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THE OBRECHKOFF TRANSFORM ON SPACES OF GENERALIZED FUNCTIONS

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ABSTRACT. In this paper we study the Obrechkoff transformation on some generalized functional spaces by employing the kernel method. Thus we extend the results of Baier and Glaeske for the Laplace transformation and of Betancor and Rodríguez-Mesa for the K transformation. Furthermore, from the results on the Obrechkoff transformation, the analogous ones for the Krätzel transformation follow as special cases.

1. Introduction. In this paper we aim to define and study the so-called Obrechkoff integral transform on some spaces of generalized functions. This transform seems to be one of the most general and effective generalizations of the Laplace transform, related to differential operators of Bessel-type, $m \in \mathbf{N}$, $\beta := m - (\alpha_0 + \alpha_1 + \cdots \alpha_m) > 0$, $\gamma_k = (\alpha_k + \alpha_{k+1} + \cdots + \alpha_m - m + k)/\beta$, $k = 1, \dots, m$,

$$(1.1) \quad B = x^{\alpha_0} \frac{d}{dx} x^{\alpha_1} \frac{d}{dx} \cdots x^{\alpha_{m-1}} \frac{d}{dx} x^{\alpha_m} = x^{-\beta} \prod_{j=1}^{m} \left(x \frac{d}{dx} + \beta \gamma_k \right),$$

also called "hyper-Bessel differential operators." As most simple cases of (1.1), the second order differential operator of Bessel and the mth order differentiation $D^m = (d/dx)^m$ appear. Many authors have introduced and studied Laplace-type integral transforms related to very special cases of (1.1). It happened that the most general transform of this kind had been introduced in 1958 by Obrechkoff [**12**], but his results remained unknown for a period of time. Dimovski [**4**] established that this transform can be successfully used for building an operational calculus for the operators (1.1) (and their linear right inverse operators, the "hyper-Bessel" integral operators L). In [**4**], [**5**], some basic properties of the Obrechkoff transform are found for

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spaces of locally integrable functions with suitable growth at zero and infinity. A deeper study, on the basis of the Meijer's G-functions and generalized fractional calculus, can be seen in Kiryakova [**6**]. Here we define the Oberchkoff transform by means of the kernel method on some spaces of generalized functions introduced by McBride [**10**]. Analyticity, boundedness and inversion theorems are established for the generalized Obrechkoff transformation, thus generalizing the work of the papers [**1**] and [**2**].

The Obrechkoff integral transformation can be defined by

(1.2)
$$\mathcal{O}\{f(t);z\} = \beta z^{-\beta(\gamma_m+1)+1}\int_0^\infty G^{m,0}_{0,m}((zt)^\beta \mid (\gamma_k+1-1/\beta)_1^m) f(t)\,dt,$$

where $G^{m,0}_{0,m}$ is a particular case of the Meijer G-function ([**6**, p. 313]).

It is well known that the Obrechkoff transformation is reduced to the K transform for $\beta = m = 2$, $\gamma_{1,2} = \pm\nu/2$, namely,

$$\mathcal{O}\{f(t);z\} = 2^{-\nu+2} z^{-1/2+\nu} K_\nu\{f(t);z\},$$

where K_ν denotes the K transform ([**13**]).

If $\beta = m$, $\gamma_k = k/m$, $k = 1,\dots,m-1$, and $\gamma_m = \nu - 1/m$, we have

$$\mathcal{O}\{f(t);z\} = m\cdot\mathcal{L}^{(m)}_\nu\{t^{m(\gamma_m+1)-1}f(t); m\cdot z\},$$

where

$$\mathcal{L}^{(m)}_\nu\{f(t);z\} = \int_0^\infty \lambda^{(m)}_\nu(zt)f(t)\,dt$$

and $\lambda^{(m)}_\nu(x)$ is the kernel studied by Krätzel ([**7**–**9**]). Note that in [**3**] this transform is studied by means of the adjoint operator on McBride's spaces.

McBride [**10**] defined the functional space $F_{p,\mu}$ as follows: let $\mu\in\mathbf{C}$,

$$F_{p,\mu} = \left\{\varphi\in\mathcal{C}^\infty(\mathbf{R}^+) : t^k\frac{d^k}{dt^k}(t^{-\mu}\varphi(t))\in L^p(\mathbf{R}^+), \forall\, k\in\mathbf{N}\right\},$$

where $1\le p<\infty$ and

$$F_{\infty,\mu} = \left\{\varphi\in\mathcal{C}^\infty(\mathbf{R}^+) : t^k\frac{d^k}{dt^k}(t^{-\mu}\varphi(t))\longrightarrow 0 \quad \text{as } x\to 0 \text{ and } x\to\infty, \forall\, k\in\mathbf{N}\right\},$$

where $p = \infty$. $F_{p,\mu}$ is a complete countable multi-normed space (Fréchet space) equipped with the topology generated by the family of semi-norms in $F_{p,\mu}$ given by

$$\gamma_k^{p,\mu}(\varphi) = \left\| t^k \frac{d^k}{dt^k}(t^{-\mu}\varphi(t)) \right\|_p, \quad k \in \mathbf{N};\ 1 \le p \le \infty,\ \mu \in \mathbf{C},$$

where $\|\cdot\|_p$ denotes the norm in the space $L^p(\mathbf{R}^+)$. By $F'_{p,\mu}$ we denote the dual space of $F_{p,\mu}$.

Throughout the paper μ denotes an arbitrary complex number, β is real and positive, $1 \le p \le \infty$ (unless the contrary is explicitly stated) while p and q are related by $1/p + 1/q = 1$.

2. The Obrechkoff transform. Next we establish a series of results in order to define the $\mathcal{O}$-transformation on these spaces, by using the kernel method.

Lemma 2.1. *Let $\beta > 1/p$. For every $z \in \mathbf{C}$ such that* $\operatorname{Re} z > 0$ *the kernel $\lambda(z,t) = z^{-\beta(\gamma_m+1)+1} G_{0,m}^{m,0}((zt)^\beta | (\gamma_k + 1 - 1/\beta)_1^m) \in F_{p,\mu}$ provided that*

$$\operatorname{Re}\mu < \beta \min_{1 \le k \le m} (\gamma_k + \beta - 1 + 1/p).$$

Proof. We can see without difficulty that

$$t^k \frac{d^k}{dt^k} \equiv \sum_{i=0}^{[k/2]} a_{i,k} \cdot t^{\beta(k-i)} \left(t^{1-\beta} \frac{d}{dt} \right)^{k-i}, \tag{2.1}$$

where $a_{0,k} = 1$ and $a_{i,k}$ are suitable real numbers, $i = 1, \dots, [k/2]$.

$$\begin{aligned} t^k \frac{d^k}{dt^k}(t^{-\mu}\lambda(z,t)) &= z^{-\beta(\gamma_m+1)+1} \\ &\qquad \cdot t^k \frac{d^k}{dt^k}[t^{-\mu} G_{0,m}^{m,0}((zt) \mid (\gamma_k + 1 - 1/\beta)_1^m)] \\ &= z^{\mu-\beta(\gamma_m+1)-1} \\ &\qquad \cdot t^k \frac{d^k}{dt^k}[(zt)^{-\mu} G_{0,m}^{m,0}((zt) \mid (\gamma_k + 1 - 1/\beta)_1^m)]. \end{aligned}$$

Using the property (A.14), page 315, [**6**], we obtain

$$t^k \frac{d^k}{dt^k}(t^{-\mu}\lambda(z,t)) = z^{\mu-\beta(\gamma_m+1)+1} \cdot t^k \frac{d^k}{dt^k}[G^{m,0}_{0,m}((zt)^\beta \mid (\gamma_k + 1 - 1/\beta - \mu/\beta)_1^m)].$$

Moreover, by (2.1) it follows that the above expression is:

$$= z^{\mu-\beta(\gamma_m+1)+1} \sum_{i=0}^{[k/2]} a_{i,k} \cdot t^{\beta(k-i)} \left(t^{1-\beta}\frac{d}{dt}\right)^{k-i} \cdot [G^{m,0}_{0,m}((zt)^\beta | (\gamma_k + 1 - 1/\beta - \mu/\beta)_1^m)].$$

Now, with the change of variables $(zt)^\beta = u$ and making use of [**6**, p. 316]

(2.2)

$$t^k \frac{d^k}{dt^k}(t^{-\mu}\lambda(z,t)) = z^{\mu-\beta(\gamma_m+1)+1} \sum_{i=0}^{[k/2]} a_{i,k} \cdot G^{m,1}_{1,m+1}\left((zt)^\beta \;\middle|\; \begin{matrix} 0 \\ (\gamma_k+1-1/\beta-\mu/\beta)_1^m \quad ; k-i \end{matrix}\right).$$

By the asymptotic behavior of the Meijer's G-functions, for $r \in \mathbf{N}$, we have that

$$\max_{0\le k\le r} \left\| t^k \frac{d^k}{dt^k}(t^{-\mu}\lambda(z,t)) \right\|_p < \infty,$$

when the conditions are verified.

Let $\beta > 1/p$ and $\operatorname{Re}\mu < \beta \min_{1\le k\le m}(\gamma_k + \beta - 1 + 1/p)$. For every $f \in F'_{p,\mu}$ the generalized Obrechkoff transform $\mathcal{O}f$ of f is defined through

$$\mathcal{O}f(z) = \langle f(t), \lambda(z,t)\rangle, \quad \operatorname{Re} z > 0.$$

In the following we establish a smoothness property of the generalized $\mathcal{O}$-transformation.

Proposition 2.1. *Let $\beta > 1/p$ and $\operatorname{Re}\mu < \beta \min_{1\le k\le m}(\gamma_k + \beta - 1 + 1/p)$. If $f \in F'_{p,\mu}$, then $(\mathcal{O}f)(z)$ is a holomorphic function on $\operatorname{Re} z > 0$. Moreover, for each $m \in \mathbf{N}$*

$$\frac{d^m}{dz^m}(\mathcal{O}f)(z) = \langle f(t), \frac{\partial^m}{\partial z^m}[\lambda(z,t)]\rangle, \quad \operatorname{Re} z > 0.$$

Proof. We see that

$$\lim_{h\to 0} \frac{(\mathcal{O}f)(z+h) - (\mathcal{O}f)(z)}{h} = \langle f(t), \frac{\partial}{\partial z}[\lambda(z,t)]\rangle, \quad \operatorname{Re} z > 0. \tag{2.3}$$

As $f \in F'_{p,\mu}$, to demonstrate (2.3) it is sufficient to see that

$$\lim_{h\to 0} \frac{\lambda(z+h,t) - \lambda(z,t)}{h} = \frac{\partial}{\partial z}[\lambda(z,t)] \tag{2.4}$$

in the sense of convergence in $F_{p,\mu}$.

Consider $z, h \in \mathbf{C}$ such that $\operatorname{Re} z > 0$, $\operatorname{Re}(z+h) > 0$. We define

$$\varphi_h(z,t) = \frac{\lambda(z+h,t) - \lambda(z,t)}{h} - \frac{\partial}{\partial z}[\lambda(z,t)], \quad t \in (0,\infty).$$

Using the Cauchy integral formula we obtain

$$\varphi_h(z,t) = \frac{h}{2\pi i}\int_\gamma \frac{\lambda(\eta,t)}{(\eta - z)^2(\eta - z - h)}\, d\eta, \quad t \in (0,\infty), \tag{2.5}$$

where the path γ can be parametrized by $\eta(\theta) = z + re^{i\theta}$, $\theta \in [0, 2\pi]$ and r, h are chosen such that $0 < |h| < r$ and $\operatorname{Re} z > r$.

Let $k \in \mathbf{N}$. By (2.2) and (2.5), we have

$$\begin{aligned}
t^k \frac{d^k}{dt^k}&(t^{-\mu}\varphi_h(z,t)) \\
&= \frac{h}{2\pi i}\int_\gamma \frac{t^k (d^k/dt^k)(t^{-\mu}\lambda(\eta,t))}{(\eta-z)^2(\eta - z - h)}\, d\eta \\
&= \frac{h}{2\pi i} \cdot z^{\mu - \beta(\gamma_m+1)+1} \sum_{i=0}^{[k/2]} a^*_{i,k} \\
&\quad \cdot \int_\gamma \frac{G^{m,1}_{1,m+1}\left((zt)^\beta \,\middle|\, \begin{matrix} 0 \\ (\gamma_k + 1 - 1/\beta - \mu/\beta)_1^m \quad ; k - i \end{matrix} \right)}{(\eta - z)^2(\eta - z - h)}\, d\eta.
\end{aligned}$$

Denote $G(z,t) = G^{m,1}_{1,m+1}\left((zt)^\beta \,\middle|\, \begin{matrix} 0 \\ (\gamma_k+1-1/\beta-\mu/\beta)_1^m \quad ; k-i \end{matrix}\right)$,
then

$$\left\|t^k \frac{d^k}{dt^k}(t^{-\mu}\varphi_h(z,t))\right\|_p \leq c|h||z^{\mu-\beta(\gamma_m+1)+1} \sum_{i=0}^{[k/2]} a^*_{i,k}\cdot \int_0^{2\pi} \frac{\|G(z,t)\|_p}{r(r-|h|)}\,d\theta.$$

Then, proceeding in a similar way to Lemma 2.1, if $\beta > 1/p$ and $\operatorname{Re}\mu < \beta \min_{1\leq k\leq m}(\gamma_k+\beta-1+1/p)$, we have for each $r\in\mathbf{N}$,

$$\max_{0\leq k\leq r}\left\|t^k\frac{d^k}{dt^k}(t^{-\mu}\varphi_h(z,t))\right\|_p \longrightarrow 0,$$

as $|h|\to 0$. With this we prove (2.4). Now the result follows by induction.

Next we establish a rule of operational calculus.

Proposition 2.2. *Let* $\beta > 1/p$ *and* $\operatorname{Re}\mu < \beta\min_{1\leq k\leq m}(\gamma_k+\beta-1+1/p)$. *For every* $f\in F'_{p,\mu}$ *and* $m\in\mathbf{N}$, *we have*

$$B_{z,(b_i)_1^m,\eta}(\mathcal{O}f)(z) = \langle f(t), (-1)^m\beta^m z^{\beta(\gamma_m+1+\eta)-1}t^{\eta\beta}\lambda(z,t)\rangle,$$
$$\operatorname{Re} z > 0,$$

where

$$B_{z,(b_i)_1^m,\eta} \equiv z^{\beta(\gamma_m+1)-1}\prod_{i=1}^m\left(\frac{d}{dz}z-b_i\right)\left(\frac{d}{dz}z-b_i-1\right) \cdots\left(\frac{d}{dz}z-b_i-\eta-1\right).$$

Proof. Denote $G^*(z,t)\equiv G^{m,0}_{0,m}((zt)^\beta|(\gamma_k+1-1/\beta-\mu/\beta)_1^m)$. By Proposition 2.1 we obtain

$$B_{z,(b_i)_1^m,\eta}(\mathcal{O}f)(z) = \left\langle f(t), \prod_{i=1}^m\left(\frac{d}{dz}z-b_i\right)\left(\frac{d}{dz}z-b_i-1\right) \cdots\left(\frac{d}{dz}z-b_i-\eta-1\right)G^*(z,t)\right\rangle.$$

Then by [**6**, Corollary B.5], we have

$$\begin{aligned}B_{z,(b_i)_1^m,\eta}(\mathcal{O}f)(z) &= \langle f(t), (-1)^m\beta^m(zt)^{\eta\beta}G^*(z,t)\rangle \\ &= \langle f(t), (-1)^m\beta^m(zt)^{\eta\beta}z^{\beta(\gamma_m+1)-1}\lambda(z,t)\rangle \\ &= \langle f(t), (-1)^m\beta^m z^{\beta(\gamma_m+1+\eta)-1}t^{\eta\beta}\lambda(z,t)\rangle.\end{aligned}$$

3. A new inversion formula. In this section we prove a new inversion formula, a uniqueness result and a boundedness property. First we recall some definitions that are useful in obtaining the inversion formula.

Definition 3.1. We define the sets $A_{p,\mu,\beta}$ and $A'_{p,\mu,\beta}$ of complex numbers by

$$A_{p,\mu,\beta} = \{\gamma : \operatorname{Re}(\beta\gamma+\mu)+\beta \neq 1/p-\beta l, \quad l=0,1,2,\dots\}, \tag{3.1}$$

$$A'_{p,\mu,\beta} = \{\gamma : \operatorname{Re}(\beta\gamma-\mu) \neq -1/p-\beta l, \quad l=0,1,2,\dots\}. \tag{3.2}$$

Definition 3.2. For $\operatorname{Re}\mu > 1/p-\beta(\gamma_k+1)$, $\gamma_k \in A_{p,\mu,\beta}$, $k = 1,\dots,m$, $\delta_k \in \mathbf{R}$, the *generalized fractional integrals* (*multiple Erdélyi-Kober operators*) are defined by

$$I_{\beta,m}^{(\gamma_k),(\delta_k)}\varphi(t) \equiv I_\beta^{\gamma_m,\delta_m}\{I_\beta^{\gamma_{m-1},\delta_{m-1}}\cdots(I_\beta^{\gamma_1,\delta_1}\varphi(t))\},$$

where $I_\beta^{\gamma,\delta}$ is the Erdélyi-Kober operator defined as in [**10**, p. 522]. Moreover, if $\delta_k > 0$, then the integral representations hold

$$\begin{aligned}I_{\beta,m}^{(\gamma_k),(\delta_k)}\varphi(t) &= \int_0^1\cdots\int_0^1\prod_{k=1}^m\frac{(1-\sigma_k)^{\delta_k-1}\sigma_k^{\gamma_k}}{\Gamma(\delta_k)} \\ &\qquad\cdot\varphi(t(\sigma_1\cdots\sigma_m)^{1/\beta})\,d\sigma_1\cdots d\sigma_m \\ &\equiv \int_0^1 G_{m,m}^{m,0}\left(\sigma\ \middle|\ \begin{matrix}(\gamma_k+\delta_k)_1^m\\ \\ (\gamma_k)_1^m\end{matrix}\right)\varphi(t\sigma^{1/\beta})\,d\sigma \\ &= t^{-\beta}\int_0^t G_{m,m}^{m,0}\left(\left(\frac{u}{t}\right)^\beta\ \middle|\ \begin{matrix}(\gamma_k+\delta_k)_1^m\\ (\gamma_k)_1^m\end{matrix}\right)\varphi(u)d(u^\beta).\end{aligned}$$

On the other hand, $I_{\beta,m}^{(\gamma_k),(\delta_k)}$ is a continuous linear mapping from $F_{p,\mu}$ into itself.

These operators have been widely studied in the classical sense in [**6**].

By the conditions stated on the parameters, the two integral representations above are equivalent. For a classical proof, see [**6**, Theorem 1.2.10, pp. 30–32].

Moreover, if $\gamma_k + \delta_k \in A_{p,\mu,\beta}$, $k = 1, \dots, m$, we define the linear left inverse operator of the generalized fractional integral $I_{\beta,m}^{(\gamma_k),(\delta_k)}$ as a generalized fractional derivative (see [**6**])

$$\begin{aligned}\left(I_{\beta,m}^{(\gamma_k),(\delta_k)}\right)^{-1} f(x)&\\ &= \left(I_{\beta}^{\gamma_m,\delta_m}\right)^{-1}\left\{\left(I_{\beta}^{\gamma_{m-1},\delta_{m-1}}\right)^{-1}\cdots\left[\left(I_{\beta}^{\gamma_1,\delta_1}\right)^{-1} f(x)\right]\right\}\\ &= I_{\beta}^{\gamma_m+\delta_m,-\delta_m}\left\{I_{\beta}^{\gamma_{m-1}+\delta_{m-1},-\delta_{m-1}}\cdots\left[I_{\beta}^{\gamma_1+\delta_1,-\delta_1} f(x)\right]\right\}\\ &= I_{\beta,m}^{(\gamma_k+\delta_k),(-\delta_k)} f(x)\end{aligned}$$

where $I_{\beta}^{\gamma_k+\delta_k,-\delta_k}$ is defined as in [**11**, p. 52].

Definition 3.3. Let $f \in F'_{p,\mu}$. For $\operatorname{Re}\mu > 1/p - \beta\gamma_k + 1$, $\gamma_k \in A'_{q,-\mu,\beta}$, $k = 1, \dots, m$, we define analogously the Weyl type (right-sided) generalized fractional integrals

$$K_{\beta,m}^{(\gamma_k),(\delta_k)} f(t) \equiv K_{\beta}^{\gamma_1,\delta_1}\{K_{\beta}^{\gamma_2,\delta_2}\cdots(K_{\beta}^{\gamma_m,\delta_m} f(t))\}$$

and, for all $\varphi \in F_{p,\mu}$,

$$\langle K_{\beta,m}^{(\gamma_k),(\delta_k)} f, \varphi\rangle = \langle f, I_{\beta}^{(\gamma_k-1+1/\beta),(\delta_k)}\varphi\rangle.$$

Under the above conditions $K_{\beta,m}^{(\gamma_k),(\delta_k)}$ is a continuous linear mapping from $F'_{p,\mu}$ into itself.

Moreover, if $\gamma_k + \delta_k \in A'_{q,-\mu,\beta}$, we define the linear left inverse operator of the generalized fractional integral $K_{\beta,m}^{(\gamma_k),(\delta_k)}$ as

$$\begin{aligned}\left\langle \left(K_{\beta,m}^{(\gamma_k),(\delta_k)}\right)^{-1} f, \varphi\right\rangle &= \langle K_{\beta,m}^{(\gamma_k+\delta_k),(-\delta_k)} f, \varphi\rangle\\ &= \left\langle f, \left(I_{\beta}^{(\gamma_k-1+1/\beta),(\delta_k)}\right)^{-1}\varphi\right\rangle\\ &= \langle f, I_{\beta,m}^{(\gamma_k+\delta_k),(-\delta_k)}\varphi\rangle,\end{aligned}$$

where $I_\beta^{(\gamma_k+\delta_k),(-\delta_k)}$ is defined as in Definition 3.2.

Definition 3.4. Let $\lambda_k = \gamma_m - \gamma_k + k/m > 0$, $\operatorname{Re}\mu > 1/p - m(\gamma_k + 1) + 1$ and $\gamma_k \in A_{p,\mu-1,m}$, $k = 1, \dots, m-1$. We define the following operator for every $\varphi \in F_{p,\mu}$:

$$
\begin{aligned}
T\varphi(t) &= t^{m-1} \int_0^{1/t} G_{m-1,m-1}^{m-1,0}\left(u^m t^m \,\middle|\, \begin{matrix} (\gamma_k+\lambda_k)_1^{m-1} \\ (\gamma_k)_1^{m-1} \end{matrix} \right) \\
&\qquad\qquad \cdot u^{-m(1+\gamma_m)} \varphi(1/u) d(u^m) \\
&= t^{-1} \cdot I_{m,m-1}^{(\gamma_k),(\lambda_k)}\{\phi(.)\}(1/t),
\end{aligned}
$$

where $\phi(u) = u^{-m(1+\gamma_m)}\varphi(1/u)$.

One can see that T is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+m(1+\gamma_m)+1}$. (Notice that if $\varphi(u) \in F_{p,\mu}$, then $\varphi(1/u) \in F_{p,\mu}$).

By means of this operator, we define for all $f \in F'_{p,\mu}$ the following "Sonine-Dimovski"-type operator T^* (see [**6**])

$$\langle T^* f(t), \varphi(t)\rangle = \langle f(t), T\varphi(t)\rangle,$$

where $\varphi \in F_{p,\mu-m(1+\gamma_m)-1}$. Namely,

$$T^* f(t) = t^{m(1+\gamma_m)} K_{m,m-1}^{(\gamma_k+1-1/m),(\lambda_k)}[g(.)](1/t),$$

where $g(u) = u \cdot f(1/u)$.

By definition, T^* is a continuous linear mapping from $F'_{p,\mu}$ into $F'_{p,\mu-m(1+\gamma_m)-1}$. Moreover, if $\gamma_m + k/m \in A_{p,\mu-1,m}$, for $k = 1, \dots, m-1$, the operator T^* admits an inverse one.

Now we record a theorem given in [**1**, p. 318].

Theorem 3.1. *Let $f \in F'_{p,\mu}$, $\operatorname{Re}\mu < 1/p$, $\operatorname{Re} z > 0$ and $c, r \in (0,\infty)$. Then*

$$f(t) = \lim_{r\to\infty} \frac{1}{(2\pi i)} \int_{c-ir}^{c+ir} \mathcal{L}\{f(t); z\} e^{tz}\, dz,$$

being $\mathcal{L}\{f(t); z\} = \langle f(t), e^{-zt}\rangle$, the Laplace transform of f, and the convergence has to be taken in the sense of $\mathcal{D}'(I)$.

To establish an inversion formula for the generalized $\mathcal{O}$-transformation, we need to prove in advance the following lemma.

Lemma 3.1. *Let* $\beta = m > 1/p$, $\lambda_k = \gamma_m - \gamma_k + k/m > 0$ *and* $f \in F'_{p,\mu}$. *If the following conditions*

(i) $\gamma_k \in A_{p,\mu-1,m}$ *when* $k = 1, \dots, m-1$;

(ii) $1/p - m(\gamma_k + 1) + 1 < \operatorname{Re}\mu < m \min_{1\le k\le m}(\gamma_k + m - 1 + 1/p)$, *for* $k = 1, \dots, m$;

are satisfied, then

$$\mathcal{L}\{T^* f(t); z\} = c_1 \cdot \mathcal{O}\{f(t); (z/m)\}, \quad \operatorname{Re} z > 0,$$

where

$$c_1 = \frac{(2\pi)^{(1-m)/2}}{m^{1/2-m}}.$$

Proof. First, we calculate

$$\begin{aligned} T\{e^{-zu^{-1}}\}(1/t) &= t \cdot I^{(\gamma_k),(\lambda_k)}_{m,m-1}\{u^{-m(1+\gamma_m)} e^{-zu^{-1}}\}(t) \\ &= t^{1-m} \int_0^t G^{m-1,0}_{m-1,m-1}\left(\frac{u^m}{t^m} \;\middle|\; \begin{matrix} (\gamma_k + \lambda_k)_1^{m-1} \\ (\gamma_k)_1^{m-1} \end{matrix} \right) \\ &\qquad \cdot u^{-m(1+\gamma_m)} e^{-zu^{-1}} \, d(u^m). \end{aligned}$$

Making the change $u = x^{-1}$ we obtain

$$\begin{aligned} &= t^{1-m} \int_{t^{-1}}^{\infty} G^{m-1,0}_{m-1,m-1}\left(\frac{t^{-m}}{x^m} \;\middle|\; \begin{matrix} (\gamma_k + \lambda_k)_1^{m-1} \\ (\gamma_k)_1^{m-1} \end{matrix} \right) x^{m(1+\gamma_m)} e^{-zx} \, dx \\ &= t^{1-m} \int_{t^{-1}}^{\infty} G^{0,m-1}_{m-1,m-1}\left(\frac{x^m}{t^{-m}} \;\middle|\; \begin{matrix} (1 - \gamma_k)_1^{m-1} \\ (1 - \gamma_k - \lambda_k)_1^{m-1} \end{matrix} \right) x^{m(1+\gamma_m)} e^{-zx} \, dx. \end{aligned}$$

Now we denote the last integral by J and substitute there

$$\begin{aligned} e^{-zx} \equiv G^{1,0}_{0,1}(zx \mid 0), G^{0,m-1}_{m-1,m-1}&\left(\frac{x^m}{t^{-m}} \;\middle|\; \begin{matrix} (1 - \gamma_k)_1^{m-1} \\ (1 - \gamma_k - \lambda_k)_1^{m-1} \end{matrix} \right) \\ &\equiv 0 \text{ for } x < t^{-1}, \end{aligned}$$

whence it takes the form

$$J = t^{1-m}\int_{t^{-1}}^{\infty} x^{m(1+\gamma_m)} G_{0,1}^{1,0}(zx \mid 0) \cdot G_{m-1,m-1}^{0,m-1}\left(\frac{x^m}{t^{-m}} \;\middle|\; \begin{matrix}(1-\gamma_k)_1^{m-1}\\ (1-\gamma_k-\lambda_k)_1^{m-1}\end{matrix}\right) dx.$$

Proceeding as in [**6**, p. 184], we achieve

$$J = \frac{(2\pi)^{(1-m)/2}}{m^{1/2-m}} \cdot z^{-m(\gamma_m+1)+1} \cdot G_{0,m}^{m,0}((z/m)^m t^{-m} \mid (\gamma_k + 1 - 1/m)_1^m).$$

Therefore, we have

$$T\{e^{-zu^{-1}}\}(t) = \frac{(2\pi)^{(1-m)/2}}{m^{1/2-m}} \cdot z^{-m(\gamma_m+1)+1} \cdot G_{0,m}^{m,0}((z/m)^m t^{-m}|(\gamma_k + 1 - 1/m)_1^m). \tag{3.3}$$

By definition we know that

$$\begin{aligned}\mathcal{L}\{T^* f(t); z\} &= \langle T^* f(t), e^{-zt}\rangle \\ &= \langle f(t), t^{-1} \cdot I_{m,m-1}^{(\gamma_k),(\lambda_k)}\{u^{-m(1+\gamma_m)}e^{-zu^{-1}}\}(1/t)\rangle \\ &= \langle f(1/t), t \cdot I_{m,m-1}^{(\gamma_k),(\gamma_k)}\{u^{-m(1+\gamma_m)}e^{-zu^{-1}}\}(t)\rangle,\end{aligned}$$

then by (3.3) it follows that

$$\begin{aligned}\mathcal{L}\{T^* f(t); z\} &= \big\langle f(1/t), c_1 \cdot z^{-m(\gamma_m+1)+1} \cdot G_{0,m}^{m,0}((z/m)^m t^{-m} \mid (\gamma_k + 1 - 1/m)_1^m)\big\rangle \\ &= \left\langle f(t), c_1 \cdot z^{-m(\gamma_m+1)+1} \cdot G_{0,m}^{m,0}\left(\left(\frac{z}{m}\right)^m t^m \mid (\gamma_k + 1 - 1/m)_1^m\right)\right\rangle \\ &= c_1\langle f(t), \lambda(z/m, t)\rangle \\ &= c_1 \cdot \mathcal{O}\{f(t); (z/m)\}.\end{aligned}$$

Thus, we obtain a new inversion theorem.

Theorem 3.2. *Let $\beta = m > 1/p$, $\lambda_k = \gamma_m - \gamma_k + k/m > 0$, $c, r \in (0,\infty)$ and $f \in F'_{p,\mu}$. If the following conditions*

(i) $\gamma_k \in A_{p,\mu-1,m}$, $\gamma_m + k/m \in A_{p,\mu-1,m}$ *for* $k = 1, \ldots, m-1$;

(ii) $1/p - m(\gamma_k + 1) + 1 < Re\,\mu < m \min_{1\le k\le m}(\gamma_k + m - 1 + 1/p)$, *when* $k = 1, \ldots, m$;

are verified, then

$$f(t) = t^{-1} \cdot \lim_{r\to\infty} \frac{1}{(2\pi i)} K_{m,m-1}^{(\gamma_m+(k+1)/m+1),(-\lambda_k)} \left\{ \int_{c-ir}^{c+ir} \phi(.,z)\,dz \right\}(1/t),$$

where $\phi(u,z) = \mathcal{O}\{f(u);(z/m)\}u^{m(1+\gamma_m)}e^{z/u}$ *and the convergence has to be taken in the sense of* $\mathcal{D}'(I)$.

Proof. Making use of Theorem 3.1 and of Lemma 3.1, we obtain

$$T^* f(t) = \lim_{r\to\infty} \frac{1}{(2\pi i)} \int_{c-ir}^{c+ir} \mathcal{O}\{f(t);(z/m)\}e^{tz}\,dz, \tag{3.4}$$

where $T^* = t^{m(1+\gamma_m)} K_{m,m-1}^{(\gamma_k+1-1/m),(\lambda_k)}[uf(1/u)](1/t)$. Since $\gamma_m + k/m \in A_{p,\mu-1,m}$, we know that the operator T^* admits an inverse. Therefore, from (3.4), we can conclude

$$t^{m(1+\gamma_m)} K_{m,m-1}^{(\gamma_k+1-1/m),(\lambda_k)}[uf(1/u)](1/t) = \lim_{r\to\infty} \frac{1}{(2\pi i)} \int_{c-ir}^{c+ir} \mathcal{O}\{f(t);(z/m)\}e^{tz}\,dz,$$

$$K_{m,m-1}^{(\gamma_k+1-1/m),(\lambda_k)}[uf(1/u)](1/t) = \lim_{r\to\infty} \frac{1}{(2\pi i)} t^{-m(1+\gamma_m)} \int_{c-ir}^{c+ir} \mathcal{O}\{f(t);(z/m)\}e^{tz}\,dz,$$

$$K_{m,m-1}^{(\gamma_k+1-1/m),(\lambda_k)}[uf(1/u)](t) = \lim_{r\to\infty} \frac{1}{(2\pi i)} t^{m(1+\gamma_m)} \int_{c-ir}^{c+ir} \mathcal{O}\{f(t);(z/m)\}e^{z/t}\,dz,$$

$$tf(1/t) = K_{m,m-1}^{(\gamma_m+(k+1)/m+1),(-\lambda_k)}\left\{\lim_{r\to\infty}\frac{1}{(2\pi i)}\int_{c-ir}^{c+ir}\phi(.,z)\,dz\right\}(t),$$

where $\phi(u,z) = \mathcal{O}\{f(u);(z/m)\}u^{m(1+\gamma_m)}e^{z/u}$, and then

$$f(t) = t^{-1}\cdot\lim_{r\to\infty}\frac{1}{(2\pi i)}K_{m,m-1}^{(\gamma_m+(k+1)/m+1),(-\lambda_k)}\left\{\int_{c-ir}^{c+ir}\phi(.,z)\,dz\right\}(1/t).$$

Remark 3.1. Notice that, as a particular case when $\beta = m$, $\gamma_k = k/m$, $k = 1,\dots,m-1$, and $\gamma_m = \nu - 1/m$, we obtain a new inversion formula for the Krätzel transformation ([**3**], [**7**]–[**9**]).

Now we establish a uniqueness result.

Proposition 3.1. *Let $\beta = m > 1/p$, $\lambda_k = \gamma_m - \gamma_k + k/m > 0$ and $f \in F'_{p,\mu}$. Suppose that the following conditions are satisfied*:

(i) $\gamma_k \in A_{p,\mu-1,m}$, $\gamma_m + k/m \in A_{p,\mu-1,m}$ *when* $k = 1,\dots,m-1$;

(ii) $1/p - m(\gamma_k+1) + 1 < Re\,\mu < m\min_{1\le k\le m}(\gamma_k + m - 1 + 1/p)$, *for* $k = 1,\dots,m$.

Then if $\mathcal{O}\{f(t);z\} = 0$ implies $f = 0$.

Proof. By Theorem 3.2, if $f \in F'_{p,\mu}$ and $\mathcal{O}\{f(t);z\} = 0$, then $\langle f,\varphi\rangle = 0$ for all $\varphi \in \mathcal{D}(I)$. Therefore, $f = 0$ since $\mathcal{D}(I)$ is dense in $F_{p,\mu}$ [**10**, Chapter 2].

Finally we demonstrate a boundedness property.

Proposition 3.2. *Let $\beta > 1/p$ and $Re\,\mu < \beta\min_{1\le k\le m}(\gamma_k + \beta - 1 + 1/p)$. If $f \in F'_{p,\mu}$, then a positive constant c exists such that*

$$|\mathcal{O}\{f(t);z\}| \le c\cdot z^{\max\{\mu-\beta(\gamma_m+1)-1-\beta,\beta(\gamma_1-\gamma_m)-2\}},\quad \operatorname{Re} z > 0.$$

Proof. Let $f \in F'_{p,\mu}$. By invoking [**13**, Theorem 1.8], $c > 0$ and $r \in \mathbf{N}$ exist such that

$$|\mathcal{O}\{f(t);z\}| \le c\cdot\max_{0\le k\le r}\Big\| t^k\frac{d^k}{dt^k}\big(t^{-\mu}z^{-\beta(\gamma_m+1)+1} \cdot G_{0,m}^{m,0}((zt)^\beta|(\gamma_k+1-1/\beta)_1^m)\big)\Big\|_p.$$

Then, proceeding as in Lemma 2.1, we obtain

$$\begin{aligned}|\mathcal{O}\{f(t);z\}| &\leq c\cdot z^{\mu-\beta(\gamma_m+1)+1}\sum_{i=0}^{[k/2]} a_{i,k}\\ &\quad\cdot\max_{0\leq k\leq r}\left\|G_{1,m+1}^{m,1}\left((zt)^\beta\middle|\begin{matrix}0\\(\gamma_k+1-1/\beta-\mu/\beta)_1^m\end{matrix}\;;k-i\right)\right\|_p\\ &\leq c\cdot z^{\max\{\mu-\beta(\gamma_m+1)-1-\beta,\beta(\gamma_1-\gamma_m)-2\}}.\end{aligned}$$

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