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SQUARES OF RIESZ SPACES

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ABSTRACT. In this paper we provide three approaches to the notion of squares of Riesz spaces and show that they are equivalent.

The study of powers of Banach lattices was initiated by Lozanovsky in [**8**] and a similar construction was studied by Krivine in [**5**], of which an account can be found in [**6**] as well. In a recent paper [**9**], partially rooted in probability theory, Szulga introduces the notion of powers of uniformly complete Riesz spaces. Avoiding a technical description, which for all of the above authors involves functional calculus, their results are exemplified by the fact that (in Szulga's notation)

$$(L^1)^2 = L^2.$$

For reasons that will become clear, we are interested in Szulga's power 1/2 rather than his power 2 and, confusing as it may seem at first, Szulga's power 1/2 will be called power 2 by us and consequently our theory will be exemplified by (in our notation)

$$(L^1)^2 = L^{1/2}.$$

Our theory develops squares of any Archimedean vector lattices. These squares of vector lattices play a role in a surprising variety of theories in functional analysis. We were motivated to investigate them while studying certain Riesz algebras and orthosymmetric operators in [**1**] and [**2**]. Given a uniformly complete f-algebra E, we defined its square in [**2**] to be

$$E^2 = \{fg : f, g \in E\},$$

which may explain the notation that we prefer. It is known that Riesz spaces can be embedded in (semi-prime) f-algebras of the type

$$C^\infty(X).$$

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A definition of E^2 for any Riesz space E naturally arises from such an embedding.

In studying the tensor product of Riesz spaces (see [**5**]) yet another candidate for E^2 emerges as a certain quotient of

$$E\bar{\otimes}E.$$

As it turns out, all these possible squares are isomorphic, and that is the main result of this paper. Our organization is as follows. In Section 1 we give the definition (via a universal property) and show (via tensor products as introduced by Fremlin in [**4**]), the existence and uniqueness of squares. In Section 2 we give an overview of results of functional calculus which we need in this paper. Our approach in Section 2 goes back essentially to Lozanovsky (see [**7**]), although we really need (in Theorem 8) the somewhat more complicated functional calculus of continuous functions $f : \mathbf{R}^N \to \mathbf{R}$ of polynomial growth for which $\lim_{t\downarrow 0} t^{-1}f(tx)$ exists uniformly on bounded subsets of $\mathbf{R}^N$ (see [**3**]). Additionally, we provide new information about the continuity of functional calculus in Theorem 7 of Section 2. In Section 3 we study the squares of Riesz spaces via the theory of f-algebras, providing a second approach to understanding squares. In Section 4 we give our version of Szulga's construction for $E^{1/2}$ (once again, E^2 in our sense) for a uniformly complete Riesz space E and show that it also satisfies the universal property introduced in Section 1.

At the end of the paper we show how several interesting features, e.g., a locally solid vector space topology, of a uniformly complete Riesz space E, carry over to the square of E.

We wish to remark that Lozanovsky's results on the power transformation of a Banach lattice are closely allied to interpolation techniques in the style of Calderón, and we intend to return to interpolation in another paper, using the novel techniques and objects of the present paper.

All Riesz spaces in this paper are Archimedean.

1. The universal property. We remind the reader of the following definitions in [**2**] and [**4**], respectively.

Definition 1. Let E and F be Riesz spaces. A bilinear map

$T : E \times E \to F$ is called *orthosymmetric* if whenever $f \wedge g = 0$ for $f, g \in E$ we have $T(f, g) = 0$.

Definition 2. Let E and F be Riesz spaces. A bilinear map $T : E \times E \to F$ is called a *bimorphism* if $f \mapsto T(f, g)$ and $g \mapsto T(f, g)$ are Riesz homomorphisms.

The most important definition of this paper, our starting point, is the following.

Definition 3. Let E be a Riesz space. $(E^{\odot}, \odot)$ is called a *square of* E if $E^{\odot}$ is a Riesz space, and if

1. $\odot : E \times E \to E^{\odot}$ is an orthosymmetric bimorphism.

2. For every Riesz space F, whenever $T : E \times E \to F$ is an orthosymmetric bimorphism, a unique Riesz homomorphism $T^{\odot} : E^{\odot} \to F$ exists such that $T^{\odot} \circ \odot = T$.

The existence and uniqueness of a square for each Riesz space are straightforward, once one is familiar with Fremlin's fundamental paper [**4**] on tensor products. Indeed, the tensor product of E with itself is obtained from the above definition by substituting $E \bar{\otimes} E$ for $E^{\odot}$ and $\otimes$ for $\odot$, while omitting the word orthosymmetric at all places. Thus, it is not the existence of a square which is surprising, but rather the consequences that we study in the next sections.

Theorem 4. *Let E be a Riesz space. Then*

1. *E has a square $(E^{\odot}, \odot)$, and*

2. *$(E^{\odot}, \odot)$ is (essentially) unique.*

Proof. Step 1. First we introduce Fremlin's tensor product $(E \bar{\otimes} E, \otimes)$. Let I be the smallest uniformly closed ideal of $E \bar{\otimes} E$ that contains $\{f \otimes g : f, g \in E \text{ and } f \wedge g = 0\}$. It follows that $E^{\odot} := E \bar{\otimes} E / I$ is an (Archimedean) Riesz space. Moreover, denoting the natural map $E \bar{\otimes} E \to E^{\odot}$ by q, we get that $\odot := q \circ \otimes$ is an orthosymmetric bimorphism. Now assume that

$$T : E \times E \longrightarrow F$$

is an orthosymmetric bimorphism. By the universal property for $E\bar{\otimes}E$, a Riesz homomorphism

$$T^{\otimes} : E\bar{\otimes}E \longrightarrow F$$

exists such that $T^{\otimes} \circ \otimes = T$. If $q(f) \wedge q(g) = 0$, then $f \otimes g \in I$ and hence $T^{\otimes}(f \otimes g) = T(f, g) = 0$. This shows that $T^{\odot} : E^{\odot} \to F$ defined by $T^{\odot}(q(f)) := T^{\otimes}(f)$ (for all $f \in E\bar{\otimes}E$) is well defined and a Riesz homomorphism. That $T^{\odot}$ is unique follows from the uniqueness of $T^{\otimes}$.

Step 2. Now the uniqueness of $E^{\odot}$. Suppose that $(E^{\bigcirc}, \bigcirc)$ is also a square. Since $\bigcirc : E \times E \to E^{\bigcirc}$ is an orthosymmetric bimorphism, there exists a unique Riesz homomorphism $S : E^{\odot} \to E^{\bigcirc}$ for which $S \circ \odot = \bigcirc$. Also, there exits a unique Riesz homomorphism $T : E^{\bigcirc} \to E^{\odot}$ with $T \circ \bigcirc = \odot$. Then

$$T \circ S \circ \odot = T \circ \bigcirc = \odot$$

but also

$$I \circ \odot = \odot$$

and hence $T \circ S = I$ and, similarly, $S \circ T = I$.

2. Review of functional calculus. Let $N \in \mathbf{N}$. We denote by $\mathcal{H}(\mathbf{R}^N)$ the Riesz space of all continuous functions $f : \mathbf{R}^N \to \mathbf{R}$ for which

$$f(tx) = tf(x) \quad \text{for all } x \in \mathbf{R}^N \quad \text{and all } t \geq 0 \quad \text{(see [\textbf{3}])}.$$

Let E be a Riesz space, $f \in \mathcal{H}(\mathbf{R}^N)$ and $a_1, \dots, a_N \in E$. We say that

$$f(a_1, \dots, a_N) \quad \text{exists in } E$$

if there is an element b of E such that

$$\omega(b) = f(\omega(a_1), \dots, \omega(a_N))$$

for every $\mathbf{R}$-valued Riesz homomorphism ω on the Riesz subspace of E generated by $a_1, \dots, a_N, b$. For any given E and f and $a_1, \dots, a_N$, at most one b exists with this property. This b is also indicated by

$$f(a_1, \dots, a_N).$$

In this situation we have the following theorem.

Theorem 5 (Lozanovsky [**7**]). *Let E be a uniformly complete Riesz space and $a_1, \dots, a_N \in E$. Then $f(a_1, \dots, a_N)$ exists for every $f \in \mathcal{H}(\mathbf{R}^N)$. The map*

$$f \longrightarrow f(a_1, \dots, a_N), \quad f \in \mathcal{H}(\mathbf{R}^N)$$

is a Riesz homomorphism $\mathcal{H}(\mathbf{R}^N) \to E$.

Remark. In a way, $f(a_1, \dots, a_N)$ is independent of E. Indeed, if D is any Riesz subspace of E that is uniformly complete and contains $a_1, \dots, a_N$, then $f(a_1, \dots, a_N)$ relative to D means the same as $f(a_1, \dots, a_N)$ relative to E. In particular, every Riesz subspace of E that is uniformly complete and contains $a_1, \dots, a_N$ must also contain $f(a_1, \dots, a_N)$.

By $\mathcal{A}(\mathbf{R}^N)$ we denote the set of all continuous functions $f : \mathbf{R}^N \to \mathbf{R}$ that are of polynomial growth and for which $\lim_{t \downarrow 0} t^{-1} f(tx)$ exists uniformly on bounded subsets of $\mathbf{R}^N$. (The latter condition is equivalent to the existence of a $g \in \mathcal{H}(\mathbf{R}^N)$ such that $f(x) = g(x) + 0(\|x\|)$ $(x \to 0)$). $\mathcal{A}(\mathbf{R}^N)$ is an f-algebra.

Let E be a semi-prime f-algebra, $f \in \mathcal{A}(\mathbf{R}^N)$ and $a_1, \dots, a_N \in E$. We say that

$$f(a_1, \dots, a_N) \quad \text{exists in } E$$

if there is a $b \in E$ with

$$\omega(b) = f(\omega(a_1), \dots, \omega(a_N))$$

for every $\mathbf{R}$-valued multiplicative Riesz homomorphism ω on the f-subalgebra of E generated by $a_1, \dots, a_N, b$. Only one such b exists, which is then called

$$f(a_1, \dots, a_N).$$

This definition is in accordance with the one we gave for $\mathcal{H}(\mathbf{R}^N)$ if $f \in \mathcal{H}(\mathbf{R}^N)$.

In this situation we have the following theorem.

Theorem 6 (see [**3**]). *Let E be a uniformly complete semi-prime f-algebra and $a_1, \dots, a_N \in E$. Then $f(a_1, \dots, a_N)$ exists for every $f \in \mathcal{A}(\mathbf{R}^N)$. The map*

$$f \longrightarrow f(a_1, \dots, a_N), \quad f \in \mathcal{A}(\mathbf{R}^N)$$

is a multiplicative Riesz homomorphism $\mathcal{A}(\mathbf{R}^N) \to E$.

The following result is new and generalizes the continuity of the functional calculus in [**9**].

Theorem 7. *Let E be a uniformly complete Riesz space, and let $f \in \mathcal{H}(\mathbf{R}^N)$. The map*

$$(a_1, \dots, a_N) \longmapsto f(a_1, \dots, a_N), \quad a_1 \dots, a_n \in E$$

is continuous

(i) *relative to relative uniform convergence and*

(ii) *relative to any locally solid vector space topology on E.*

Proof. We will write

$$\|x\| = |x_1| \vee \cdots |x_N|, \quad x \in E^N$$

and

$$\|s\| = |s_1| \vee \cdots |s_N|, \quad s \in \mathbf{R}^N.$$

Step 1. Let $\varepsilon > 0$. We will prove that a number C_ε exists such that

$$(*) \qquad |f(a+x) - f(a)| \le \varepsilon\|a\| + C_\varepsilon\|x\|, \quad (a, x \in E^N).$$

Set

$$A = \{(s,t) \in \mathbf{R}^N \times \mathbf{R}^N : \|s\| \vee \|t\| = \mathbf{1}, |f(s+t) - f(s)| \ge \varepsilon\|s\|\}.$$

A is a compact subset of $\mathbf{R}^{2N}$ and for all $(s,t) \in A$ we have that $\|t\| \ne 0$. Hence there is a number $C_\varepsilon > 0$ for which

$$\frac{|f(s+t) - f(s)| - \varepsilon\|s\|}{\|t\|} \le C_\varepsilon \quad \text{for all } (s,t) \in A.$$

Then $|f(s+t) - f(s)| \leq \varepsilon\|s\| + C_\varepsilon\|t\|$ for all $(s,t) \in \mathbf{R}^{2N}$ with $\|s\| \vee \|t\| = 1$, and even for all $(s,t) \in \mathbf{R}^{2N}$. Thus $(*)$ follows.

Step 2. (Proof of part (i) of the theorem.) Let $a \in E^N$, and let $u \in E^+$. We show that for every $\varepsilon > 0$ there exists a $\delta > 0$ with

$$x_1, \dots, x_N \in [-\delta u, \delta u] \Longrightarrow |f(a+x) - f(a)| \leq \varepsilon(\|a\| + u).$$

Indeed, take C_ε as above and $\delta := \varepsilon C_\varepsilon^{-1}$.

Step 3. (Proof of part (ii) of the theorem.) Let τ be a locally solid vector space topology on E. Let $a \in E^N$, and let U be a solid τ-neighborhood of 0. We prove the existence of a τ-neighborhood W of 0 with

$$x_1, \dots, x_N \in W \Longrightarrow |f(a+x) - f(a)| \in U.$$

Indeed, choose a solid neighborhood W_0 of 0 with $W_0 + \cdots + W_0 \subset U$ where there are $N+1$ terms in the lefthand side of the latter inclusion. Choose $\varepsilon > 0$ such that $\varepsilon\|a\| \in W_0$. Take C_ε as above and define $W := C_\varepsilon^{-1} W_0$. If $x_1, \dots, x_N \in W$, then

$$|f(a+x) - f(a)| \leq \varepsilon\|a\| + C_\varepsilon\|x\|.$$

But the righthand side of the previous inequality is in $W_0 + \cdots + W_0$ ($N+1$ terms) and hence in U. Thus $|f(a+x) - f(a)| \in U$.

3. Squares and f-algebras. We now study the connection between f-algebras and squares of uniformly complete Riesz spaces.

Theorem 8. *Let E be a uniformly complete Riesz subspace of an Archimedean semi-prime f-algebra G whose multiplication is indicated by a period $\bullet$. Put $E^2 := \{x \bullet y : x, y \in E\}$. Then E^2 is a Riesz subspace of G and $(E^2, \bullet)$ is a square of E.*

Proof. By Lemma 8 of [**2**], E^2 is a Riesz subspace of G. Considered as a map $E \times E \to E^2$, the multiplication is an orthosymmetric bimorphism. Let F be a Riesz space, and let $T : E \times E \to F$ be an orthosymmetric bimorphism. By Lemma 4 in [**2**], a unique increasing linear map

$$T^\bullet : E^2 \longrightarrow F$$

exists with $T^\bullet(x \bullet y) = T(x,y)$ for all x, y. We need to prove that $T^\bullet$ actually is a Riesz homomorphism, i.e.,

$$|T^\bullet(x \bullet y)| = T^\bullet(|x \bullet y|) \quad \text{for all } x, y.$$

Now functional calculus comes in. Take $x, y \in E$. The function

$$f(s,t) = \sqrt{|st|}\,\mathrm{sgn}\,(st), \quad s, t \in \mathbf{R}$$

is in $A(\mathbf{R}^2)$ and $f(s,t)|f(s,t)| = st$, $s, t \in \mathbf{R}$. Thus, there is an element a in E such that $a = f(x,y)$. But then $a \bullet |a| = x \bullet y$. The map

$$z \longmapsto T(z, |a|)$$

of E into F is a Riesz homomorphism, thus

$$\begin{aligned} |T^\bullet(x \bullet y)| &= |T^\bullet(a \bullet |a|)| = |T(a, |a|)| \\ &= T(|a|, |a|) = T^\bullet(|a| \bullet |a|) = T^\bullet(|x \bullet y|). \end{aligned}$$

4. Szulga's construction. In this section E is a uniformly complete Riesz space. Consequently, we can apply the functional calculus results of Section 2. For the special case of Banach lattices, the following construction can be found in [**6**], with the fundamental ideas going back to Lozanovsky [**8**] and Krivine [**5**], while in the case of uniformly complete Riesz spaces it is due to Szulga [**9**]. For the reader's convenience we provide the details of the construction.

The function $\vartheta : t \mapsto t|t|$ is an order isomorphism of $\mathbf{R}$. We define $H, J : \mathbf{R}^2 \to \mathbf{R}$ by

$$H(s,t) = \vartheta^{-1}(\vartheta(s) + \vartheta(t))$$

and

$$J(s,t) = \vartheta^{-1}(st) \quad \text{for all } s, t \in \mathbf{R}.$$

Then $H, J \in \mathcal{H}(\mathbf{R}^2)$. Thus $H(x,y)$ and $J(x,y)$ exist for all $x, y \in E$. We will use the map H to define an addition $\tilde{+}$ and a scalar multiplication $\cdot$ on E. Then we show that the resulting vector space is a Riesz space that satisfies the universal property of Section 1.

Theorem 9. *Let E be a uniformly complete Riesz space. Define an addition $\tilde{+}$ and a scalar multiplication $\cdot$ on E by*

$$x \tilde{+} y := H(x,y) \quad \textit{and} \quad \lambda \cdot x := \vartheta^{-1}(\lambda)x$$
$$\textit{for all } x,y \in E \quad \textit{and all } \lambda \in \mathbf{R}.$$

(i) *Under these operations and with the given ordering, E is a Riesz space $E^\bullet$ and J, considered as a map form $E \times E$ into $E^\bullet$, is a surjective orthosymmetric bimorphism.*

(ii) *If F is any Riesz space and if $T : E \times E \to F$ is bilinear, orthosymmetric and order bounded, there exists a unique $T^\bullet : E^\bullet \to F$ with $T = T^\bullet J$; this $T^\bullet$ is linear and order bounded. $T^\bullet$ is positive if and only if T is bipositive. $T^\bullet$ is a Riesz homomorphism if and only if T is a bimorphism.*

(iii) *In particular, $(E^\bullet, J)$ is a square of E.*

Proof. For any Riesz space F and any orthosymmetric and order bounded bilinear map $T : E \times E \to F$, we define $T^\bullet : E^\bullet \to F$ by

$$T^\bullet(x) := T(x, |x|), \quad x \in E.$$

For $u \in E^+$, let E_u be the principal ideal generated by u. As E_u is uniformly complete, it follows that $x \tilde{+} y \in E_u$ for all $x, y \in E_u$, whereas, of course, $\lambda \cdot x \in E_u$ if $\lambda \in \mathbf{R}$ and $x \in E_u$. By $E_u^\bullet$ we denote the underlying set of E_u provided with the addition $\tilde{+}$, the scalar multiplication $\cdot$ and the ordering inherited from the given ordering of E.

Let F, T, u be as above. We intend to prove:

(1) $E_u^\bullet$ is a Riesz space; $x \mapsto J(x,u)$ is a Riesz isomorphism of E_u onto $E_u^\bullet$.

(2) The restriction of J is an orthosymmetric map $E_u \times E_u \to E_u^\bullet$.

(3) The restriction of $T^\bullet$ is an order bounded linear map $E_u^\bullet \to F$; it is a Riesz homomorphism if T is a bimorphism.

(From this the theorem follows easily.)

For (1), (2) and (3), the part of E that is outside E_u is irrelevant; we may as well assume $E = E_u$. By the Kakutani-Krein Representation

Theorem for Riesz spaces with strong order unit, we may even assume that $E = C(X)$ for some compact Hausdorff space X and $u = 1$.

Thus, let $E = C(X)$, $u = 1$. Proving (1), (2) and (3) is now a matter of bookkeeping. The map

$$P : f \longmapsto \vartheta \circ f = f|f|, \quad f \in C(X)$$

is an order isomorphism of $C(X)$ onto $C(X)$. For all $f, g \in C(X)$ we have

$$\begin{aligned} H(f,g) &= H \circ (f,g) = \vartheta^{-1} \circ (\vartheta \circ f + \vartheta \circ g) \\ &= P^{-1}(P(f) + P(g)) \\ J(f,g) &= J \circ (f,g) = \vartheta^{-1} \circ (fg) = P^{-1}(fg). \end{aligned}$$

Hence, if $f, g \in C(X)$ then

$$\begin{aligned} P(f \tilde{+} g) &= P(H(f,g)) = P(f) + P(g), \\ J(f,u) &= J(f,1) = P^{-1}(f), \end{aligned}$$

whereas for all $\lambda \in R$, since ϑ is multiplicative

$$P(\lambda \cdot f) = \vartheta^{-1} \circ (\vartheta^{-1}(\lambda) f) = \lambda P(f).$$

It follows from Theorem 1 in [**1**] that $T(f,g) = T(fg,1)$ for all $f, g \in C(X)$, so that

$$\begin{aligned} T^{\bullet}(f) &= T(f,|f|) = T(f|f|,1) = T(P(f),1), \\ T(f,g) &= T(fg,1) = T(PJ(fg),1) = T^{\bullet}J(f,g). \end{aligned}$$

(1), (2) and (3) follow.

As a corollary we obtain several surprising facts. It is easy to see (and well-known) that $l^{\infty}\bar{\otimes}l^{\infty}$ is not Dedekind complete even though, of course, l^{∞} is. But as an ordered set (and we emphasize the word set, i.e., we are *not* talking about the lattice ordered linear structure), the square of a uniformly complete Riesz space is indistinguishable from that Riesz space itself. Thus, we have the following result.

Corollary 10. *If E is Dedekind complete (or σ-Dedekind complete, or laterally complete) then so is its square.*

A lot harder to prove is the fact that, actually

$$c_0 \bar{\otimes} c_0$$

is not even uniformly complete. For contrast we offer the following result.

Corollary 11. *If E is uniformly complete then so is its square.*

The proof of this corollary follows the exact same lines as the proof of Corollary 12 below and is left to the reader.

In light of the equality $(L^1)^2 = L^{1/2}$ we cannot expect a locally convex and locally solid topology on E to carry over to a locally convex and locally solid topology on the square of E. In spite of that, we offer the following.

Corollary 12. *Let E be a uniformly complete Riesz space, let $E^\bullet$ be its square as constructed in Theorem 9, and let τ be a locally solid vector space topology on E. Then τ defines a locally solid vector space topology on $E^\bullet$ such that the identity map $E \to E^\bullet$ is a topological isomorphism.*

Proof. For a net f_α in $E^\bullet$ and $g \in E^\bullet$, define $f_\alpha \to g$ in $E^\bullet$ if $H(f_\alpha, -g) \to 0$ in E. This convergence defines a locally solid vector space topology on $E^\bullet$. Half of the assertion now follows from Theorem 7. On the other hand, $H(H(f,g),-g) = f$ for all $f, g \in E$. Thus we infer from Theorem 7 as well that $H(f_\alpha, -g) \to 0$ in E implies that $H(H(f_\alpha, -g), g) \to H(0,g) = g$ in E, hence that $f_\alpha \to g$ in E. This proves the Corollary.

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