ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 2001

# HOMOGENEOUS ALGEBRAIC DISTRIBUTIONS

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ABSTRACT. Vertical distributions on a vector bundle admitting a system of homogeneous algebraic vector fields of the same degree are characterized.

**1. Introduction.** The goal of this paper is to provide a characterization of homogeneous algebraic distributions on vector bundles. The starting point is the classical result according to which a vector field $X$ on $\mathbf{R}^m$ is homogeneous algebraic of degree $d$ if and only if $[\chi, X] = (d-1)X$, $\chi$ being the Liouville vector field. If one wants to involve exclusively the module structure spanned by a vector field $X$ in characterizing algebraic vector fields, then one is led to study the equation $[\chi, X] = fX$. In this case a first result (cf. 4.7 below) states that the function $f$ should be constant along the zero section of the vector bundle $p : E \to M$ on which $X$ is defined, and this constant should be an integer $\geq -1$. Our main result is a generalization of the above statement to distributions of arbitrary rank: it is stated that a vertical distribution locally spanned by $X_1, \dots, X_r$ is homogeneous algebraic of degree $d$ if and only if an $r \times r$ matrix $A = (a_{ij})$, $a_{ij} \in C^\infty(E)$, exists which is equal to $d-1$ times the identity matrix along the zero section of $E$, and such that $[\chi, X_j] = \sum_{i=1}^r a_{ij} X_i$, for $j = 1, \dots, r$ (cf. 4.6).

Algebraic distributions play a role in several fields of real and complex geometry such as singularities of vector fields, the moduli problem for differential forms, calculation of differential invariants of a Lie group action, etc. (e.g., see [**6**], [**10**], [**14**], [**16**], [**22**]). Thus, it seems interesting to obtain a characterization of these differential systems. Linear representations of families of Lie groups on vector bundles give rise to such distributions in a natural way. Families of Lie groups and specially Lie group fiber bundles naturally appear in the field theory

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1991 AMS *Mathematics Subject Classification.* Primary 13N10, Secondary 12H05, 58A30.
*Key words and phrases.* Adjoint bundle, algebraic morphism of vector bundles, algebraic vector field, involutive distribution, gauge algebra, linear representation, Lie group bundle, Liouville's vector field.
Research supported by CICYT (Spain) under grant no. PB95-0124.
Received by the editors on November 21, 1997.

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and gauge theories as well as in differential geometry. For example, the gauge group of a principal bundle $P \to M$ can be obtained as the global sections of the adjoint bundle of $P$ (cf. [**1**]; see also [**2**], [**3**], [**4**], [**9**], [**17**], [**18**], [**19**]). This bundle is endowed with a natural structure of Lie group fiber bundle. Similarly, the gauge algebra of $P$ is obtained by taking global sections in a Lie algebra bundle attached to $P$. For Lie algebra bundles this is indeed an extension to nontrivial bundles of the well-known process of gauging a Lie algebra $\mathfrak{g}$ with functions of a manifold $M$, which corresponds to consider the infinite Lie algebra $C^\infty(M) \otimes \mathfrak{g}$ with the bracket $[f \otimes A, g \otimes B] = fg \otimes [A, B]$, $f, g \in C^\infty(M)$, $A, B \in \mathfrak{g}$, as $C^\infty(M) \otimes \mathfrak{g}$ may be viewed as the global sections of the trivial bundle $\mathrm{pr}_1 : M \times \mathfrak{g} \to M$. These techniques also apply to Supersymmetry ([**15**]). In Section 5 we introduce the general definition of these structures showing the usual settings in which they are commonly found, and we characterize (see 5.8) the distributions induced by a linear representation of a family of Lie groups.

We define an algebraic distribution as a sheaf of submodules of the sheaf of germs of $p$-vertical vector fields on a vector bundle $p : E \to M$, which is locally spanned by a finite number of algebraic vector fields. Note that, according to this definition, distributions may be singular and they usually are in the algebraic case but in any case the rank of the distribution is kept to be locally constant on a dense open subset. This seems to be a suitable general setting in order to introduce algebraic distributions as a vector bundle is endowed with a canonical structure of algebraic scheme over $M$, independently of the class of functions ($C^\infty$, $C^\omega$ or complex-analytic) that we consider on the base manifold, although we formulate our results in the $C^\infty$ category for the sake of simplicity.

**2. Algebraic morphisms of vector bundles.** In this section we shall briefly introduce some notations and preliminary results which we shall use throughout this paper.

**2.1 Vector bundle charts.** Let $p : E \to M$ be a vector bundle over a $C^\infty$ manifold $M$. Assume $U \subseteq M$ is an open subset such that $E|_U$ is trivial, and let $s_1, \dots, s_m$ be a basis of sections of $E$ over $U$. Each vector $e \in p^{-1}(x)$, $x \in U$, can be uniquely written as $e = \lambda_1 s_1(x) + \cdots + \lambda_m s_m(x)$ for some scalars $\lambda_i \in \mathbf{R}$, $1 \leq i \leq m$.

Hence, we can define $m$ functions $y_i : p^{-1}(U) \to \mathbf{R}$, $y_i(e) = \lambda_i$. In addition, assume that $U$ is a coordinate domain with coordinates $(x_1, \dots, x_n)$ with $n = \dim M$. Then it is not difficult to prove that $(x_j; y_i)$, $1 \le i \le m$, $1 \le j \le n$, is a coordinate system on $p^{-1}(U)$ for the manifold $E$. Such systems of coordinates will be called *vector bundle charts*.

**Definition 2.2.** Let $p : E \to M$, $p' : E' \to M'$ be two vector bundles over the $C^\infty$ manifolds $M, M'$, respectively. A differentiable mapping $\Phi : E \to E'$ is said to be an *algebraic morphism* if $\Phi$ satisfies the following two conditions:

1. $\Phi$ is a bundle map over $M$, i.e., a differentiable map $\phi : M \to M'$ exists making the following diagram commutative

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{\phi} & M' \end{array}$$

2. For every $x \in M$, there exist open coordinate neighborhoods $U, U'$ of $x$ and $\phi(x)$, respectively over which $E|_U$ and $E'|_{U'}$ are trivial, and vector bundle charts $(U; x_j, y_i), (U'; x'_{j'}, y'_{i'})$, $1 \le j \le n$, $1 \le i \le m$, $1 \le j' \le n'$, $1 \le i' \le m'$ for $E$ and $E'$, respectively such that $\phi(U) \subseteq U'$ and

$$y'_{i'}(\Phi(v)) = P_{i'}(\phi(p(v)))(y_1(v), \dots, y_m(v)),$$
$$1 \le i' \le m', \quad \forall\, v \in p^{-1}(U),$$

where $P_{i'}$ are polynomials in the fiber variables with coefficients in the ring of differentiable functions of $U'$, i.e., $P_{i'} \in C^\infty(U')[t_1, \dots, t_m]$, $t_i$ being $m$ indeterminates.

The morphism $\Phi$ is said to be *homogeneous of degree* $(r_1, \dots, r_{m'})$ if $P_{i'}$ is a homogeneous polynomial of degree $r_{i'}$, $1 \le i' \le m'$. If all polynomials $P_{i'}$ have the same degree $r$, we shall say that $\Phi$ is homogeneous of degree $r$. If $M = M'$ and $\phi$ is the identity map, then $\Phi$ is said to be a *vertical morphism*. Note that the definition makes sense as it does not depend on the vector bundle charts chosen. In fact, if $(U; \bar{x}_j, \bar{y}_i)$ is another vector bundle chart of $E$ on $U$, then there exists an invertible $m\times m$ matrix $(a_{ij}(x))_{i,j=1}^m$, $a_{ij}\in C^\infty(U)$ such that

$$\bar{y}_i=\sum_{j=1}^m a_{ij}(x)y_j,\quad 1\le i\le m,$$

and hence if $P(\bar{y}_1,\dots,\bar{y}_m)$ is a polynomial of degree $r$ in $C^\infty(U')[t_1,\dots,t_m]$, then

$$P(\phi(x))\bigg(\sum_{j=1}^m a_{1j}(x)y_j,\dots,\sum_{j=1}^m a_{mj}(x)y_j\bigg)$$

is also a polynomial in $C^\infty(U')[t_1,\dots,t_m]$ of the same degree, and similarly in changing vector bundle charts in the bundle $E'$.

**Notations 2.3.** Given a vector bundle $p:E\to M$, we denote by $S^r(E)$ the vector bundle of its $r$th symmetric power and $S^\bullet(E)=\oplus_{r=0}^\infty S^r(E)$ stands for the direct sum of such bundles endowed with the standard $\mathbf{Z}$-graded algebra structure.

**Proposition 2.4.** *The vertical homogeneous algebraic morphisms of degree $r$ from $p:E\to M$ into $p':E'\to M$ can be identified with the global sections of the bundle $S^r(E^*)\otimes E'\to M$.*

*Proof.* Every section $\sigma:M\to S^r(E^*)\otimes E'$ gives rise to a vertical algebraic morphism of degree $r$, $F(\sigma):E\to E'$ by setting $F(\sigma)(e)=\sigma(p(e))(e,\dots,e)$. Conversely, if $\Phi:E\to E'$ is a vertical algebraic morphism of degree $r$, taking into account that $\mathbf{R}[y_1,\dots,y_m]$, can be identified to $S^\bullet(\mathbf{R}^m)^*$, for every $x\in M$ there exists a unique symmetric mapping $F'(\Phi)(x)\in S^r(E_x^*)\otimes E'_x$ such that

$$\Phi(e)=F'(\Phi)(x)(e,\overset{(r}{\dots},e),\quad \forall\, e\in E_x,$$

where $E_x=p^{-1}(x)$. Moreover, it follows from Section 2.1 that the mapping $x\mapsto F'(\Phi)(x)$ is differentiable, thus proving that $F(F'(\Phi))=\Phi$. Similarly, we have $F'(F(\sigma))=\sigma$. □

**3. Algebraic vector fields.** Let $p:E\to M$ be a vector bundle. We set $S^\bullet(E^*)=\Gamma(M,S^\bullet(E^*))$. We have a natural inclusion of

$S^\bullet(E^*)$ as a subalgebra $S^\bullet(E^*) \subset C^\infty(E)$; that is, the *sections of the symmetric algebra of* $E^*$ *can be interpreted as differentiable functions on the manifold* $E$, simply by setting

$$e \in E \longmapsto \sigma(e, \overset{(r}{\ldots}, e) \in \mathbf{R}, \quad \forall\, e \in E, \forall\, \sigma \in \mathcal{S}^r(E^*).$$

Accordingly, a $C^\infty$ vector bundle $p : E \to M$ is endowed with a natural compatible algebraic structure since the elements of $S^\bullet(E^*)$ are understood to be polynomial functions on $E$. In fact, $\operatorname{Spec} S^\bullet(E^*)$ is a $C^\infty(M)$-scheme in the sense of [**8**, I. Definition 2.6.1], see also [**8**, I.9.4]. Moreover, from the Stone-Weierstrass-Nachbin theorem (see [**21**]) it follows that $S^\bullet(E^*)$ is a dense subalgebra of $C^\infty(E)$ with respect to the natural $C^\infty$ topology in such a way that the ring of the algebraic functions on $E$ approximates that of differentiable functions.

**Definition 3.1.** A vector field $X \in \mathfrak{X}(E)$ is said to be *algebraic* if it is $p$-vertical and leaves invariant the subalgebra $S^\bullet(E^*) \subset C^\infty(E)$, i.e., $X(S^\bullet(E^*)) \subseteq \mathcal{S}^\bullet(E^*)$. An algebraic vector field $X$ is +said to be *homogeneous of degree* $r$ if $X(\mathcal{S}^k(E^*)) \subseteq \mathcal{S}^{k+r-1}(E^*)$, for all $k \in \mathbf{N}$.

*Remark* 3.1.1. It is not difficult to prove that algebraic vector fields on $E$ can be identified with $\operatorname{Der}_{C^\infty(M)}(S^\bullet(E^*))$.

**Proposition 3.2.** 1) *A vector field* $X \in \mathfrak{X}(E)$ *is algebraic if and only if on every vector bundle chart* $(U; x_j, y_i)$ *we have*

$$X|_U = \sum_{i=1}^m P_i \frac{\partial}{\partial y_i}, \quad P_i \in C^\infty(U)[y_1, \ldots, y_m].$$

2) $X$ *is homogeneous of degree* $r$ *if and only if* $P_1, \ldots, P_m$ *are homogeneous polynomials of the same degree* $r$.

3) *There is a canonical isomorphism of* $C^\infty(M)$*-modules*

$$\lambda_E : \mathcal{S}^\bullet(E^*) \otimes \mathcal{S}^1(E) \longrightarrow Der_{C^\infty(M)}(\mathcal{S}^\bullet(E^*)).$$

4) *For every* $x \in M$, *there is a unique algebraic structure* $[\,,\,]$ *on* $S^\bullet(E_x^*) \otimes E_x$ *such that*

$$(*) \qquad [t_k \otimes e, t'_{k'} \otimes e'] = (i_{e'} t_k) \cdot t'_{k'} \otimes e - (i_e t'_{k'}) \cdot t_k \otimes e',$$

*for every* $t_k \in S^k(E_x^*)$, $t'_{k'} \in S^{k'}(E_x^*)$, $e, e' \in E_x$ *with respect to which* $\lambda_E$ *is an anti-isomorphism, where* $\mathrm{i_e}$ *stands for the interior product by* $e$ *on the symmetric algebra* (*cf.* [**5**, III.11.6]).

*Proof.* Parts 1) and 2) follow from the definition taking into account that the sheaf of sections of $S^\bullet(E^*)$ is quasi-flasque in the sense of [**24**, V.6. Appendice]. As for 3), the definition of $\lambda_E$ is as follows. A section of $S^r(E^*) \otimes E$ can be identified to a symmetric multilinear mapping $f : E \times_M \dots \times_M E \to E$ (i.e., $f_x : E_x \times \cdots \times E_x \to E_x$ is a symmetric multilinear map which smoothly depends on $x \in M$), and this mapping gives rise to a family of one-parameter vertical diffeomorphisms $\varphi_t : E \to E$, $t \in \mathbf{R}$, defined by $\varphi_t(e) = e + t \cdot f(e, \dots, e)$. As $\varphi_0$ is the identity of $E$, $\varphi_t$ induces a vertical vector field by simply taking derivatives at $t = 0$. Note, however, that $\varphi_s \circ \varphi_t \neq \varphi_{s+t}$, in general; in other words, $\varphi_t$ is not the flow associated to $\lambda_E(f)$. The rest of the properties are easily checked. $\square$

*Remark* 3.2.1. The first homogeneous component of $S^\bullet(E^*) \otimes E$ was introduced by Lecomte in studying the Lie algebra of infinitesimal automorphisms of a vector bundle (see [**11**] and references therein).

*Remark* 3.2.2. The bracket defined by the formula $(*)$ on $\mathcal{S}^\bullet(E^*) \otimes \mathcal{S}^1(E)$ is similar to the Richardson-Nijenhuis bracket (cf. [**23**, II.c. Remarque 1], [**12**]), although our bracket does not take into account the graded structure of the algebra $\mathcal{S}^\bullet(E^*) \otimes \mathcal{S}^1(E)$, as we wish $\lambda_E$ to be an anti-isomorphism.

## 4. Algebraic distributions.

**Notations 4.1.** We denote by $\mathfrak{X}_M$ the sheaf of germs of vector fields on $M$. Let $p : E \to M$ be a fibered manifold, i.e., $p$ is a surjective submersion. We denote by $\mathfrak{X}_E^v$ the subsheaf of $p$-vertical vector fields in $\mathfrak{X}_E$.

**Definition 4.2.** A *vertical distribution* over a fibered manifold $p : E \to M$ is a subsheaf of $C^\infty_E$-modules $\mathcal{D} \subset \mathfrak{X}^v_E$. A vertical distribution is said to be *locally finitely generated* if for every $e \in E$ there exist an open neighborhood $U$ of $x = p(e)$ in $M$ and vector fields $X_1, \dots, X_r \in \mathfrak{X}^v_E(p^{-1}(U))$ such that $\mathcal{D}(p^{-1}(U))$ is spanned as a $C^\infty(p^{-1}(U))$-module by $X_1, \dots, X_r$. If $p : E \to M$ is a vector bundle and the generators are homogeneous algebraic vector fields (respectively, homogeneous algebraic vector fields of the same degree), we say that $\mathcal{D}$ is an *algebraic distribution* (respectively, a *homogeneous algebraic distribution*).

*Remark* 4.2.1. Note that, according to the above definition, a distribution may be singular, i.e., the rank of the vector space $\mathcal{D}_e = \{X_e \mid X \in \mathcal{D}(V)\}$, $e \in V$, need not be locally constant. In fact, this is usually the case for algebraic distributions. The above notion of a distribution corresponds to that of a generalized distribution given in [**13**, Appendix 3].

*Remark* 4.2.2. We shall repeatedly use the fact that, if $X_1, \dots, X_r$ span $\mathcal{D}(V)$, then they also span $\mathcal{D}(W)$ for each open subset $W \subset V$. This is an easy consequence of the fact that $\mathfrak{X}^v_E$ is a quasi-flasque sheaf (cf. [**24**]).

**Proposition 4.3.** *For each algebraic distribution $\mathcal{D}$ on $p : E \to M$, there exists a dense open subset $O \subset E$ such that $\mathcal{D}|_O$ is homogeneous.*

*Proof.* For every $x \in M$ there exist an open neighborhood $U$ and algebraic vector fields $X_1, \dots, X_r \in \mathfrak{X}^v_E(p^{-1}U)$ spanning $\mathcal{D}(p^{-1}U)$. Shrinking $U$, we can also assume that $E|_U$ is trivial and $U$ is a coordinate domain of $M$. Then, on a vector bundle chart $(U; x_j, y_i)$, $1 \le j \le n$, $1 \le i \le m$, we have

$$X_k = \sum_{i=1}^{m} P_{ik} \frac{\partial}{\partial y_i}, \quad 1 \le k \le r,$$

where $P_{ik} \in C^\infty(U)[y_1, \dots, y_m]$ are homogeneous polynomials of degree $d_k = \deg P_{ik}$, $1 \le i \le m$. Reordering these vector fields, we can assume $d_1 \leq \cdots \leq d_r$. Then the vector fields

$$\tilde{X}_k = y_1^{d_r - d_k} \cdot X_k, \quad 1 \leq k \leq r,$$

have the same degree and span $\mathcal{D}$ over $p^{-1}(U) - \{y_1 = 0\}$, which is a dense open subset of $p^{-1}(U)$. □

**Definition 4.4.** Let $p : E \to M$ be a vector bundle. The *Liouville vector field* on $E$ is the infinitesimal generator $\chi \in \mathfrak{X}^v(E)$ of the one-parameter group of homotheties, $\varphi_t : E \to E$, $\varphi_t(e) = \exp(t) \cdot e$, for all $t \in \mathbf{R}$, for all $e \in E$.

*Remark* 4.4.1. It is easy to see that, on each vector bundle chart $(U; x_j, y_i)$, $1 \leq j \leq n$, $1 \leq i \leq m$, we have

$$\chi|_U = \sum_{i=1}^{m} y_i \frac{\partial}{\partial y_i}.$$

**Lemma 4.5.** *A vector field $X \in \mathfrak{X}^v(E)$ is a homogeneous algebraic vector field of degree $d$ if and only if*

$$[\chi, X] = (d-1)X.$$

*Proof.* If $X$ is a homogeneous algebraic vector field of degree $d$, locally we have (cf. Section 3.2) $X = \sum_{i=1}^{m} P_i(\partial/\partial y_i)$ with $P_i \in C^\infty(U)[y_1, \dots, y_m]$, $\deg P_i = d$, $1 \leq i \leq m$, and the result follows taking into account that

$$\left[\chi, \frac{\partial}{\partial y_i}\right] = -\frac{\partial}{\partial y_i} \quad \text{and} \quad \chi(y_1^{\alpha_1} \cdots y_m^{\alpha_m}) = (\alpha_1 + \cdots + \alpha_m) y_1^{\alpha_1} \cdots y_m^{\alpha_m}.$$

Conversely, assume $[\chi, X] = (d-1)X$. If $X = \sum_{i=1}^{m} f_i(\partial/\partial y_i)$ is the local expression of $X$ on a vector bundle chart, then the above condition yields $\chi(f_i) = df_i$, $1 \leq i \leq m$, and we can conclude by simply applying the parametric form of the Euler theorem characterizing homogeneous polynomials on $\mathbf{R}^m$. In fact, as $p^{-1}(U) \simeq U \times \mathbf{R}^m$, for every $y \in \mathbf{R}^m$

we can define a function $\varphi_t : U \times \mathbf{R} \to \mathbf{R}$, $\varphi_i(x,t) = f_i(x,ty)$, which satisfies

$$t\frac{\partial \varphi_i}{\partial t}(x,t) = \chi(f_i)(x,ty) = d \cdot \varphi_i(x,t).$$

Hence $f_i(x,ty) = f_i(x,y) \cdot t^d$, i.e., $f_i$ is a homogeneous function of degree $d$ on the variables $y = (y_i, \dots, y_m)$, globally defined on $U \times \mathbf{R}^m$, and accordingly it is an algebraic form of degree $d$ in $C^\infty(U)[y_1, \dots, y_m]$. □

**Theorem 4.6.** *Let $\mathcal{D}$ be a vertical distribution over a vector bundle $p : E \to M$, and let $X_1, \dots, X_r$ be $p$-vertical vector fields on $p^{-1}(U)$ spanning $\mathcal{D}|_{p^{-1}(U)}$, and assume that $E|_U$ is trivial. Then a matrix $C \in GL(r; C^\infty(p^{-1}U))$, $C = (c_{ij})_{i,j=1}^r$, exists so that the vector fields*

$$Y_j = \sum_{i=1}^r c^{ij} X_i, \quad 1 \le j \le r,$$

*are homogeneous algebraic of degree $d$, with $(c^{ij}) = C^{-1}$, if and only if an $r \times r$ matrix $A = (a_{ij})$, $a_{ij} \in C^\infty(p^{-1}U)$, exists such that*

1. *$A(0_x) = (d-1)I$, for all $x \in U$, where $I$ stands for the $r \times r$ identity matrix,*
2. *$[\chi, X_j] = \sum_{i=1}^r a_{ij} X_i$, $1 \le j \le r$.*

*Proof.* Assume that $C$ exists. Then we have

$$X_j = \sum_{i=1}^r c_{ij} Y_i, \quad 1 \le j \le r.$$

If we compute the bracket of $X_j$ with the vector field $\chi$, from the previous lemma we obtain

$$\begin{aligned}
[\chi, X_j] &= \sum_{i=1}^r \chi(c_{ij}) Y_i + \sum_{i=1}^r c_{ij}[\chi, Y_i] \\
&= \sum_{i=1}^r \chi(c_{ij}) Y_i + \sum_{i=1}^r (d-1) c_{ij} Y_i \\
&= \sum_{i,k=1}^r \chi(c_{ij}) c^{ki} X_k + (d-1) X_j,
\end{aligned}$$

and setting $A = C^{-1} \cdot \chi(C) + (d-1)I$, where $\chi(C)$ stands for the matrix $(\chi(c_j))_{i,j=1}^r$, taking into account that $\chi(f)(0_x) = 0$, for all $f \in C^\infty(E)$ and for all $x \in M$, we can conclude.

Conversely, assume that $A$ exists. Set $B = A - (d-1)I$, and let us fix a point $e \in p^{-1}U$. The mapping $(t, e) \mapsto B(te)/t$ is $C^\infty$ on $\mathbf{R} \times p^{-1}U$ as $B \in C^\infty(p^{-1}U)$ and $B(0_x) = 0$, for all $x \in M$. Hence, the following $r \times r$ matrix valued ordinary differential system

$$\varphi'(t) = \varphi(t) \cdot \frac{B(te)}{t},$$

is not singular at $t = 0$. The theorems of existence, uniqueness and differentiable dependence on parameters for linear systems (e.g., see [**7**, II.2.2, II.3.6, II.3.7]) thus ensure the existence of a unique differentiable mapping $\Phi : \mathbf{R} \times p^{-1}U \to \mathrm{GL}\,(r; \mathbf{R})$ such that

$$\text{(i)} \qquad \frac{\partial \Phi}{\partial t}(t, e) = \Phi(t, e) \cdot \frac{B(te)}{t},$$

$$\text{(ii)} \qquad \Phi(0, e) = I, \quad \forall\, e \in p^{-1}U.$$

The matrix $\Phi(t, e)$ is invertible for all $t, e$ since we have (cf. [**7**, II. Proposition 2.3.1])

$$\det(\Phi(t, e)) = \exp \int_0^t \operatorname{trace}\left(\frac{B(\tau e)}{\tau}\right) d\tau,$$

and the righthand side of the above equation never vanishes.

Set $C(e) = \Phi(1, e)$ for all $e \in p^{-1}U$. Let us consider the map $\Phi_u(t, e) = \Phi(tu, u^{-1}e)$, $u \in \mathbf{R}^*$. It is easy to check that the matrix $\Phi_u$ verifies (i)–(ii). Thus $\Phi_u = \Phi$ by virtue of the uniqueness of the solution. If we take the derivative with respect to $u$ in the vector bundle chart $(U; x_j, y_i)$, the chain rule yields

$$\frac{\partial \Phi}{\partial t}(tu, u^{-1}e)t - \sum_{i=1}^m \frac{\partial \Phi}{\partial y_i}(tu, u^{-1}e) \cdot y_i \cdot u^{-2} = 0.$$

Letting $u = 1$, we obtain $t(\partial\Phi/\partial t) = \chi(\Phi)$, which implies $\chi(\Phi) = \Phi \cdot B$. In particular, $\chi(C) = C \cdot B$. Finally, we set $Y_j = \sum_{i=1}^r c^{ij} X_i$, $1 \leq j \leq r$.

Hence,

$$\begin{aligned}[\chi, Y_j] &= \sum_{i=1}^{r} \chi(c^{ij}) X_i + \sum_{i,k=1}^{r} c^{ij} a_{ki} X_k \\ &= \sum_{i,l=1}^{r} (c_{li}\chi(c^{ij}) + c_{lk} a_{ki} c^{ij}) Y_l,\end{aligned}$$

and computing into the expression of the matrix $D = C \cdot \chi(C^{-1}) + C \cdot A \cdot C^{-1}$, we have

$$\begin{aligned}D &= C \cdot (-C^{-1} \cdot \chi(C) \cdot C^{-1}) + C \cdot (B + (d-1)I) \cdot C^{-1} \\ &= -C \cdot B \cdot C^{-1} + C \cdot B \cdot C^{-1} + (d-1)I \\ &= (d-1)I.\end{aligned}$$

By applying Lemma 4.5, it follows that the vector fields $Y_j$, $1 \le j \le r$, are degree-$d$ homogeneous algebraic vector fields, thus finishing the proof. □

*Remark* 4.6.1. As an easy consequence of the above theorem, we can conclude that if $f$ is a first integral of an algebraic distribution $\mathcal{D}$, then $\chi(f)$ also is a first integral of $\mathcal{D}$, but unfortunately this method does not provide new algebraic first integral as we have $\chi(f) = d \cdot f$, if $f$ is a homogeneous polynomial of degree $d$.

*Remark* 4.6.2. For rank-1 distributions, the condition 1 in the above theorem can be weakened. More precisely,

**Proposition 4.7.** *Let $X$ be a $p$-vertical vector field on $p : E \to M$ such that* $\operatorname{support} X = E$. *Furthermore, assume that there exists a function $f \in C^\infty(E)$ such that $[\chi, M] = fX$. If $M$ is connected, then there exist an integer $d \ge 0$ and an invertible function $g \in C^\infty(E)$ such that*

1. $f(0_x) = d - 1$ *for all* $x \in M$,
2. $g^{-1}X$ *is a homogeneous algebraic vector field of degree* $d$.

*Proof.* Let $X = \sum_{i=1}^{m} h_i(\partial/\partial y_i)$ be the local expression of $X$ in the vector bundle chart $(U; x_j, y_i)$. Then we have

$$[\chi, X] = \sum_{i=1}^{m} \chi(h_i)\frac{\partial}{\partial y_i} - \sum_{i=1}^{m} h_i \frac{\partial}{\partial y_i},$$

which implies $\chi(h_i) = (f+1)h_i$, $1 \leq i \leq m$. Let $k = f+1$. Let us fix a vector $e \in p^{-1}U$, $x = p(e)$. If we consider the function $\varphi_i(t) = h_i(te)$, $t \in \mathbf{R}$, by using the chain rule we obtain

$$(**) \qquad \varphi_i'(t) = \frac{1}{t}\sum_{s=1}^{m} \frac{\partial h_i}{\partial y_s}(te) \cdot t y_s = k(te)\,\frac{\varphi_i(t)}{t}, \quad \forall\, t \in \mathbf{R}^+.$$

Then it is not difficult to prove that the function

$$\varphi_i(t) = \lambda_i \cdot t^{k(0_x)} \exp\left(\int_0^t \frac{k(\tau e) - k(0_x)}{\tau}\, d\tau\right),$$

with

$$\lambda_i = \frac{h_i(e)}{g(e)}, \quad g(e) = \exp\left(\int_0^1 \frac{k(\tau e) - k(0_x)}{\tau}\, d\tau\right),$$

is the unique solution of the differential equation $(**)$ with initial value $\varphi_i(1) = h_i(e)$. Suppose that $h_i(e) \neq 0$. As

$$(\dagger) \qquad t^{k(0_x)} = \varphi_i(t) \cdot \lambda_i^{-1} \exp\left(-\int_0^t \frac{k(\tau e) - k(0_x)}{\tau}\, d\tau\right), \quad \forall\, t > 0,$$

and $\varphi_i(t) = h_i(te)$, $k(te)$ are $C^\infty$ on $\mathbf{R}$, the limits of all derivatives $d^r(t^{k(0_x)})/dt^r$ must exist as $t \to 0^+$. Hence $d = k(0_x) \geq 0$. Let $r = [d] + 1$, and suppose that $d$ is not an integer. Taking derivatives $r$ times in $(\dagger)$ and letting $t \to 0^+$, it follows that the lefthand side goes to infinity, which is not possible. Hence, $d \in \mathbf{Z}$ and $f(0_x) = d - 1$ on a dense open subset of $U$, and by virtue of our hypotheses we thus obtain condition 1 in the statement. Consider the functions $P_i(e) = h_i(e)/g(e)$, $1 \leq i \leq m$. Computing $P_i(te)$, we have

$$\begin{aligned} P_i(te) = \frac{h_i(te)}{g(te)} &= \varphi_i(t) \exp\left(-\int_0^t \frac{k(\tau e) - k(0_x)}{\tau}\, d\tau\right) \\ &= \lambda_i \cdot t^d = t^d P_i(e), \end{aligned}$$

thus proving that $P_i$ is a homogeneous function in the variables $y_1, \dots, y_m$ of degree $d$, globally defined. Hence $P_i$, $1 \le i \le m$, are homogeneous polynomials in $C^\infty(E)[y_1, \dots, y_m]$ of degree $d$, thus finishing the proof. $\square$

## 5. Examples and applications.

**Example 5.1.** Let $f_1, \dots, f_r$ be real analytic functions on $\mathbf{R}^m$, and let $\mathcal{D} \subset \mathfrak{X}_{\mathbf{R}^m}$ be the involutive distribution of all vector fields $X$ in $\mathbf{R}^m$ such that $Xf_i = 0$, $1 \le i \le r$. The distribution $\mathcal{D}$ is finitely generated. In fact, let $\mathcal{O}_{\mathbf{R}^m}$ be the sheaf of germs of real analytic functions, and let $\mathfrak{X}^\omega_{\mathbf{R}^m}$ be the sheaf of germs of analytic vector fields. We have an exact sequence of sheaves of $\mathcal{O}_{\mathbf{R}^m}$-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathfrak{X}^\omega_{\mathbf{R}^m} \xrightarrow{f^\omega} (\mathcal{O}_{\mathbf{R}^m})^r,$$

where $\mathcal{K} = \ker f^\omega$ and $f^\omega(X) = (Xf_1, \dots, Xf_r)$. From the Noetherian properties of stalks of $\mathcal{O}_{\mathbf{R}^m}$ and $\mathfrak{X}^\omega_{\mathbf{R}^m}$ (e.g., see [**24**, II. Théorème 1.5] and remark that $\mathfrak{X}^\omega_{\mathbf{R}^m} \cong (\mathcal{O}_{\mathbf{R}^m})^m$) it follows that $\mathcal{K}$ is a finitely generated sheaf of $\mathcal{O}_{\mathbf{R}^m}$-modules. Moreover, we have a natural isomorphism of sheaves of $C^\infty_{\mathbf{R}^m}$-modules, $C^\infty_{\mathbf{R}^m} \otimes \mathfrak{X}^\omega_{\mathbf{R}^m} \cong \mathfrak{X}_{\mathbf{R}^m}$. Tensoring the above exact sequence over $C^\infty_{\mathbf{R}^m}$ we obtain an exact sequence of sheaves of $C^\infty_{\mathbf{R}^m}$-modules,

$$0 \longrightarrow C^\infty_{\mathbf{R}^m} \otimes_{\mathcal{O}_{\mathbf{R}^m}} \mathcal{K} \longrightarrow \mathfrak{X}_{\mathbf{R}^m} \xrightarrow{f^\infty} (C^\infty_{\mathbf{R}^m})^r,$$

where $f^\infty$ is given by $f^\infty(X) = (Xf_1, \dots, Xf_r)$ for every $X \in \mathfrak{X}(\mathbf{R}^m)$, as follows from Malgrange's division theorem taking into account that $\mathcal{O}_{\mathbf{R}^m} \hookrightarrow C^\infty_{\mathbf{R}^m}$ is a flat ring extension (see [**24**, VI. Corollaire 1.3]). Accordingly, from the very definition of $\mathcal{D}$, we have a canonical isomorphism $\mathcal{D} \cong C^\infty_{\mathbf{R}^m} \otimes_{\mathcal{O}_{\mathbf{R}^m}} \mathcal{K}$. Hence $\mathcal{D}$ is locally generated by a number of analytic vector fields and, recalling that $\mathcal{D}$ is a quasi-flasque sheaf by using [**24**, V. Proposition 6.4] and a partition of unity, we conclude that $\mathcal{D}$ is finitely generated.

Moreover, if $f_1, \dots, f_r$ are homogeneous polynomials of common degree $d$, then it can be proved that $\mathcal{D}$ is a homogeneous algebraic distribution of degree $(d-1)r$, where we further assume that at least one of the $r \times r$ determinants of the Jacobian matrix $(\partial f_i / \partial y_j)$, $1 \le i \le r$,

$1 \leq j \leq m$, does not vanish identically. We also remark that the above result can be generalized to the vertical bundle of an arbitrary analytic morphism between $C^\omega$ manifolds.

Next we confine ourselves to the case of homogeneous algebraic distributions of degree 1, which correspond to linear representations of Lie group families. Precisely,

**Definition 5.2.** A *family of Lie groups* on a manifold $M$ is a surjective submersion $\pi : \mathcal{G} \to M$ endowed with two differentiable mappings $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $\iota : \mathcal{G} \to \mathcal{G}$, such that for every $x \in M$ the mappings $\mu_x : \mathcal{G}_x \times \mathcal{G}_x \to \mathcal{G}_x$, $\iota_x : \mathcal{G}_x \to \mathcal{G}_x$ induced on the fiber, define on $\mathcal{G}_x$ a structure of Lie group.

**Definition 5.3.** An *operation of a family of Lie groups* $\pi : \mathcal{G} \to M$ on a fibered manifold $p : E \to M$ is a differentiable map $\lambda : \mathcal{G} \times_M E \to E$ such that

1. $\lambda(1_x, y) = y$, for all $x \in M$, for all $y \in \pi^{-1}(x)$,

2. $\lambda(g', \lambda(g'', y)) = \lambda(g', g'', y)$, for all $x \in M$, for all $g', g'' \in \pi^{-1}(x)$, for all $y \in p^{-1}(x)$.

**Example 5.4.** Let us consider an operation of a family of Lie groups $\pi : \mathcal{G} \to M$ on a fibered manifold $p : E \to M$. A vector bundle of Lie algebras $\bar{\pi} : \mathfrak{G} \to M$ is defined by setting $\bar{\pi}^{-1}(x) = \mathfrak{G}_x =$ Lie algebra of $\pi^{-1}(x) = \mathcal{G}_x$. Each section $\xi : U \to \mathfrak{G}$ of $\bar{\pi}$ induces a $p$-vertical vector field $\tilde{\xi} \in \mathfrak{X}^v_E(p^{-1}(U))$ whose flow is given by

$$\tau_t(e) = \lambda(\exp(t\xi(p(e))), e), \quad \forall\, e \in p^{-1}(U).$$

Then the given operation $\lambda$ induces a $p$-vertical distribution $\mathcal{D}$ on $E$ defined as follows. For every open subset $V \subseteq E$, $\mathcal{D}(V)$ is the $C^\infty(V)$-module spanned by the vector fields $\tilde{\xi}$, where $\xi \in \Gamma(p(V), \mathfrak{G})$. Note that $\mathcal{D}$ is involutive as we have $[\xi_1, \xi_2]^\sim = [\tilde{\xi}_1, \tilde{\xi}_2]$ for every $\xi_1, \xi_2 \in \Gamma(\mathfrak{G})$.

**Example 5.5.** In the previous example assume that $p : E \to M$ is a vector bundle and also that, for every $x \in M$, the operation induced on the fiber $\lambda_x : \mathcal{G}_x \times E_x \to E_x$ is a linear representation. Then for every

$\xi \in \Gamma(\mathfrak{G})$, we have $[\chi, \tilde{\xi}] = 0$, and hence the associated distribution is algebraic of degree 1. In fact, $\tilde{\xi}$ is linear as its flow is a one-parameter group of linear automorphisms of the vector bundle $E$.

**Lemma 5.6.** *With the above hypotheses and notations, let $\mathcal{D}$ be the $p$-vertical distribution on $E$ defined by $\lambda : \mathcal{G} \times_M E \to E$. Assume that on an open subset $U \subseteq M$ the following holds true*

$$\dim \mathfrak{G}_x = \max_{e \in E_x} (\operatorname{rk} \mathcal{D}_e), \quad \forall\, x \in U.$$

*If $\xi_1, \dots, \xi_r$ is a basis of sections of $\bar{\pi} : \mathfrak{G} \to M$ over $U$, then $\tilde{\xi}_1, \dots, \tilde{\xi}_r$ are linearly independent on a dense open subset in $p^{-1}(U)$.*

*Proof.* According to our assumptions, we have

$$r = \dim \mathfrak{G}_x = \max_{e \in E_x} (\operatorname{rk} \mathcal{D}_e), \quad \forall\, x \in U.$$

We set $C = \{e \in p^{-1}(U) \mid \operatorname{rk} \mathcal{D}_e < r\}$. It follows that a vector $e \in p^{-1}(U)$ belongs to $C$ if and only if $\tilde{\xi}_1(e) \wedge \cdots \wedge \tilde{\xi}_r(e) = 0$. Hence, $C$ is a closed subset in $p^{-1}(U)$ and, for every $x \in U$, the fiber $C_x$ is an algebraic submanifold in $E_x$. Consequently, we only need to prove that $C_x \neq E_x$, for all $x \in U$. In fact, if $e \in E_x$ is a vector such that $\dim \mathfrak{G}_x = \operatorname{rk} \mathcal{D}_e$ (this vector exists by virtue of the hypothesis), then $e \notin C_x$. $\square$

**Proposition 5.7.** *If $\lambda : \mathcal{G} \times_M E \to E$ is a linear representation of $\pi : \mathcal{G} \to M$ on a vector bundle $p : E \to M$ and*

$$\dim \mathfrak{G}_x = \max_{e \in E_x} (\operatorname{rk} \mathcal{D}_e) = r, \quad \forall\, x \in M,$$

*then a linear vector field $X \in \mathfrak{X}^v(E)$ belongs to the distribution $\mathcal{D}$ induced from $\lambda$ if and only if a vector field $\xi \in \Gamma(M, \mathfrak{G})$ exists such that $X = \tilde{\xi}$.*

*Proof.* If $X = \tilde{\xi}$, then $X$ is linear (see Example 5.5) and trivially $X \in \mathcal{D}$. Conversely, assume $X$ is a linear vector field of $\mathcal{D}$. Locally we have $X = \sum_{j=1}^{r} f_j \tilde{\xi}_j$, where $\xi_1, \dots, \xi_r$ is a basis of $\Gamma(U, \mathfrak{G})$. Hence, as

$X$ is linear, we have $0 = [\chi, X] = \sum_{j=1}^{r} \chi(f_j)\tilde{\xi}_j$ (cf. Lemma 4.5 and Example 5.5) and according to Lemma 5.5, the functions $\chi(f_j)$ vanish on an open dense subset and thus they identically vanish. Therefore, every $f_j$ is constant. □

**Proposition 5.8.** *Let $\mathcal{D}$ be a homogeneous algebraic involutive distribution of degree* 1 *over a vector bundle $p : E \to M$. Then a family of Lie groups $\pi : \mathcal{G} \to M$ and a linear representation $\lambda : \mathcal{G} \times_M E \to E$ exist such that $\mathcal{D}$ is the distribution induced by $\lambda$.*

*Proof.* Let $\pi : \mathrm{Aut}\,(E) \to M$ be the family of Lie groups with fibers $\pi^{-1}(x) = \mathrm{Aut}\,(E_x)$, and let $\bar{\pi} : \mathrm{End}\,(E) \to M$ be the associated vector bundle of Lie algebras $(\bar{\pi}^{-1}(x) = \mathrm{End}\,(E_x))$. From Proposition 3.2–3) we have a one-to-one correspondence between linear vector fields on $E$ and sections of $S^1(E^*) \otimes E = E^* \otimes E = \mathrm{End}\,(E)$. Consider the subset $\mathfrak{G}_x \subseteq \mathrm{End}\,(E_x)$ of all linear vector fields $X \in \mathcal{D}$ defined on $p^{-1}(U)$ where $U$ is an open neighborhood of $x$. As $\mathcal{D}$ is an involutive distribution, it is easy to check that $\mathfrak{G}_x$ is a Lie subalgebra of $\mathrm{End}\,(E_x)$. Let $\mathcal{G}_x$ be the connected Lie subgroup of $\mathrm{Aut}\,(E_x)$ corresponding to $\mathfrak{G}_x \subseteq \mathrm{End}\,(E_x)$. The space $\mathcal{G} = \cup\mathcal{G}_x \subseteq \mathrm{Aut}\,(E)$, endowed with the standard $C^\infty$ structure, is a Lie group family with a natural linear representation on $E$ which induces the distribution $\mathcal{D}$. □

Families of Lie groups appear in a natural way in the framework of differential geometry and field theory. Let us describe some examples of such a structure which are closely related to our present work.

**Example 5.9.** Each vector bundle $p : E \to M$ defines a Lie group fiber bundle, precisely the bundle of its fibered automorphisms $\pi : \mathrm{Aut}\,(E) \to M$ whose standard fiber is $\mathrm{Aut}\,(E_x) \cong GL(m; \mathbf{R})$, $m = \mathrm{rk}\, E$. Several Lie group subfamilies in $\mathrm{Aut}\,(E)$ are also interesting when different geometric structures on $E$ are considered. Also note that $\mathrm{Aut}\,(E)$ admits a natural linear representation on $E$.

**Example 5.10.** Given a manifold $M$, let $\pi_r : G^r(M) \to M$ be the bundle of Lie groups whose fiber over $x \in M$ is the group of $r$-jets at $x$ of differentiable mappings $f : M \to M$ such that $f(x) = x$ and

$f_* : T_x M \to T_x M$ is an isomorphism. Then, $G^r(M)$ is a Lie group fiber bundle whose standard fiber is $G^r(n)$, $n = \dim M$, the $r$th order linear group (see [**10**, 12.6]). Let $p : \mathcal{M} \to M$ be the bundle of metrics of a prescribed signature. An operation of the Lie group fiber bundle $G^{r+1}(M)$ on $J^r(\mathcal{M})$ is defined by setting

$$j_x^{r+1} f \cdot j_x^r g = j_{f(x)}^r (\bar{f} \circ g \circ f^{-1}),$$

for every $j_x^{r+1} f \in G^{r+1}(M)$ and every local section $g$ of $p : \mathcal{M} \to M$, where $\bar{f} : \mathcal{M} \to \mathcal{M}$ stands for the natural lifting of $f$ to the metric bundle, i.e., $\bar{f} \cdot g_x = (f^{-1})^* g_x$ for $g_x \in \mathcal{M}_x = p^{-1}(x)$. Note that $\mathcal{M}$ is a convex open subset in $S^2 T^*(M)$ and the above operation is the restriction of the natural linear representation of $\pi_{r+1} : G^{r+1}(M) \to M$ on $p_r : J^r(S^2 T^*(M)) \to M$. It can be proved (cf. [**20**]) that the determination of the $r$th order $\mathrm{Diff}(M)$-invariant Lagrangians on the metric bundle is reduced to calculate $C^\infty$ functions on the $r$-jet bundle which are invariant under the above operation of the $(r+1)$th order linear group fiber bundle.

**Example 5.11.** The basic groups in the field theory are the group of diffeomorphisms of a manifold and the group of vertical automorphisms of a principal bundle. The previous example shows how $\mathrm{Diff}(M)$ gives rise to natural operations of a family of Lie groups. Next we consider the gauge groups. Let $\pi : P \to M$ be a principal bundle with structure group $G_0$, and let us consider an operation of $G_0$ on another Lie group $F$ by acting on the left by automorphisms of $F$ i.e., $g \cdot (f_1 f_2) = (g \cdot f_1)(g \cdot f_2)$ for all $g \in G_0$ for all $f_1, f_2 \in F$. Then the associated bundle $\pi_F : \mathcal{G} = P \times^{G_0} F \to M$ is endowed with a natural structure of Lie group fiber bundle, uniquely determined by imposing $[u, f_1] \cdot [u, f_2] = [u, f_1 f_2]$ for every $u \in P$, $f_1, f_2 \in F$, where $[u, f]$ stands for the coset defined by the pair $(u, f) \in P \times F$ in the quotient manifold $\mathcal{G} = (P \times F)/G_0$.

**Example 5.12.** The above situation is obtained when $G_0$ acts onto itself by conjugation, i.e., $g \cdot f = g f g^{-1}$ for all $f, g \in G_0$. In this case the associated bundle is called the *adjoint bundle* of $P$ and is denoted by $\pi_{G_0} : \mathrm{Ad}\, P \to M$. Its sections can be identified with the gauge group, $\Gamma(M, \mathrm{Ad}\, P) = \mathrm{Gau}\, P$. Similarly, if we consider the adjoint representation of $G_0$ onto its Lie algebra $\mathfrak{g}_0$, then the associated fiber bundle $\bar{\pi} = \pi_{\mathfrak{g}_0} : \mathfrak{G} = \mathrm{ad}\, P = P \times^{G_0} \mathfrak{g}_0 \to M$ is endowed with a natural structure of Lie algebra fiber bundle given by $[[u, \xi_1], [u, \xi_2]] = [u, [\xi_1, \xi_2]]$ for $u \in P$, $\xi_1, \xi_2 \in \mathfrak{g}_0$. With such a structure, $\mathfrak{G} = \mathrm{ad}\, P$ can be identified with the Lie algebra bundle of $\mathcal{G} = \mathrm{Ad}\, P$ in the sense of Example 5.4. Also note that the global sections of $\bar{\pi}$ can be identified with the gauge algebra of $P$ (see [**4**], [**9**]). Let $p : \mathcal{C}(P) \to M$ be the bundle of connections on $P$ (here a connection is understood to be a splitting of the Atiyah sequence; cf. [**1**], [**17**]). According to Utiyama's theorem (see [**3**], [**4**]), the determination of the gauge invariant Lagrangians on $J^1(\mathcal{C}(P))$ can be reduced to calculate $C^\infty$ functions on the "curvature bundle," $\wedge^2 T^*(M) \otimes \mathrm{ad}\, P$ which are invariant under the natural representation of the adjoint bundle $\pi_{G_0} : \mathrm{Ad}\, P \to M$ on $\wedge^2 T^*(M) \otimes \mathrm{ad}\,(P)$.

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