \dots, k$. *Moreover,* $0 < ||y_1|| < p_1 < ||y_2|| < p_2 < \cdots < ||y_{n-1}|| < p_{n-1} < ||y_n||$.

Theorem 3.5 (Any even number of solutions). *The* BVP (1.5), (1.6) *has at least* n *positive solutions where* $n = 2k$, $k \in \mathbf{N}$ *provided* (3.1) *holds and there are* $0 < p_1 < p_2 < \cdots < p_{n-1}$ *such that* (C_1) *holds at* p_{2i-1}, $i = 1, \dots, k$, *while at the same time* (C_2) *holds at* p_{2i}, $i = 1, \dots, k-1$. *Moreover,* $0 < ||y_1|| < p_1 < ||y_2|| < p_2 < \cdots < ||y_{n-1}|| < p_{n-1} < ||y_n||$.

Theorem 3.6 (Any even number of solutions). *The* BVP (1.5), (1.6) *has at least* n *positive solutions where* $n = 2k$, $k \in \mathbf{N}$ *provided* (3.2) *holds and there are* $0 < p_1 < p_2 < \cdots < p_{n-1}$ *such that* (C_2) *holds at* p_{2i-1}, $i = 1, \dots, k$, *while at the same time* (C_1) *holds at* p_{2i}, $i = 1, \dots, k-1$. *Moreover,* $0 < ||y_1|| < p_1 < ||y_2|| < p_2 < \cdots < ||y_{n-1}|| < p_{n-1} < ||y_n||$.

4. Symmetric solutions. As mentioned in the introduction, one motivation for considering BVP's of the form (1.5), (1.6) is that these are generalizations of the Lidstone BVP's; see [**1**]. Symmetry plays an important role in [**1**]. We now address the symmetry of solutions of these higher order Sturm-Liouville problems.

For each fixed y, if $f(t, y)$ is symmetric about $t = 1/2$, we may take

$C[0,1]$ as our Banach space and define the cone $K \subset C[0,1]$ by

$$K := \big\{\varphi \in C[0,1] : \varphi(t) \geq 0 \text{ is symmetric, and } \min_{1/4 \leq t \leq 3/4} \varphi(t) \geq \Sigma ||\varphi||\big\}.$$

We remark that an equivalent formulation of (2.1) is given by

$$G_j(t,s) = \int_0^1 G_{j-1}(t,r) G(r,s)\, dr, \quad 2 \leq j \leq m,$$

where $G_1(t,s) := G(t,s)$ is the Green's function for (1.7), (1.6). This implies that $G_m(t,s)$ is in fact symmetric and leads to the following variation of Theorem 3.3. Note that symmetric versions of Theorems 3.4–3.6 could be stated as well.

Theorem 4.1 (Any odd number of symmetric solutions). *Suppose that for all fixed* y, $f(t,y)$ *is symmetric about* $t = 1/2$. *Then the* BVP (1.5), (1.6) *has at least* n *positive, symmetric solutions where* $n = 2k+1$, $k \in \mathbf{N}$ *provided* (C_3) *in Corollary* 2.1 *holds and there are* $0 < p_1 < p_2 < \cdots < p_{n-1}$ *such that* (C_2) *holds at* p_{2i-1}, $i = 1, \ldots, k$, *while at the same time* (C_1) *holds at* p_{2i}, $i = 1, \ldots, k$. *Moreover,* $0 < ||y_1|| < p_1 < ||y_2|| < p_2 < \cdots < ||y_{n-1}|| < p_{n-1} < ||y_n||$.

5. Examples.

Example 1. Consider the BVP

$$\begin{aligned} (-1)^m y^{(2m)}(t) &= p(t) f(y), \quad 0 < t < 1, \\ \alpha y^{(2i)}(0) - \beta y^{(2i+1)}(0) &= 0, \quad 0 \leq i \leq m-1, \\ \gamma y^{(2i)}(1) + \delta y^{(2i+1)}(1) &= 0, \quad 0 \leq i \leq m-1. \end{aligned}$$

As an example, let $\alpha = \beta = \gamma = \delta = 1$ so that $\rho := \gamma\beta + \alpha\gamma + \alpha\delta = 3$ and let

$$\begin{aligned} p(t) &= \lambda t(1-t), \quad 0 \leq t \leq 1, \\ f(y) &= \sinh y. \end{aligned}$$

Then

$$\begin{aligned} h_0(t) &:= \lim_{y \to 0^+} \frac{p(t) \sinh y}{y} = p(t), \\ h_\infty(t) &:= \lim_{y \to \infty} \frac{p(t) \sinh y}{y} = \infty, \quad 1/4 \leq t \leq 3/4. \end{aligned}$$

Since $p(t)$ is symmetric (with respect to $t = 1/2$), we will obtain symmetric solutions. As for the other constants, we get

$$\begin{aligned}
I &:= \int_0^1 G(s,s)\,ds = \frac{13}{18} \\
J &:= \int_{1/4}^{3/4} G(s,s)\,ds = \frac{107}{288} \\
\sigma &:= \min\left\{\frac{\gamma+4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\right\} = \frac{5}{8} \\
\eta &:= \frac{1}{I^m} = \left(\frac{18}{13}\right)^m \\
\mu &:= \frac{1}{\sigma^m J^m} = \left(\frac{2304}{535}\right)^m \\
\Sigma &:= \frac{\sigma^m J^{m-1}}{I^{m-1}} = \left(\frac{5}{8}\right)^m \left(\frac{107}{288}\right)^{m-1} \left(\frac{18}{13}\right)^{m-1}.
\end{aligned}$$

Suppose now that $m = 2$ and as a result, we are dealing with the fourth order BVP

$$\begin{aligned}
y^{(4)}(t) = \lambda t(1-t)\sinh y, &\quad 0 < t < 1, \\
y(0) - y'(0) &= 0, \\
y(1) + y'(1) &= 0, \\
y''(0) - y'''(0) &= 0, \\
y''(1) + y'''(1) &= 0.
\end{aligned}$$

The first part of Corollary 2.1 yields the existence of a positive solution provided

$$h_0(t) = p(t) = \lambda t(1-t) < \eta = \left(\frac{18}{13}\right)^2.$$

Since $p(t) \le p(1/2) = \lambda/4$, we are guaranteed a positive symmetric solution if

$$\frac{\lambda}{4} < \left(\frac{18}{13}\right)^2 \quad \Longleftrightarrow \quad \lambda < 7.6686\ldots$$

(Here, $h_\infty(t) = \infty$ as observed earlier.)

Example 2. Consider the fourth order BVP

$$\begin{aligned} y^{(4)}(t) &= \lambda t(1-t)g(y), \quad 0<t<1,\\ y(0)-y'(0)&=0,\\ y(1)+y'(1)&=0,\\ y''(0)-y'''(0)&=0,\\ y''(1)+y'''(1)&=0. \end{aligned}$$

where g is a continuous function such that

$$g(y)=\begin{cases} ay & 0\le y\le 1,\\ \varepsilon(0,a] & 1\le y\le q_1,\\ \text{by} & y\ge \hat{q}_2,\\ \text{arbitrary} \ge 0 & \text{otherwise}, \end{cases}$$

where q_1 and $\hat{q}_2$ are chosen below. For this case,

$$\begin{aligned} h_0(t)&=a\lambda t(1-t),\\ h_\infty(t)&=b\lambda t(1-t), \quad \frac{1}{4}\le t\le \frac{3}{4}. \end{aligned}$$

We need the following conditions satisfied.

(C_1) There is a $p>0$ such that $f(t,y)\le \eta p$ for $0\le t\le 1$ and $0\le y\le p$.

(C_2) There is a $q>0$ such that $f(t,y)\ge \mu q$ for $(1/4)\le t\le (3/4)$ and $\Sigma q\le y\le q$.

For $m=2$ we have

$$\begin{aligned} \eta &= \left(\frac{18}{13}\right)^2\\ \mu &= \left(\frac{2304}{535}\right)^2\\ \Sigma &= \left(\frac{5}{8}\right)^2\left(\frac{107}{288}\right)\left(\frac{18}{13}\right). \end{aligned}$$

In this example,

$$\begin{aligned} f(t,y)&=\lambda t(1-t)g(y)\\ &\le \lambda t(1-t)a \quad \text{for } 0\le y\le q_1,\ 0\le t\le 1\\ &\le \frac{\lambda a}{4} \quad \text{for } 0\le y\le q_1,\ 0\le t\le 1. \end{aligned}$$

Hence, if $(\lambda a/4) \leq \eta q_1$, then (C_1) holds for $p = q_1$. That is, we need $q_1 > 1$ so that

$$q_1 \geq \frac{a\lambda}{4\eta} = \left(\frac{13}{18}\right)^2 \frac{a\lambda}{4}.$$

Therefore, given $a, \lambda > 0$ choose $q_1 \geq (13/18)^2(a\lambda/4)$ and then (C_1) will hold for $p = q_1$.

On the other hand, for $1/4 \leq t \leq 3/4$,

$$\lambda t(1-t) \geq \lambda \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{3\lambda}{16}$$

which implies

$$f(t, y) = \lambda t(1-t)g(y) \geq \frac{3\lambda}{16}\, by \quad \text{for } y \geq \hat{q}_2.$$

Therefore, (C_2) will hold for $q = q_2$ provided

$$\frac{3\lambda}{16}\, by \geq \mu q_2, \quad \Sigma q_2 \leq y \leq q_2,\ y \geq \hat{q}_2.$$

We need

$$\frac{3\lambda}{16}\, b\Sigma q_2 \geq \mu q_2, \quad \text{where } \Sigma q_2 = \hat{q}_2.$$

If

$$\lambda b \geq \left(\frac{\mu}{\Sigma}\right) \cdot \frac{16}{3} = 492.2$$

then (C_2) holds at $q = q_2 = \hat{q}_2/\Sigma$ where $\hat{q}_2 > q_1$.

To summarize, suppose $a, \lambda > 0$ are given. Then (C_1) holds for $p = q_1$ provided

$$\frac{q_1}{a} \geq \left(\frac{13}{18}\right)^2 \frac{\lambda}{4}$$

and (C_2) holds for $q = q_2$ provided

$$\lambda \geq \frac{492.2}{b}.$$

To elaborate even further, if $a = 1$ then choose $q_1 \geq (13/18)^2(\lambda/4)$ so that (C_1) holds for $p = q_1$. Setting $\hat{q}_2 = \Sigma q_2 > q_1$, then (C_2) holds for $q = q_2$ provided $b\lambda \geq 492.2$.

For the above example with $a, b, \lambda > 0$, if

$$7.69\left(\frac{q_1}{a}\right) \geq \lambda \geq \frac{492.2}{b}$$

then (C_1) and (C_2) both hold. Theorem 2.2 guarantees the existence of a positive solution y such that

$$q_1 \leq ||y|| \leq q_2 = \frac{\hat{q}_2}{\Sigma}.$$

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