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COMPOSITION OPERATORS FROM THE SPACE OF CAUCHY TRANSFORMS INTO ITS HARDY-TYPE SUBSPACES

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ABSTRACT. This paper studies the boundedness and compactness of composition operators from the space of Cauchy transforms into its Hardy-type subspaces. They are characterized by the behavior of the Cauchy kernel composed by inducing self-maps of the unit disk.

1. Introduction. Let $\mathbf{D}$ be the open unit disk and T the unit circle in the complex plane. A holomorphic function f on $\mathbf{D}$ is said to belong to K, the space of all Cauchy transforms, if it admits a representation $f(z) = \int_T 1/(1-\bar{\eta}z)\, d\mu(\eta)$ where μ is a complex Borel measure on T. The following inclusion relations between the class K and Hardy spaces are well known: $H^1 \subset K \subset \cap_{p<1} H^p$. See [**2**] and [**9**].

Now let φ be a holomorphic self-map of $\mathbf{D}$. It was known that the composition operator $C_\varphi(f) = f \circ \varphi$ acts as a bounded operator on the Hardy spaces [**12**], [**10**] and on the space K [**2**]. The compactness of C_φ on the Hardy spaces was completely characterized in terms of the behavior of Nevanlinna counting function by Shapiro [**14**]. A few years later another equivalent characterization, the so-called Sarason's condition, was obtained by Sarason, Shapiro and Sundberg [**13**], [**15**]. Recently, Cima and Matheson [**4**] considered the problem of characterizing the compactness of C_φ on K and have established that C_φ is compact on K if and only if it is compact on H^2.

The purpose of this paper is to study composition operators C_φ which map the space K into some of its subspaces. Indeed, we shall characterize those holomorphic self-maps φ of $\mathbf{D}$ that induce bounded or compact composition operators from the space K to H^p, $p \geq 1$,

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BMOA, VMOA and the Dirichlet space. The characterizations are given by the behavior, in the target spaces, of the Cauchy kernel composed by φ. Some related remarks are also included.

Throughout this paper, the symbol φ will be used to denote a holomorphic self-map of $\mathbf{D}$.

2. Prerequisites. The materials in this section are well known and summarized shortly.

2.1 Spaces K, K_a, H^p. As mentioned in the introduction, the space K is the set of all holomorphic functions f on $\mathbf{D}$ of the form

$$f(z) = \int_T \frac{1}{1-\bar{\eta}z}\, d\mu(\eta) \tag{2.1}$$

where $\mu \in M$, the space of all complex Borel measures on T. This class K becomes a Banach space under the norm

$$\|f\|_K = \inf\{\|\mu\| : \mu \text{ satisfies } (2.1)\},$$

where $\|\mu\|$ denotes the total variation of the measure μ. The space K_a is the subclass of K consisting of all Cauchy transforms of absolutely continuous complex Borel measures on T. It is known that the space K_a is a closed subspace of K [**2**].

For $1 \le p < \infty$, the Hardy space H^p is the Banach space of all holomorphic functions f on $\mathbf{D}$ for which

$$\begin{aligned}\|f\|_{H^p} &\equiv \sup_{0\le r<1}\left(\int_T |f(r\zeta)|^p\, d\sigma(\zeta)\right)^{1/p}\\ &= \left(\int_T |f^*(\zeta)|^p\, d\sigma(\zeta)\right)^{1/p} < \infty,\end{aligned}$$

where $d\sigma$ denotes the Lebesgue measure on T of total mass 1 and $f^*(\zeta) = \lim_{r\nearrow 1} f(r\zeta)$ is the radial limit which exists almost everywhere $\zeta \in T$.

2.2 BMOA, VMOA and the Dirichlet space $\mathfrak{D}$. The space BMOA consists of those functions $f \in H^2$ for which

$$\|f\|_* \equiv \sup_{a\in\mathbf{D}} \|f\circ\varphi_a - f(a)\|_{H^2} < \infty,$$

where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is the conformal automorphism of $\mathbf{D}$ that interchanges a with 0. BMOA is a Banach space under the norm $\|f\|_{\mathrm{BMOA}} = |f(0)| + \|f\|_*$. VMOA is a closed subspace of BMOA which consists of all functions $f \in H^2$ for which

$$\lim_{|a|\to 1^-} \|f \circ \varphi_a - f(a)\|_{H^2} = 0.$$

It is well known that if f is holomorphic on $\mathbf{D}$, then $f \in$ BMOA if and only if

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)\, dx\, dy < \infty, \tag{2.2}$$

and $f \in$ VMOA if and only if

$$\lim_{|a|\to 1^-} \int_{\mathbf{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)\, dx\, dy = 0, \tag{2.3}$$

where $z = x + iy$. See [**1**] and [**6**]. The Dirichlet space $\mathfrak{D}$ is the collection of functions f holomorphic on $\mathbf{D}$ for which

$$\|f\|_{\mathfrak{D}}^2 \equiv |f(0)|^2 + \int_{\mathbf{D}} |f'(z)|^2\, dx\, dy < \infty.$$

From the definitions of these spaces, it is clear that the following inclusion relations hold among these spaces: $\mathfrak{D} \subset$ VMOA $\subset$ BMOA.

2.3 Boundedness and compactness. Let X and Y be two Banach spaces with respective norms $\| \ \|_X$ and $\| \ \|_Y$. As usual, a linear operator T from X to Y is said to be bounded if a positive constant B exists such that

$$\|Tx\|_Y \le B\|x\|_X \quad \text{for all } x \in X.$$

This bounded operator T is said to be compact if T maps the closed unit ball $\{x : \|x\|_X \le 1\}$ about the origin in X into a relatively compact set in Y. Equivalently, $T : X \to Y$ is compact if and only if for every sequence $\{x_n\}$ in X which is bounded by one, $\{Tx_n\}$ has a convergent subsequence in Y.

3. $C_\varphi : K \to H^p$ with $p \ge 1$. In this section we characterize the boundedness and compactness of $C_\varphi : K \to H^p$, $p \ge 1$. The results in this section may be compared with those in [**3**]. The characterizations are given in terms of the behavior of the Cauchy kernel composed by φ. It seems rather easy to formulate the criterion for boundedness, but not at all obvious even to expect the one for compactness. To get an idea what the one might be, we need to mention the following result for C_φ on K. The equivalence of (ii) and (iii) in the following theorem was recently proved by Cima and Matheson in [**4**]. The equivalence of (i) and (ii), or (iii), is interesting itself and it is also contained there. We add to the list a new condition (iv) which will serve as a prototype of our idea.

Theorem 3.1. *For a holomorphic self-map φ of $\mathbf{D}$, the following conditions are equivalent*:

(i) $C_\varphi(K) \subset K_a$.

(ii) C_φ *is compact on* K.

(iii) φ *satisfies Sarason's condition for the compactness of* C_φ *on* H^2, *that is,*

$$\int_T \frac{1-|\varphi^*(\zeta)|^2}{|\eta-\varphi^*(\zeta)|^2}\,d\sigma(\zeta) = \frac{1-|\varphi(0)|^2}{|\eta-\varphi(0)|^2} \tag{3.1}$$

for every $\eta \in T$.

(iv) $|\varphi^*(\zeta)| < 1$ *almost everywhere* $\zeta \in T$ *and the integral*

$$\int_T \frac{1-|\varphi^*(\zeta)|^2}{|\eta-\varphi^*(\zeta)|^2}\,d\sigma(\zeta) \tag{3.2}$$

is a continuous function of $\eta \in T$.

Proof. We only have to prove the equivalence of (iii) and (iv). First we assume (iii) holds. Applying Fubini's theorem to the integral of (3.1) against the measure $d\sigma(\eta)$,

$$\int_T \int_T \frac{1-|\varphi^*(\zeta)|^2}{|\eta-\varphi^*(\zeta)|^2}\,d\sigma(\eta)\,d\sigma(\zeta) = \int_T \frac{1-|\varphi(0)|^2}{|\eta-\varphi(0)|^2}\,d\sigma(\eta) = 1.$$

The inner integral on the lefthand side is zero if $|\varphi^*(\zeta)| = 1$ and is one if $|\varphi^*(\zeta)| < 1$. Therefore, we can conclude that $|\varphi^*(\zeta)| < 1$ almost everywhere $\zeta \in T$. It is obvious that the condition (3.1) implies the continuity of the integral (3.2).

To prove the reverse implication, we note that there is a Poisson integral representation of the positive harmonic function: for every $\eta \in T$,

$$\frac{1-|\varphi(z)|^2}{|\eta-\varphi(z)|^2} = \int_T \frac{1-|z|^2}{|\xi-z|^2}\left(\frac{1-|\varphi^*(\xi)|^2}{|\eta-\varphi^*(\xi)|^2}\,d\sigma(\xi) + d\mu_\eta(\xi)\right)$$

where μ_η is a positive singular measure on T. By taking $z=0$, we have

$$\mu_\eta(T) = \frac{1-|\varphi(0)|^2}{|\eta-\varphi(0)|^2} - \int_T \frac{1-|\varphi^*(\xi)|^2}{|\eta-\varphi^*(\xi)|^2}\,d\sigma(\xi). \tag{3.3}$$

By the continuity of the integral (3.2), $\mu_\eta(T)$ is a continuous function of $\eta \in T$. Integrating both sides of (3.3) with respect to $d\sigma(\eta)$ and applying Fubini's theorem,

$$\begin{aligned}\int_T \mu_\eta(T)\,d\sigma(\eta) &= \int_T \frac{1-|\varphi(0)|^2}{|\eta-\varphi(0)|^2}\,d\sigma(\eta)\\ &\quad - \int_T\int_T \frac{1-|\varphi^*(\xi)|^2}{|\eta-\varphi^*(\xi)|^2}\,d\sigma(\eta)\,d\sigma(\xi)\\ &= 1-1=0,\end{aligned}$$

as above. Therefore, we have $\mu_\eta(T)=0$ for every $\eta \in T$. This gives (3.1), and the proof is complete. □

Now we have the following characterization of the boundedness of $C_\varphi : K \to H^p$, $p \geq 1$. As indicated in the beginning of this section, the proof is quite standard. But we include it for the sake of completeness. Throughout this section we always assume $1 \leq p < \infty$.

Theorem 3.2. *For a holomorphic self-map φ of* $\mathbf{D}$, *the following conditions are equivalent*:

(i) $C_\varphi : K \to H^p$ *is bounded.*

(ii) $|\varphi^*(\zeta)| < 1$ *almost everywhere $\zeta \in T$ and a positive constant B exists such that*

$$\sup_{\eta\in T}\left(\int_T \left|\frac{1}{1-\bar\eta\varphi^*(\zeta)}\right|^p d\sigma(\zeta)\right)^{1/p} \leq B < \infty. \tag{3.4}$$

(iii) *The family* $\{1/(1-\bar{\eta}\varphi) : \eta \in T\}$ *is a norm-bounded subset of* H^p, *that is, a positive constant* B *exists such that*

$$\sup_{\eta\in T}\left\|\frac{1}{1-\bar{\eta}\varphi}\right\|_{H^p} \le B < \infty. \tag{3.5}$$

Proof. (i) $\Rightarrow$ (ii). Since $C_\varphi(K) \subset H^p \subset K_a$, we have $|\varphi^*| < 1$ almost everywhere by Theorem 3.1. Choose a family $f_\eta(z) = 1/(1-\bar{\eta}z)$, $\eta \in T$, in K. Then $\|f_\eta\|_K = 1$. See, for example, [**2**, p. 468]. By the boundedness of $C_\varphi : K \to H^p$, a positive constant B exists such that

$$\left(\int_T |f_\eta \circ \varphi^*(\zeta)|^p \, d\sigma(\zeta)\right)^{1/p} = \|f_\eta \circ \varphi\|_{H^p} \le B\|f_\eta\|_K = B$$

for any $\eta \in T$. This is (3.4).

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). If $f \in K$, a $\mu \in M$ exists with $\|\mu\| = \|f\|_K$ such that

$$f(z) = \int_T \frac{1}{1-\bar{\eta}z} \, d\mu(\eta).$$

Composing with φ and applying Jensen's inequality, we get

$$|f \circ \varphi(r\zeta)|^p \le \|\mu\|^{p-1} \int_T \left|\frac{1}{1-\bar{\eta}\varphi(r\zeta)}\right|^p d|\mu|(\eta). \tag{3.6}$$

Integrating both sides of (3.6) with respect to $d\sigma(\zeta)$ and then applying Fubini's theorem, we have by (3.5)

$$\begin{aligned}\int_T |f \circ \varphi(r\zeta)|^p \, d\sigma(\zeta) &\le \|\mu\|^{p-1} \int_T \int_T \left|\frac{1}{1-\bar{\eta}\varphi(r\zeta)}\right|^p d\sigma(\zeta) \, d|\mu|(\eta) \\ &\le B^p\|\mu\|^p = B^p\|f\|_K^p.\end{aligned}$$

That is, $\|f \circ \varphi\|_{H^p} \le B\|f\|_K$. □

Remark 1. If $p = 1$, the condition (3.5) is easily seen to be equivalent to the condition

$$\frac{\eta+\varphi}{\eta-\varphi} \in H^1 \quad \text{for every } \eta \in T. \tag{3.7}$$

Since $\operatorname{Re}[(\eta+\varphi)/(\eta-\varphi)] \geq 0$, (3.7) is again equivalent to

$$\frac{1-|\varphi^*|^2}{|\eta-\varphi^*|^2} \in L\log L \quad \text{for every } \eta \in T,$$

by the famous $L\log L$-theorem of Zygmund [**7**, pp. 135–136]. Consequently, we have $C_\varphi : K \to H^1$ is bounded if and only if $(1-|\varphi^*|^2)/|\eta-\varphi^*|^2 \in L\log L$ for every $\eta \in T$.

Remark 2. Since $H^1 \subset K_a$ if $C_\varphi : K \to H^1$ is bounded then C_φ maps K into K_a, so C_φ is compact on H^2 by Theorem 3.1. We mention that the converse is not true. Indeed, the self-map

$$\varphi(z) = ir\frac{\sqrt{i(1+iz)/(1-iz)}-i}{\sqrt{i(1+iz)/(1-iz)}+i}, \quad 0<r<1,$$

can be shown to have $\int_T (1/|1-\varphi^*|)\,d\sigma = \infty$ so that $C_\varphi : K \to H^1$ is not bounded by Theorem 3.2. This map φ can be found in [**5**, pp. 147–148], where they showed that φ induces a compact composition operator C_φ on H^2 but the operator is not Hilbert-Schmidt. We can easily adapt the argument used by them to establish $\int_T (1/|1-\varphi^*|)\,d\sigma = \infty$.

Next we turn to the compactness of $C_\varphi : K \to H^p$ which is one of our main concerns in this paper. The continuity condition (ii) of the integral in the following theorem is, in the spirit, analogous to the condition (iv) for the compactness of C_φ on K in Theorem 3.1.

Theorem 3.3. *The following conditions are equivalent*:

(i) $C_\varphi : K \to H^p$ *is compact.*

(ii) $|\varphi^*(\zeta)| < 1$ *almost everywhere* $\zeta \in T$, *and the integral*

$$\int_T \left|\frac{1}{1-\bar{\eta}\varphi^*(\zeta)}\right|^p d\sigma(\zeta) \tag{3.8}$$

is a continuous function of $\eta \in T$.

(iii) $|\varphi^*(\zeta)| < 1$ *almost everywhere* $\zeta \in T$ *and the family of measures* $\{\nu_\eta : \eta \in T\}$ *defined by*

$$\nu_\eta(E) = \int_E \left|\frac{1}{1-\bar{\eta}\varphi^*(\zeta)}\right|^p d\sigma(\zeta) \tag{3.9}$$

is equi-absolutely continuous, with respect to η*. That is, given* $\varepsilon > 0$*, a* $\delta > 0$ *exists such that* $\nu_\eta(E) < \varepsilon$ *for all* $\eta \in T$ *whenever* $\sigma(E) < \delta$*.*

Proof. Before we start, let us first observe that, because of (2.1) the unit ball of K becomes a normal family of holomorphic functions. A standard normal family argument then shows that $C_\varphi : K \to H^p$ is compact if and only if whenever $\{f_n\}$ is a sequence in K with $\|f_n\|_K \le 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$, then $\|f_n \circ \varphi\|_{H^p} \to 0$. Now we proceed with the proof of Theorem 3.3.

(i) $\Rightarrow$ (ii). Since $C_\varphi(K) \subset H^p \subset K_a$, we have $|\varphi^*| < 1$ almost everywhere by Theorem 3.1.

Let $\eta_n \in T$ with $\eta_n \to \eta$, $n \to \infty$, and let $f_{\eta_n}(z) = 1/(1-\overline{\eta_n} z)$. Then $\|f_{\eta_n}\|_K = 1$ and $f_{\eta_n} \to f_\eta$ uniformly on compact subsets of $\mathbf{D}$ with the obvious notation for f_η. By the compactness of $C_\varphi : K \to H^p$, $\|f_{\eta_n} \circ \varphi - f_\eta \circ \varphi\|_{H^p} \to 0$, $n \to \infty$. Since $C_\varphi : K \to H^p$ is bounded, there is a constant $B > 0$ such that

$$\|f_\eta \circ \varphi\|_{H^p} \le B\|f_\eta\|_K = B, \quad \text{for all } \eta \in T.$$

Applying Exercise 24(b) on page 74 of [**11**], we have

$$\begin{aligned}\int_T \Big| |f_{\eta_n} \circ \varphi^*|^p - |f_\eta \circ \varphi^*|^p \Big| \, d\sigma & \\ &\le 2pB^{p-1}\bigg(\int_T |f_{\eta_n} \circ \varphi^* - f_\eta \circ \varphi^*|^p \, d\sigma\bigg)^{1/p} \\ &= 2pB^{p-1}\|f_{\eta_n} \circ \varphi - f_\eta \circ \varphi\|_{H^p} \longrightarrow 0, \quad n \to \infty.\end{aligned}$$

In particular, we have

$$\int_T |f_{\eta_n} \circ \varphi^*|^p \, d\sigma \to \int_T |f_\eta \circ \varphi^*|^p \, d\sigma, \quad n \to \infty.$$

That is,

$$\int_T \left|\frac{1}{1-\overline{\eta_n}\varphi^*}\right|^p d\sigma \longrightarrow \int_T \left|\frac{1}{1-\bar{\eta}\varphi^*}\right|^p d\sigma, \quad n \to \infty,$$

which shows the continuity of the integral (3.8).

(ii) $\Rightarrow$ (iii). If not, a sequence $\{\eta_k\}$ in T with $\eta_k \to \eta$ and a sequence of Borel sets $E_k \subset T$ exist such that $\sigma(E_k) \to 0$, $k \to \infty$, but $\nu_{\eta_k}(E_k) \geq c > 0$ for all $k = 1, 2, 3, \ldots$. Note that

$$\nu_{\eta_k}(E_k) \leq \int_{E_k} \left| \left| \frac{1}{1 - \overline{\eta_k}\varphi^*} \right|^p - \left| \frac{1}{1 - \bar{\eta}\varphi^*} \right|^p \right| d\sigma + \nu_\eta(E_k) \equiv \text{(I)} + \text{(II)}.$$

Obviously, (II) $= \nu_\eta(E_k) \to 0$, $k \to \infty$. Also, by another application of Exercise 24(b) in [**11**, p. 74] and by applying Exercise 17(b) of [**11**, p. 73], we have

$$\begin{aligned}
\text{(I)} &\leq \int_T \left| \left| \frac{1}{1 - \overline{\eta_k}\varphi^*} \right|^p - \left| \frac{1}{1 - \bar{\eta}\varphi^*} \right|^p \right| d\sigma \\
&\leq 2pR^{p-1} \left(\int_T \left| \frac{1}{1 - \overline{\eta_k}\varphi^*} - \frac{1}{1 - \bar{\eta}\varphi^*} \right|^p d\sigma \right)^{1/p} \longrightarrow 0, \quad k \to \infty,
\end{aligned}$$

where

$$R = \sup_{\eta \in T} \left(\int_T \left| \frac{1}{1 - \bar{\eta}\varphi^*} \right|^p d\sigma \right)^{1/p}.$$

Therefore, $\nu_{\eta_k}(E_k) \to 0$, $k \to \infty$. This contradiction shows (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). Suppose $\|f_n\|_K \leq 1$ for $n = 1, 2, \ldots$, and $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. We have to show $\|f_n \circ \varphi\|_{H^p} \to 0$, $n \to \infty$. For each n, we can find a $\mu_n \in M$ with $\|\mu_n\| = \|f_n\|_K$ such that $f_n(z) = \int_T 1/(1 - \bar{\eta}z)\, d\mu_n(\eta)$. Therefore,

$$f_n \circ \varphi^*(\zeta) = \int_T \frac{1}{1 - \bar{\eta}\varphi^*(\zeta)}\, d\mu_n(\eta) \quad \text{for a.e. } \zeta \in T.$$

Given $\varepsilon > 0$, we now choose $\delta > 0$ such that $\nu_\eta(E) < \varepsilon$ for all $\eta \in T$ whenever $\sigma(E) < \delta$. Since $|\varphi^*| < 1$ almost everywhere on T, we can choose a positive number $\rho < 1$ and a compact set $F \subset T$ such that $|\varphi^*(\zeta)| < \rho$ for all $\zeta \in F$ and $\sigma(T \backslash F) < \delta$. On $T \backslash F$, Jensen's inequality and Fubini's theorem yield

$$\begin{aligned}
(3.10) \quad & \int_{T \backslash F} |f_n \circ \varphi^*(\zeta)|^p\, d\sigma(\zeta) \\
&\leq \|\mu_n\|^{p-1} \int_T \int_{T \backslash F} \left| \frac{1}{1 - \bar{\eta}\varphi^*(\zeta)} \right|^p d\sigma(\zeta)\, d|\mu_n|(\eta) \\
&\leq \|f_n\|_K^{p-1} \varepsilon \int_T d|\mu_n|(\eta) \\
&= \varepsilon \|f_n\|_K^p \leq \varepsilon.
\end{aligned}$$

On F, $f_n \circ \varphi^* \to 0$ uniformly as $n \to \infty$. Hence,

$$\int_F |f_n \circ \varphi^*(\zeta)|^p \, d\sigma(\zeta) \longrightarrow 0, \quad n \to \infty. \tag{3.11}$$

Therefore, $\|f_n \circ \varphi\|_{H^p} \to 0$, $n \to \infty$, by (3.10) and (3.11). $\square$

We give a sufficient condition for $C_\varphi : K \to H^p$ to be compact.

Theorem 3.4. *If $\int_T 1/(1 - |\varphi^*|)^p \, d\sigma < \infty$, then $C_\varphi : K \to H^p$ is compact.*

Proof. It suffices to show that φ satisfies the condition (ii) of Theorem 3.3. It is obvious that the hypothesis implies $|\varphi^*| < 1$ almost everywhere. Now let $\eta_n \in T$ with $\eta_n \to \eta$, $n \to \infty$. Then, as $n \to \infty$,

$$\left| \frac{1}{1 - \overline{\eta_n}\varphi^*(\zeta)} \right|^p \longrightarrow \left| \frac{1}{1 - \bar{\eta}\varphi^*(\zeta)} \right|^p \quad \text{for a.e. } \zeta \in T.$$

Since $|1/(1 - \overline{\eta_n}\varphi^*(\zeta))|^p \leq 1/(1 - |\varphi^*|)^p \in L^1(T)$, we have by the dominated convergence theorem

$$\int_T \left| \frac{1}{1 - \overline{\eta_n}\varphi^*(\zeta)} \right|^p d\sigma(\zeta) \longrightarrow \int_T \left| \frac{1}{1 - \bar{\eta}\varphi^*(\zeta)} \right|^p d\sigma(\zeta), \quad n \to \infty,$$

which shows the continuity of the integral $\int_T |1/(1 - \bar{\eta}\varphi^*(\zeta))|^p \, d\sigma(\zeta)$. Therefore, C_φ is compact by Theorem 3.3. $\square$

It seems probably that the converse of Theorem 3.4 above fails to be true, but we have not been able to find such an example. However, the following self-map φ of Lotto [**8**, pp. 93–95] can be shown to have $\int_T 1/(1-|\varphi^*|)^p \, d\sigma = \infty$ but it induces a bounded composition operator $C_\varphi : K \to H^p$. We could not show that it is compact.

Example 3.5. Let $\varphi(z) = 1/\{1 - i(i(1 - z)/(1 + z))^{1/2p}\}$ where $1 \leq p < \infty$. Then $\int_T 1/(1 - |\varphi^*(e^{i\theta})|)^p \, d\theta = \infty$, but $C_\varphi : K \to H^p$ is bounded.

Proof. We give the proof only for the case $p = 1$. A tedious but similar argument works for the case $p > 1$. Let us first show $\int_{-\pi}^{\pi} d\theta/(1 - |\varphi^*(e^{i\theta})|) = \infty$. Let $h(z) = \sqrt{i(1-z)/(1+z)}$. If $0 < \theta < \pi$, $i(1 - e^{i\theta})/(1 + e^{i\theta}) = \tan(\theta/2)$ is positive. We put $t = h(e^{i\theta}) = \sqrt{\tan(\theta/2)}$. Then

$$\varphi^*(e^{i\theta}) = \frac{1}{1 - it},$$

and so

$$\frac{1}{1 - |\varphi^*(e^{i\theta})|^2} = \frac{1}{t^2} + 1 = \cot(\theta/2) + 1.$$

Therefore,

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 - |\varphi^*(e^{i\theta})|} \geq \int_0^{\pi} \frac{d\theta}{1 - |\varphi^*(e^{i\theta})|^2} \geq \int_0^{\pi} \cot(\theta/2)\, d\theta = \infty.$$

Next we will show that $C_\varphi : K \to H^1$ is bounded. We have to show that φ satisfies the condition (ii) of Theorem 3.2. Since φ fixes the point 1 and sends every other point of T into $\mathbf{D}$, it suffices to show that, for a fixed small $\varepsilon > 0$,

$$\sup_{-\sqrt{\varepsilon} \leq \alpha \leq \sqrt{\varepsilon}} \int_{-2\varepsilon}^{2\varepsilon} \frac{d\theta}{|e^{i\alpha} - \varphi^*(e^{i\theta})|} < \infty.$$

We split the integral into two parts, the first over $[0, 2\varepsilon]$ and the second over $[-2\varepsilon, 0]$. For $0 < \theta \leq 2\varepsilon$, we can calculate as above

$$\frac{1}{|e^{i\alpha} - \varphi^*(e^{i\theta})|} = \frac{\sqrt{1 + \tan(\theta/2)}}{\sqrt{(\sqrt{\tan(\theta/2)} - \sin\alpha)^2 + (1 - \cos\alpha)^2}},$$

so that

(3.12)
$$\sup_{-\sqrt{\varepsilon} \leq \alpha \leq \sqrt{\varepsilon}} \int_0^{2\varepsilon} \frac{d\theta}{|e^{i\alpha} - \varphi^*(e^{i\theta})|} \leq 2\sqrt{1 + \tan\varepsilon} \sup_{0 < \alpha \leq \sqrt{\varepsilon}} \int_0^{\varepsilon} \frac{d\theta}{\sqrt{(\sqrt{\tan(\theta)} - \sin\alpha)^2 + (1 - \cos\alpha)^2}}.$$

Using the change of variable $\sqrt{\tan\theta} - \sin\alpha = u(1-\cos\alpha)$, we see that the last integral of (3.12) becomes

$$\int_{-(\sin\alpha/(1-\cos\alpha))}^{(\sqrt{\tan\varepsilon}-\sin\alpha)/(1-\cos\alpha)} \frac{2(\sin\alpha + (1-\cos\alpha)u)}{\sqrt{1+u^2}[1+\{(\sin\alpha+(1-\cos\alpha)u\}^4]}\,du$$

which is obviously less than

$$\begin{aligned}
&2\sin\alpha\int_{-(\sin\alpha/(1-\cos\alpha))}^{((\sqrt{\tan\varepsilon}-\sin\alpha)/(1-\cos\alpha))} \frac{du}{\sqrt{1+u^2}}\\
&\quad + 2(1-\cos\alpha)\int_{-(\sin\alpha/(1-\cos\alpha))}^{((\sqrt{\tan\varepsilon}-\sin\alpha)/(1-\cos\alpha))} \frac{|u|}{\sqrt{1+u^2}}\,du\\
&= 2\sin\alpha\left(\int_{-(\sin\alpha/(1-\cos\alpha))}^{0} + \int_0^{((\sqrt{\tan\varepsilon}-\sin\alpha)/(1-\cos\alpha))}\right)\frac{du}{\sqrt{1+u^2}}\\
&\quad + 2(1-\cos\alpha)\int_{-(\sin\alpha/(1-\cos\alpha))}^{((\sqrt{\tan\varepsilon}-\sin\alpha)/(1-\cos\alpha))} \frac{|u|}{\sqrt{1+u^2}}\,du\\
&\equiv (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).
\end{aligned}$$

For $0 < \alpha \le \sqrt{\varepsilon}$, it is easy to see that

$$\begin{aligned}
(\mathrm{III}) &\le 4\Big[\sqrt{(1-\cos\alpha)^2 + (\sqrt{\tan\varepsilon}-\sin\alpha)^2} + \sqrt{(1-\cos\alpha)^2+\sin^2\alpha}\Big]\\
&\le 4\Big[\sqrt{1+\tan\varepsilon} + \sqrt{2}\Big]
\end{aligned}$$

and (I) is dominated by 4.

We also see that, since $1+u^2 \ge 2u$,

$$\begin{aligned}
(\mathrm{II}) &\le 2\sin\alpha\int_0^{(\sqrt{\tan\varepsilon}/(1-\cos\alpha))}\frac{du}{\sqrt{2u}}\\
&= 2\sqrt{2}\sqrt[4]{\tan\varepsilon}\frac{\sin\alpha}{\sqrt{1-\cos\alpha}}\\
&= 4\sqrt[4]{\tan\varepsilon}\cos(\alpha/2) \le 4\sqrt[4]{\tan\varepsilon}.
\end{aligned}$$

From these estimates, we have

$$\sup_{-\sqrt{\varepsilon}\le\alpha\le\sqrt{\varepsilon}}\int_0^{2\varepsilon}\frac{d\theta}{|e^{i\alpha}-\varphi^*(e^{i\theta})|} < \infty.$$

It remains to prove that

$$\sup_{-\sqrt{\varepsilon}\le\alpha\le\sqrt{\varepsilon}}\int_{-2\varepsilon}^{0}\frac{d\theta}{|e^{i\alpha}-\varphi^*(e^{i\theta})|}<\infty.$$

For this, we note that for $-2\varepsilon \le \theta < 0$, $h(e^{i\theta}) = it$ where $t = \sqrt{\tan(|\theta|/2)} > 0$. Hence for such θ, we have $\varphi^*(e^{i\theta}) = 1/(1+t)$ so that

$$\frac{1}{|e^{i\alpha}-\varphi^*(e^{i\theta})|}=\frac{1+t}{\sqrt{1-2(1+t)\cos\alpha+(1+t)^2}}\le 1+\frac{1}{t},$$

and thus

$$\sup_{-\sqrt{\varepsilon}\le\alpha\le\sqrt{\varepsilon}}\int_{-2\varepsilon}^{0}\frac{d\theta}{|e^{i\alpha}-\varphi^*(e^{i\theta})|}\le\int_{-2\varepsilon}^{0}\left(1+\frac{1}{\sqrt{\tan(|\theta|/2)}}\right)d\theta<\infty$$

and the proof is complete. □

4. $C_\varphi : K \to$ BMOA, VMOA or $\mathfrak{D}$. In this section we consider the boundedness and compactness of $C_\varphi : K \to$ BMOA, VMOA or the Dirichlet space $\mathfrak{D}$. They are also connected with the behavior of the Cauchy kernel composed with φ. For the boundedness of C_φ we have the following theorem which is analogous to Theorem 3.2.

Theorem 4.1. *For φ a holomorphic self-map of $\mathbf{D}$, we have*

(a) $C_\varphi : K \to$ BMOA *is bounded* $\Leftrightarrow$ *The family* $\{1/(1-\bar\eta\varphi) : \eta \in T\}$ *is a norm-bounded subset of* BMOA*, that is, $B > 0$ exists such that*

$$\sup_{\eta\in T}\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}\frac{|\varphi'(z)|^2}{|1-\bar\eta\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy\le B<\infty. \tag{4.1}$$

(b) $C_\varphi : K \to$ VMOA *is bounded* $\Leftrightarrow$

$$\lim_{|a|\to 1^-}\int_{\mathbf{D}}\frac{|\varphi'(z)|^2}{|1-\bar\eta\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy=0 \tag{4.2}$$

for every $\eta \in T$.

(c) $C_\varphi : K \to \mathfrak{D}$ *is bounded* $\Leftrightarrow$ *The family* $\{1/(1-\bar\eta\varphi) : \eta \in T\}$ *is a norm-bounded subset of* $\mathfrak{D}$*, that is,* $B > 0$ *exists such that*

$$\sup_{\eta\in T}\int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar\eta\varphi(z)|^4}\,dx\,dy \le B < \infty. \tag{4.3}$$

Proof. We shall only prove (b), since the proof of (a) or (c) is similar to that of (b).

($\Rightarrow$). As noted in the proof of Theorem 3.2, we know $\|1/(1-\bar\eta z)\|_K = 1$ for each $\eta \in T$. Thus, by the boundedness of $C_\varphi : K \to \mathrm{VMOA}$, we have, in particular,

$$\frac{1}{1-\bar\eta\varphi} \in \mathrm{VMOA} \quad \text{for every } \eta \in T.$$

Due to (2.3), this says

$$\lim_{|a|\to 1^-}\int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar\eta\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy = 0 \quad \text{for every } \eta \in T.$$

($\Leftarrow$). If $f \in K$, there is a $\mu \in M$ such that

$$f(z) = \int_T \frac{1}{1-\bar\eta z}\,d\mu(\eta).$$

Composing with φ, taking derivatives and applying Jensen's inequality, we have

$$|(f\circ\varphi)'(z)|^2 \le \|\mu\|^2 \int_T \frac{|\varphi'(z)|^2}{|1-\bar\eta\varphi(z)|^4}\frac{d|\mu|(\eta)}{|\mu\|}. \tag{4.4}$$

Integrating (4.4) with respect to $(1-|\varphi_a(z)|^2)\,dx\,dy$ and applying Fubini's theorem yield

$$\begin{aligned}\int_{\mathbf{D}} &|(f\circ\varphi)'(z)|^2(1-|\varphi_a(z)|^2)\,dx\,dy \\ &\le \|\mu\|\int_T\left[\int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar\eta\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy\right]d|\mu|(\eta).\end{aligned} \tag{4.5}$$

By (4.2), the inner integral in the second term of (4.5) tends to zero as $|a| \to 1^-$ for every $\eta \in T$. Also we see that the inner integral of (4.5) is at most

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy,$$

which is dominated by

$$B\left\|\frac{1}{1-\bar{\eta}z}\right\|_K = B$$

because the condition (4.2) implies the boundedness of $C_\varphi : K \to$ BMOA. Here B is a positive constant, independent of $\eta \in T$. Thus by the bounded convergence theorem, the second term of (4.5) tends to zero as $|a| \to 1^-$, so that

$$\lim_{|a|\to 1^-} \int_{\mathbf{D}} |(f\circ\varphi)'(z)|^2(1-|\varphi_a(z)|^2)\,dx\,dy = 0.$$

We conclude that if $f \in K$, then $C_\varphi(f) \in$ VMOA. The boundedness of $C_\varphi : K \to$ VMOA follows from the closed graph theorem. □

For compactness, we have the following theorem which is analogous to Theorem 3.3.

Theorem 4.2. *For φ a holomorphic self-map of $\mathbf{D}$, we have*

(a) $C_\varphi : K \to$ BMOA *is compact $\Leftrightarrow$ The quantity*

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy \tag{4.6}$$

is a continuous function of $\eta \in T$.

$\Leftrightarrow$ *The family of measures $\{\nu_\eta : \eta \in T\}$ defined by*

$$\nu_\eta(E) = \sup_{a\in\mathbf{D}} \int_{E} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy$$

is equi-absolutely continuous.

(b) $C_\varphi : K \to \mathrm{VMOA}$ *is compact* $\Leftrightarrow$

$$\lim_{|a|\to 1^-} \sup_{\eta\in T} \int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}(1-|\varphi_a(z)|^2)\, dx\, dy = 0. \tag{4.7}$$

(c) $C_\varphi : K \to \mathfrak{D}$ *is compact* $\Leftrightarrow$ *The integral*

$$\int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}\, dx\, dy \tag{4.8}$$

is a continuous function of $\eta \in T$.
$\Leftrightarrow$ *The family of measures* $\{\nu_\eta : \eta \in T\}$ *defined by*

$$\nu_\eta(E) = \int_E \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}\, dx\, dy$$

is equi-absolutely continuous.

The equivalence (a) or (c) can be proved by a similar method as in the proof of Theorem 3.3. The details of the proof are left to the reader. For the proof of (b), we require the following result which gives a characterization of compact composition operators whose range is a subset of VMOA. We quote this as a lemma, which can be found in [**16**, Theorem 3.11].

Lemma. *Let* φ *be a holomorphic self-map of* $\mathbf{D}$ *and* X *a Möbius invariant Banach space, that is, if* $f \in X$ *then* $f \circ \varphi_a \in X$ *for every* $a \in \mathbf{D}$. *Then* $C_\varphi : X \to \mathrm{VMOA}$ *is compact if and only if*

$$\lim_{|a|\to 1^-} \sup_{\substack{\|f\|_X \le 1 \\ f\in X}} \int_{\mathbf{D}} |(f\circ\varphi)'(z)|^2 (1-|\varphi_a(z)|^2)\, dx\, dy = 0.$$

Proof of (b). We first recall that K is a Möbius invariant Banach space. See, for example, [**2**]. Suppose that $C_\varphi : K \to \mathrm{VMOA}$ is compact. Take $f_\eta(z) = 1/(1-\bar{\eta}z)$, $\eta \in T$. Then $f_\eta \in K$ and $\|f_\eta\|_K = 1$. Now (4.7) follows from the lemma since

$$|(f_\eta \circ \varphi)'(z)| = \frac{|\varphi'(z)|}{|1-\bar{\eta}\varphi(z)|^2}.$$

Conversely, we are assuming (4.7) and will show that $C_\varphi : K \to$ VMOA is compact. From the lemma again, it suffices to show

$$\lim_{|a|\to 1^-} \sup_{\substack{\|f\|_K \le 1\\ f\in K}} \int_{\mathbf{D}} |(f\circ\varphi)'(z)|^2(1-|\varphi_a(z)|^2)\,dx\,dy = 0.$$

Let $f \in K$ with $\|f\|_K \le 1$. Then there is a $\mu \in M$ with $\|\mu\| = \|f\|_K$ such that $f(z) = \int_T 1/(1-\bar{\eta}z)\,d\mu(\eta)$. Then, as in the proof of Theorem 4.1, we have

$$\begin{aligned}
\int_{\mathbf{D}} &|(f\circ\varphi)'(z)|^2(1-|\varphi_a(z)|^2)\,dx\,dy\\
&\le \|\mu\| \int_T \left[\int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy\right] d|\mu|(\eta)\\
&\le \left(\sup_{\eta\in T}\int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}(1-|\varphi_a(z)|^2\,dx\,dy\right)\cdot\|\mu\|^2\\
&\le \sup_{\eta\in T}\int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{\substack{\|f\|_K \le 1\\ f\in K}} &\int_{\mathbf{D}} |(f\circ\varphi)'(z)|^2(1-|\varphi_a(z)|^2)\,dx\,dy\\
&\le \sup_{\eta\in T}\int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{|1-\bar{\eta}\varphi(z)|^4}(1-|\varphi_a(z)|^2)\,dx\,dy.
\end{aligned}\tag{4.9}$$

By (4.7), the second term of (4.9) tends to zero as $|a| \to 1^-$ and therefore the first term goes to zero as $|a| \to 1^-$. This establishes our claim. $\square$

Finally, we have the following sufficient conditions for compactness. (a) or (c) follows directly from the line of argument in Theorem 3.4 and (b) is obvious.

Theorem 4.3. *Suppose that φ is a holomorphic self-map of* $\mathbf{D}$.

(a) *If* $\sup_{a\in\mathbf{D}} \int_{\mathbf{D}}\{|\varphi'(z)|^2/1-|\varphi(z)|^4\}(1-|\varphi_a(z)|^2)\,dx\,dy<\infty$, *then* $C_\varphi : K \to \mathrm{BMOA}$ *is compact.*

(b) *If* $\lim_{|a|\to 1^-} \int_{\mathbf{D}}\{|\varphi'(z)|^2/(1-|\varphi(z)|)^4\}(1-|\varphi_a(z)|^2)\,dx\,dy = 0$, *then* $C_\varphi : K \to \mathrm{VMOA}$ *is compact.*

(c) *If* $\int_{\mathbf{D}} |\varphi'(z)|^2/(1-|\varphi(z)|)^4\,dx\,dy < \infty$, *then* $C_\varphi : K \to \mathfrak{D}$ *is compact.*

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