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SOLUTION OF INTEGRAL EQUATIONS USING PADÉ TYPE APPROXIMANTS

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ABSTRACT. We consider the use of Padé-type methods to accelerate the convergence of the Neumann series of linear integral equations. The effectiveness of the improved squared Padé approximant for approximating the characteristic value and the characteristic function of a linear integral equation is illustrated. The estimates of characteristic function derived using the improved squared Padé approximants are found to be substantially more accurate than the original squared Padé approximants. Also the family of the Padé-type methods are reviewed specifically to demonstrate the consistency of the methods.

1. Introduction. The purpose of this paper is to demonstrate the improvement of squared Padé approximants, which was initially introduced in [**20**]. We adopt the same principle used in [**7**], [**14**], [**20**], which is expressing the denominator polynomial of the new squared Padé approximants in terms of zeros obtained by the denominator polynomial of the original squared Padé approximants. This modification is defined in the improved squared Padé approximants section. We review the recently introduced methods together with classical methods for accelerating the convergence of sequence of functions. The effectiveness of the methods is examined by determining the characteristic value and the characteristic function of several linear Fredholm linear equations. This paper is actually a continuation of the previous study [**20**]. The extension of this investigation is based on the improvement of the squared Padé approximants and showing the magnitude of the error obtained by each of the methods.

The prime motive of the development of the Padé type methods was to overcome the essential difficulty encountered by the well-established methods, such as the classical Padé approximants and the functional

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Padé approximants [5], [7], [14], [20]. It was noticed that the problem with the classical Padé approximants and the functional Padé approximants methods was the use of the minimal sensitivity principle [1], [2] and the presence of superfluous zeros in the denominator of the rational function [6]–[8], [12], [14]. Consequently, we found that the integral Padé approximant is the most effective of the Padé-type methods considered, and this is a good alternative to functional Padé approximant. We actually begin by describing the fundamentals of the denominator for each of the recently introduced Padé-type methods, and the numerator is determined naturally. In order to construct these Padé-type methods, we use a similar procedure to classical Padé approximant.

The solution of the Fredholm integral equation of the second kind is based on generating a series of functions, given as

$$f(x, \lambda) = \sum_{i=0}^{\infty} C_i(x) \lambda^i, \tag{1}$$

in which $C_i(x) \in L_2[a, b]$ are given and $[a, b]$ is the domain of definition of $C_i(x)$ in some natural sense. We also suppose that $f(x, \lambda)$ is holomorphic as a function of λ at the origin $\lambda = 0$. Then (1) converges for values of $|\lambda|$ which are sufficiently small. In this paper we see how the methods of Padé-type approximation can be used to accelerate the convergence of a series having the form (1).

The structure of this paper is as follows. In Sections 2 to 4 we define three recently introduced methods, namely, the integral Padé approximant, the improved Padé approximant and the modified Padé approximant, respectively, and in Section 5 we define the improved squared Padé approximant. In Sections 6 and 7 we also review two well-established methods, namely, the classical Padé approximant and the functional Padé approximant, respectively. In Section 8 we examine the effectiveness of methods based on the Padé-type approximants for determining the characteristic values and the characteristic functions of two linear Fredholm integral equations. The technique utilized for solving the integral equation is based on successive substitution, which is an iterative procedure, yielding a sequence of approximations leading to an infinite power series solution. We make two distinct comparisons of the estimates derived from the row sequences of the Padé-type approximants. First, we compare the estimates derived from the row sequences of the integral Padé approximant of type $(n, 1)$ with corresponding estimates derived from the improved Padé approximant of type (n), the modified Padé approximant of type $(n, 1)$, the improved squared Padé approximant of type $(n, 1)$, the functional Padé approximant of type $(n, 2)$, and the classical Padé approximant of type $(n, 1)$. In the second comparison the estimates are based on another row sequence of the integral Padé approximant of type $(n, 2)$ with corresponding estimates derived from the improved Padé approximant of type $(n + 1)$, the modified Padé approximant of type $(n, 2)$, the improved squared Padé approximant of type $(n, 2)$, the functional Padé approximant of type $(n, 4)$ and the classical Padé approximant of type $(n, 2)$. Finally, in Section 9 we illustrate the precision of a particular characteristic function of the integral Padé approximant. The effectiveness of these Padé-type methods for accelerating the convergence of a sequence of functions is investigated in the context of the Neumann series of several linear Fredholm integral equations of the second kind. In this paper and in the previous study [**20**] the integral Padé approximants method is proved to be the most effective of the methods considered. However, the improvement of the squared Padé approximant is effective when compared to the original squared Padé approximant.

2. The integral Padé approximants method (IPA). We define a rational function $r(x, \lambda)$ to be an integral Padé approximant of type (n, k) for $f(x, \lambda)$ if

$$r(x,\lambda) = \frac{N(x,\lambda)}{D(\lambda)} = \frac{\sum_{i=0}^{n} N_i(x)\lambda^i}{\sum_{i=0}^{k} D_i\lambda^i}, \tag{2}$$

where

(3) (i) $N(x, \lambda)$ is holomorphic as a function of λ,

(ii) $N(x, \lambda) \in L_2[a, b]$ as a function of x,

(iii) $D(\lambda)$ is a holomorphic function of λ,

(iv) $\partial\{N\} \le n$, $\partial\{D\} \le k$,

(v) $D(0) = 1$,

(vi) $N(x, \lambda) - D(\lambda)f(x, \lambda) = 0(\lambda^{n+1})$.

If $N(x, \lambda), D(\lambda)$ satisfy axiom (3), then there exists a unique $r(x, \lambda)$ defined by (1). The proofs of existence and uniqueness are similar to that for the classical Padé approximant [**4**], [**5**], and the rate of convergence is similar to hybrid functional Padé approximant [**15**]. This is evident from this investigation and in all other test examples performed.

We express the denominator polynomial of an integral Padé approximant of type (n,k) as

(4)
$$D(\lambda)=\begin{vmatrix} \int_a^b C_{n+k-1}(x)C_{n-k}(x)\,dx & \int_a^b C_{n+k-1}(x)C_{n-k+1}(x)\,dx & \cdots & \int_a^b C_{n+k-1}(x)C_{n}(x)\,dx \\ \int_a^b C_{n+k-1}(x)C_{n-k+1}(x)\,dx & \int_a^b C_{n+k-1}(x)C_{n-k+2}(x)\,dx & \cdots & \int_a^b C_{n+k-1}(x)C_{n+1}(x)\,dx \\ \vdots & \vdots & \cdots & \vdots \\ \int_a^b C_{n+k-1}(x)C_{n-1}(x)\,dx & \int_a^b C_{n+k-1}(x)C_{n}(x)\,dx & \cdots & \int_a^b C_{n+k-1}(x)C_{n+k-1}(x)\,dx \\ \lambda^k & \lambda^{k-1} & \cdots & 1 \end{vmatrix},$$

provided $D(0)\neq 0$ and $C_i(x)$ are the coefficients of (1).

We take the appropriate roots of denominator polynomial, given by (4), as our estimates of the characteristic value for the integral Padé approximant. Once the denominator polynomial has been specified, the actual approximants are constructed using the accuracy-through order principle, and this also applies to the other Padé-type approximants.

Naturally, the numerator polynomial $N(x,\lambda)$ follows from (3)(vi) as

$$N(x,\lambda)=[D(\lambda)f(x,\lambda)]_0^n, \tag{5}$$

where this notation, now and in the sequel, indicates that truncation at degree n in λ has been effected. If, in the representation (4), $D(0)\neq 0$, then $r(x,\lambda)$ defined by (2), (4) and (5) is an integral Padé approximant of type (n,k) for $f(x,\lambda)$.

Theorem 1. Integral Padé approximant. *Let*

$$f(x,\lambda)=r(x,\lambda)=N(x,\lambda)\div D(\lambda) \tag{6}$$

be a meromorphic function with precisely k finite poles. Then, for all n sufficiently large, there exists a unique rational function $r(x,\lambda)$ of type (n,k) which interpolates to $f(x,\lambda)$. Hence,

$$\lim_{n\to\infty} \frac{N_n(x,\lambda)}{D_n(\lambda)} = f(x,\lambda). \tag{7}$$

The rational fractions of the integral Padé approximants can be laid out in a table:

$$\begin{matrix} r^{(0,0)} & r^{(0,1)} & r^{(0,2)} & \cdots \\ r^{(1,0)} & r^{(1,1)} & r^{(1,2)} & \cdots \\ r^{(2,0)} & r^{(2,1)} & r^{(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \tag{8}$$

This concept and Theorem 1 are similar to classical Padé approximants [**5**] and may be expressed for other Padé-type approximants, also see [**20**].

Theorem 2. Convergence theorem for integral Padé approximant. *Let $f(x,\lambda)$ be a meromorphic function expressed as*

$$f(x,\lambda) = N(x,\lambda)/D(\lambda), \tag{9}$$

where

(i) *$N(x,\lambda)$ is holomorphic in $|\lambda_i| < \Omega$, $i = 1,\dots,k$,*

(ii) *$D(\lambda)$ is a monic real polynomial of degree k having k zeros and $|\lambda_i| < \Omega$,*

(iii) $D(\lambda_i) \neq 0$, $i = 1,\dots,k$.

(iv) $x,\lambda \in \forall$.

Thus $f(x,\lambda)$ is holomorphic in the ring

$$R_\Omega := \{\lambda : |\lambda| < \Omega\} - \bigcup_{i=1}^{k} \{\lambda_i\}, \tag{10}$$

where the singularities, i.e., zeros, are removed.

The rate of convergence is formally expressed as

$$\lim_{n\to\infty} |f - N_n/D_n|^{1/n} \leq \alpha. \tag{11}$$

The denominator converges according to

$$\lim_{n\to\infty} D_n(\lambda) = D(\lambda), \tag{12}$$

and the rate of convergence is given by

$$\lim_{n\to\infty} |D_n - D|^{1/n} \leq \beta, \tag{13}$$

where α and β are the asymptotic error constants.

It was shown in [**20**] that the integral Padé approximant is identical to the hybrid functional Padé approximant for the first row sequence of type $(n, 1)$, and therefore justifies part of the proof. Furthermore, the error obtained by the integral Padé approximant is numerically less than that of the functional Padé approximant for the row sequence type (n, k) when k is equal to or greater than 2. The results of these findings are illustrated in the numerical examples and in [**20**]. The proof of the convergence of the functional Padé approximant is given in [**5**], [**9**]–[**11**], [**13**], and the proof of the associate method, the hybrid functional Padé approximant, is given in [**15**].

3. The improved Padé approximant method (IMPA). We define a rational function $r(x, \lambda)$ to be an improved Padé approximant of type (n, k) for $f(x, \lambda)$ if

$$r(x, \lambda) = P(x, \lambda)/Q(\lambda), \tag{14}$$

where $P(x, \lambda), Q(\lambda)$ are polynomials in λ, $P(x, \lambda) \in L_2[a, b]$ as a function of x and

(15) (i) $\partial\{P\} \leq n$, $\partial\{Q\} \leq k$,

(ii) $Q(0) = 1$,

(iii) $P(x, \lambda) - Q(\lambda) f(x, \lambda) = 0(\lambda^{n+1})$.

If $P(x,\lambda), Q(\lambda)$ satisfy axiom (15), then there exists a unique $r(x,\lambda)$ defined by (1).

We define the denominator polynomial of an improved Padé approximant of type (n,k) as

$$Q(\lambda) = \begin{vmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & & \vdots \\ M_{k1} & M_{k2} & \cdots & M_{kk} \end{vmatrix}, \tag{16}$$

provided $Q(0) \neq 0$ and where

$$M_{ij} = \lambda \int_a^b \sum_{l=0}^{i-1} x^l \, dx \int_a^b k(x,y) \sum_{l=0}^{j-1} y^l \, dy - \int_a^b \sum_{l=0}^{i-1} x^l \sum_{l=0}^{j-1} x^l \, dx. \tag{17}$$

We take the zeros of (16) as our estimates of the characteristic values for the improved Padé approximant.

Naturally, the numerator polynomial $P(x,\lambda)$ follows from (15)(iii) as

$$P(x,\lambda) = [Q(\lambda) f(x,\lambda)]_0^n. \tag{18}$$

If, in the representation (16), $Q(0) \neq 0$, then $r(x,\lambda)$ defined by (14), (16) and (18) is an improved Padé approximant of type (n,k) for $f(x,\lambda)$.

4. The modified Padé approximant method (MPA). A modified Padé approximant of type (n,k) for the given power series (1) is the rational function

$$r(x,\lambda) = A(x,\lambda)/B(\lambda), \tag{19}$$

where $A(x,\lambda), B(\lambda)$ are polynomials in λ, $A(x,\lambda) \in L_2[a,b]$ as a function of x and

(20) (i) $\partial\{A\} \leq n$, $\partial\{B\} \leq k$,

(ii) $B(0) = 1$,

(iii) $A(x,\lambda) - B(\lambda) f(x,\lambda) = 0(\lambda^{n+1})$.

The construction of the denominator polynomial of the modified Padé approximant of type (n,k) is given as

$$(21)\quad B(\lambda) = \begin{vmatrix} \int_a^b C_{n-k}(x)\,dx & \int_a^b C_{n-k+1}(x)\,dx & \cdots & \int_a^b C_n(x)\,dx \\ \int_a^b C_{n-k+1}(x)\,dx & \int_a^b C_{n-k+2}(x)\,dx & \cdots & \int_a^b C_{n+1}(x)\,dx \\ \vdots & \vdots & & \vdots \\ \lambda^k & \lambda^{k-1} & \cdots & 1 \end{vmatrix}$$

provided $B(0) \neq 0$ and $C_i(x)$ are the coefficients of (1).

Naturally, the numerator polynomial $A(x,\lambda)$ follows from (20)(iii) as

$$(22)\qquad A(x,\lambda) = [B(\lambda)f(x,\lambda)]_0^n.$$

Each approximant of the sequence of (n,k)-type modified Padé approximant has precisely k poles. If, in the representation (21), $B(0) \neq 0$, then $r(x,\lambda)$ defined by (19), (21) and (22) is the modified Padé approximant of type (n,k) for $f(x,\lambda)$. We take the zeros of (21) as our estimates of the characteristic value for the modified Padé approximant.

5. The improved squared Padé approximant method (ISPA). An improved squared Padé approximant of type (n,k) for the given power series (1) is the rational function

$$(23)\qquad r(x,\lambda) = G(x,\lambda)/H(\lambda),$$

where $G(x,\lambda), H(\lambda)$ are polynomials in λ, $G(x,\lambda) \in L_2[a,b]$ as a function of x and

(24) (i) $\partial\{G\} \leq n$, $\partial\{H\} \leq k$,

(ii) $H(0) = 1$,

(iii) $G(x,\lambda) - H(\lambda)f(x,\lambda) = 0(\lambda^{n+1})$.

The roots of the denominator polynomial of the improved squared Padé approximant of type (n,k) are evaluated from the following determinant

$$L(\lambda) = \begin{vmatrix} \int_a^b C_{n-k}^2(x)\,dx & \int_a^b C_{n-k+1}^2(x)\,dx & \cdots & \int_a^b C_n^2(x)\,dx \\ \int_a^b C_{n-k+1}^2(x)\,dx & \int_a^b C_{n-k+2}^2(x)\,dx & \cdots & \int_a^b C_{n+1}^2(x)\,dx \\ \vdots & \vdots & & \vdots \\ \lambda^k & \lambda^{k-1} & \cdots & 1 \end{vmatrix} \tag{25}$$

provided $L(0) \neq 0$ and $C_i(x)$ are the coefficients of (1). Then we take the square root of the zeros given by (25) and express the denominator polynomial of the improved squared Padé approximant of type (n,k) as

$$H(\lambda) = \prod_{i=1}^{k} (\lambda - \sqrt{|\lambda_i|}). \tag{26}$$

This improvement is based on the same principle as the hybrid functional Padé approximant [**7**], [**14**]. The numerator polynomial $G(x,\lambda)$ follows from (24)(iii) as

$$G(x,\lambda) = [H(\lambda) f(x,\lambda)]_0^n. \tag{27}$$

Each approximant of the sequence of (n,k)-type Padé approximant has precisely k poles. If, in the representation (25), $L(0) \neq 0$, then $r(x,\lambda)$ defined by (23), (25), (26) and (27) is the improved squared Padé approximant of type (n,k) for $f(x,\lambda)$. Actually we take the square root of the zeros formed by (25) as our estimates of the characteristic value.

In addition, it has been established in [**20**] that the rational function for the squared Padé approximant is expressed as

$$r(x,\lambda) = M(x,\lambda) \div L(\lambda). \tag{28}$$

The axiom (24) also applies here with the exception of replacing the notations of G for M and H for L.

The denominator polynomial of the squared Padé approximant of type (n,k) is identical (25), and the numerator polynomial is given by

$$M(x,\lambda) = [L(\lambda) f(x,\lambda)]_0^n. \tag{29}$$

Similarly, in the representation (25), $L(0) \neq 0$, then $r(x, \lambda)$ defined by (25), (28) and (29) is the squared Padé approximant of type (n, k) for $f(x, \lambda)$.

6. The classical Padé approximant method (CPA). A classical Padé approximant of type (n, k) for the given power series (1) is the rational function

$$r(x, \lambda) = U(x, \lambda)/V(x, \lambda), \tag{30}$$

where $U(x, \lambda), V(x, \lambda)$ are polynomials in λ, $U(x, \lambda) \in L_2[a, b]$ as a function of x and

(31) (i) $\partial\{U\} \leq n$, $\partial\{V\} \leq k$,

(ii) $V(0) = 1$,

(iii) $U(x, \lambda) - V(x, \lambda)f(x, \lambda) = 0(\lambda^{n+k+1})$.

The construction of the denominator polynomial of classical Padé approximant of type (n, k) is given as

$$V(x, \lambda) = \begin{vmatrix} C_{n-k}(x) & C_{n-k+1}(x) & \cdots & C_n(x) \\ C_{n-k+1}(x) & C_{n-k+2}(x) & \cdots & C_{n+1}(x) \\ \vdots & \vdots & & \vdots \\ \lambda^k & \lambda^{k-1} & \cdots & 1 \end{vmatrix}, \tag{32}$$

provided $V(x, 0) \neq 0$ and $C_i(x)$ are the coefficients of (1).

Naturally, the numerator polynomial $U(x, \lambda)$ follows from (31)(iii) as

$$U(x, \lambda) = [V(\lambda)f(x, \lambda)]_0^n. \tag{33}$$

Each approximant of the sequence of (n, k)-type Padé approximant has precisely k poles. To determine these zeros, in order to estimate the characteristic value, we must assign a particular value of x in the Neumann series, and this is usually done using the principle of minimal sensitivity [**1**], [**2**]. The proof of convergence, existence, uniqueness and other related definitions of the classical Padé approximants are treated in [**4**], [**5**] and many other texts.

7. The functional Padé approximant method (FPA). We define a rational function $r(x, \lambda)$ to be a functional Padé approximant of type $(n, 2k)$ for $f(x,\lambda)$ if

$$r(x,\lambda) = p(x,\lambda)/q(\lambda), \tag{34}$$

where $p(x,\lambda), q(\lambda)$ are polynomials in λ, $p(x,\lambda) \in L_2[a,b]$ as a function of x, and

(35) (i) $\begin{cases} \partial\{p\} \leq n-\alpha \\ \partial\{q\} \leq 2k-2\alpha \end{cases}$ for $\alpha \geq 0$,

(ii) $q(\lambda) \mid \int_a^b |p(x,\lambda)|^2\,dx$,

(iii) $q(\lambda) = q^*(\lambda)$,

(iv) $q(\lambda) \neq 0$,

(v) $p(x,\lambda) - q(\lambda)f(x,\lambda) = 0(\lambda^{n+1})$.

The asterisk in (35)(iii) denotes the functional complex conjugate.

If $p(x,\lambda), q(\lambda)$, satisfy (35)(i)–(35)(v), then $r(x,\lambda)$ defined by (1) is unique; the questions of existence, uniqueness and degeneracy are treated in [**11**]. The explicit formula for the denominator polynomial is given by

$$q(\lambda) = \begin{vmatrix} 0 & M_{01} & \cdots & M_{0,2k-1} & M_{0,2k} \\ -M_{01} & 0 & \cdots & M_{1,2k-1} & M_{1,2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -M_{0,2k-1} & -M_{1,2k-1} & \cdots & 0 & M_{2k-1,2k} \\ \lambda^{2k} & \lambda^{2k-1} & \cdots & \lambda & 1 \end{vmatrix}. \tag{36}$$

The elements of (36) are defined by

$$M_{ij} := \sum_{l=0}^{j-i-1} \int_a^b C_{l+i+n-2k+1}(x)[C_{j-l+n-2k}(x)]^*\,dx, \tag{37}$$

for $i = 0, 1, \ldots, 2k$ and $j = i + l, i + 2, \ldots 2k$ and taking $C_j(x) := 0$ if $j < 0$.

We know that the polynomials produced by (36) are strictly positive for $\lambda \in \Re$ and their zeros occur in complex conjugate pairs close to the real axis [**6**], [**7**], [**9**], [**14**]. We take the real parts of the zeros of $q(\lambda)$ as our estimates of the characteristic value λ_c. The proof of convergence of the characteristic value of the functional Padé approximants are discussed in [**5**], [**8**], [**13**], [**15**].

The numerator polynomial $p(x,\lambda)$ follows from (35)(v) as

$$p(x,\lambda) = [q(\lambda)f(x,\lambda)]_0^n. \tag{38}$$

If, in the representation (36), $q(0) \neq 0$, then $r(x,\lambda)$ defined by (34), (36) and (38) is the functional Padé approximant of type $(n, 2k)$ for $f(x,\lambda)$.

In addition, the denominator polynomial of a hybrid functional Padé approximant is defined in terms of the roots of the denominator of the corresponding functional Padé approximant [**6**], [**7**], [**9**], [**14**]. We actually take the real parts of the roots of the functional Padé approximant and express the hybrid functional Padé approximant of type (n,k) as

$$q^H(\lambda) = \prod_{i=1}^{k}(\lambda - \lambda_i^{\Re}). \tag{39}$$

The associated numerator polynomial is defined as

$$p^H(x,\lambda) = [f(x,\lambda)q^H(\lambda)]_0^n. \tag{40}$$

Since n and k govern the degree of the numerator and denominator, respectively, we express the hybrid functional Padé approximant of type (n,k) as

$$r^H(x,\lambda) = p^H(x,\lambda)/q^H(\lambda). \tag{41}$$

We have found that (41) is identical to the integral Padé approximant (2) for $k = 1$. Also it has been shown that the hybrid functional Padé approximant is dependent on functional Padé approximant and therefore requires much more numerical computation that the integral Padé approximant [**20**]. We can easily prove this by comparing the dimensions of the determinant and the degree of the denominator polynomial of the functional Padé approximant and the integral Padé approximant, which are given by (35) and (4), respectively.

8. Application to linear integral equations. To demonstrate the performance of each of the Padé-type methods, we take two familiar linear Fredholm integral equations of the second kind. We determine the consistency and stability of the results by examining the convergence of the Padé-type methods for two particular types of row sequence. The findings are generalized by illustrating the effectiveness of the Padé-type methods for determining the characteristic values and the characteristic function of two linear integral equations. Consequently, we shall demonstrate the improvement of the squared Padé approximant and the error obtained by each of the Padé-type methods.

The following procedure is common to both of the illustrative numerical examples. We illustrate the convergence of the methods described by making two distinct comparisons of the estimates based on two particular types of row sequence. The first of the comparisons is based on the estimates obtained by the row sequence of the improved squared Padé approximant of type $(n, 1)$ with corresponding estimates derived from the integral Padé approximant of type $(n, 1)$, the classical Padé approximant of type $(n, 1)$, the modified Padé approximant of type $(n, 1)$, the improved Padé approximant of n-dimensional system (given by (16)) and the functional Padé approximant of type $(n, 2)$. The second comparison is based on the row sequence of the improved squared Padé approximant of type $(n, 2)$ with corresponding estimates derived from the integral Padé approximant of type $(n, 2)$, the classical Padé approximant of type $(n, 2)$, the modified Padé approximant of type $(n, 2)$, the improved Padé approximant of $(n + 1)$-dimensional system and the functional Padé approximant of type $(n, 4)$. In each case the comparisons with other methods were made using a similar amount of data, that is, using a similar number of terms of the Neumann series, with the exception of the improved Padé approximant which is constructed differently from the other Padé-type approximant.

8.1 Numerical example 1. We investigate the convergence of the Padé-type methods for the Neumann series solution of the linear integral equation of the form

$$f(x, \lambda) = g(x) + \lambda \int_0^1 k(x, y) f(y, \lambda)\, dy, \tag{42}$$

where

$$g(x) = x^2 \quad \text{and} \quad k(x, y) = \begin{cases} y(2 - x) & 0 \le y \le x \le 1, \\ x(2 - y) & 0 \le x \le y \le 1. \end{cases}$$

This integral equation is a linear inhomogeneous Fredholm of the second kind with a nondegenerate kernel. As a standard procedure, the analytic solution of (42) can be found by converting it to a second order ordinary differential equation [**3**], and the explicit solution of (42) is given by

$$f(x,\lambda) = \frac{1}{\lambda} - \frac{\cos(\mu x)}{\lambda} - \frac{[1-\cos(\mu)+\sin(\mu)\mu - 3\lambda]\sin(\mu x)}{\lambda\sin(\mu)+\lambda\mu\cos(\mu)} \tag{43}$$

where $\mu = \sqrt{2\lambda}$. It is easily found that the denominator of (43) has the following characteristic values $\lambda_1 = 2.05793\ldots$, $\lambda_2 = 12.06967\ldots$, $\lambda_3 = 31.82955\ldots$, etc.

The Neumann series solution for (42) is convergent for $|\lambda| < \lambda_c$ [**16**], [**19**], and the first few terms of this series, given by iteration of (42), are

$$\begin{aligned} f(x,\lambda) &= \sum_{i=0}^{\infty} C_i(x)\lambda^i \\ &= x^2 + \left[\frac{5}{12}x - \frac{1}{6}x^4\right]\lambda + \left[\frac{43}{180}x - \frac{5}{36}x^3 + \frac{1}{90}x^6\right]\lambda^2 + \cdots . \end{aligned} \tag{44}$$

In Table 1 we compare the estimates of the first characteristic value for each of the iterative methods described. We know that the estimates of the improved squared Padé approximant and the original squared Padé approximant are identical because both of these methods use the same determinantal formula given by (25). We also know that the estimates of the integral Padé approximant and the functional Padé approximant are identical for the first row sequence [**20**]. We find that the estimates from the integral Padé approximant give better approximations than the other similar methods. Moreover, we list the errors obtained by the Padé-type methods in Table 2. The results in Tables 3a and 3b are also the estimates of the first characteristic value derived by the second row sequence. Here also, we see that the row sequence of the integral Padé approximants give better approximation than the other Padé-type methods. Again we list the errors obtained by these methods in Table 4. The other estimates produced by the second row sequence are the second characteristic value, and we list these estimates in Table 5. Accordingly, the errors that occurred by these methods are listed in Table 6.

TABLE 1. Estimates showing the precision of the characteristic value λ_1 derived using the six methods described. The exact value of $\lambda_1 = 2.057929182\ldots$.

n	**IPA**$(n,1)$ =**FPA**$(n,2)$ λ_1	**ISPA**$(n,1)$ =**SPA**$(n,1)$ λ_1	**MPA**$(n,1)$ λ_1	**IMPA**(n) λ_1	**CPA**$(n,1)$ λ_1
1	2.125	2.3	1.9	2.4	1.3
2	2.0598	2.064	2.03	2.106	1.94
3	2.05798	2.0581	2.052	2.0585	2.04
4	2.057931	2.057935	2.0570	2.05795	2.055
5	2.05792923	2.0579293	2.0578	2.05792929	2.0574

TABLE 2. Errors occurring in the estimates, shown in Table 1, of the six methods described.

n	**IPA=FPA** λ_1	**ISPA=SPA** λ_1	**MPA** λ_1	**IMPA** λ_1	**CPA** λ_1
1	-0.0674	-0.242	-0.153	-0.342	0.795
2	-0.188(-2)	-0.652(-2)	-0.0303	-0.0481	0.122
3	-0.545(-4)	-0.187(-3)	-0.550(-2)	-0.589(-3)	0.0196
4	-0.158(-5)	-0.543(-5)	-0.963(-3)	-0.249(-4)	-0.325(-2)
5	-0.460(-7)	-0.158(-6)	-0.166(-3)	-0.104(-6)	-0.547(-3)

TABLE 3a. Estimates showing the precision of the characteristic value λ_1 derived using the six methods described. The exact value of $\lambda_1 = 2.05792918284726141\ldots$.

n	**IPA**$(n,2)$ λ_1	**MPA**$(n,2)$ λ_1	**CPA**$(n,2)$ λ_1
1	2.0599	2.04	1.82
2	2.0579294	2.0588	2.0583
3	2.0579291837	2.05798	2.0578
4	2.057929182851	2.057931	2.057917
5	2.05792918284728	2.0579294	2.057928

TABLE 3b. Estimates showing the precision of the characteristic value λ_1 derived using the six methods described. The exact value of $\lambda_1 = 2.05792918284726141\ldots$.

n	**FPA**$(n,4)$ λ_1	**ISPA**$(n,2)$=**SPA**$(n,2)$ λ_1	**IMPA**$(n+1)$ λ_1
1	1.96	2.099	2.106
2	2.057931	2.05795	2.0585
3	2.057929187	2.05792925	2.05795
4	2.057929182861	2.0579291831	2.05792928
5	2.05792918284732	2.0579291828483	2.057929185

TABLE 4. Errors occurring in the estimates, shown in Tables 3a and 3b, of the six methods described.

n	**IPA** λ_1	**FPA** λ_1	**ISPA** λ_1	**MPA** λ_1	**IMPA** λ_1	**CPA** λ_1
1	0	0	0	0	0	0
2	-0.205(-6)	-0.208(-5)	-0.229(-4)	-0.914(-3)	-0.589(-3)	-0.329(-3)
3	-0.804(-9)	-0.373(-8)	-0.703(-7)	-0.573(-4)	-0.249(-4)	0.106(-3)
4	-0.331(-11)	-0.134(-10)	-0.270(-9)	-0.383(-5)	-0.104(-6)	0.124(-4)
5	-0.138(-13)	-0.541(-13)	-0.110(-11)	-0.256(-6)	-0.194(-8)	0.100(-5)

TABLE 5. Estimates showing the precision of the characteristic value λ_2 derived using the six methods described. The exact value of $\lambda_2 = 12.069671015222778\ldots$.

n	**IPA**$(n,2)$ λ_2	**FPA**$(n,2)$ λ_2	**ISPA**$(n,2)$ =**SPA** λ_2	**MPA**$(n,2)$ λ_2	**IMPA**$(n+2)$ λ_2	**CPA**$(n,2)$ λ_2
1	–	–	–	–	12.96	–
2	12.13	12.69	12.95	10.0	12.14	12.53
3	12.077	12.14	12.16	11.2	12.76	12.77
4	12.0707	12.079	12.08	11.7	12.0699	12.46
5	12.06983	12.071	12.071	11.93	12.069681	12.24

TABLE 6. Errors occurring in the estimates, shown in Table 5, of the six methods described.

n	**IPA** λ_2	**FPA** λ_2	**ISPA=SPA** λ_2	**MPA** λ_2	**IMPA** λ_2	**CPA** λ_2
1	–	–	–	–	-0.890	–
2	-0.0560	-0.6201	-0.8836	2.0747	-0.8885	-0.4647
3	-0.0076	-0.0715	-0.0880	0.8514	-0.0656	-0.6985
4	-0.0011	-0.0097	-0.0116	0.3444	-0.0065	-0.3888
5	-0.0002	-0.0014	-0.0016	0.1362	-0.0002	-0.1725

8.2. Precision of the approximate solution. In Figure 1 we display the exact (analytic) solution and its approximations obtained using the improved squared Padé approximant of type (2,2), the original squared Padé approximant of type (2,2), the integral Padé approximant of type (2,2), the modified Padé approximant of type (2,2), the improved Padé approximant of type (2,2) and the functional Padé approximant of type (2,4). Also in Figure 1 we see a remarkable precision of the improved squared Padé approximant and other Padé-type approximants, where graphically there is no significant difference between the improved squared Padé approximant and the exact solution, whereas the original squared Padé approximant is visibly different from the exact solution. We have found that the precision of the characteristic function for the improved squared Padé approximant has been improved when compared to the original squared Padé approximant. Therefore, in Table 7 we show the errors incurred by the improved squared Padé approximant and the original squared Padé approximant together with the other Padé type approximants for $x = 0(0.25)1$ in the solution of (42). For a particular value of the characteristic value, we list the appropriate rational functions displayed and we observe the precision of coefficients of x for each of the Padé-type approximants with the exact solution $f(x, \lambda)$ given as

$$\begin{aligned} f(x, 0.5) = 0.8884x + x^2 - 0.2961x^3 - 0.1667x^4 \\ + 0.02961x^5 + 0.0111x^6 - \cdots . \end{aligned} \tag{45}$$

Solution of the integral equation (42) using the improved squared Padé

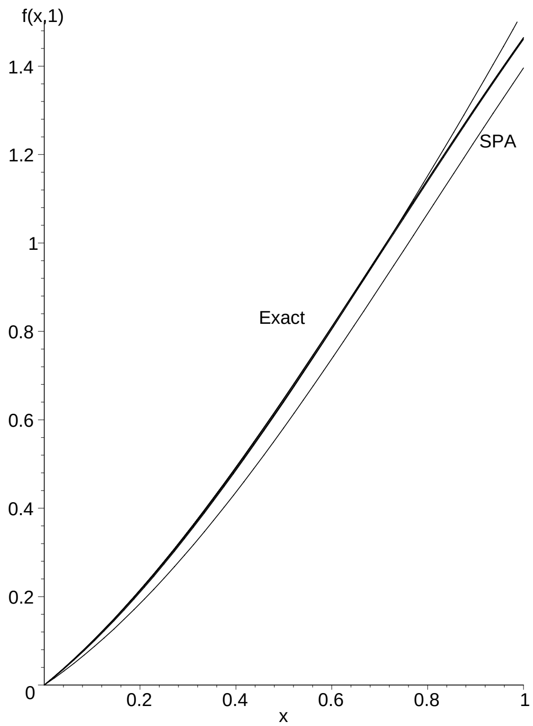

FIGURE 1. The analytic solution (exact) of (42) for $\lambda = 1$. The curves IPA, ISPA, FPA, IMPA and MPA are indistinguishable from the exact curve whereas the curve SPA is perceptibly different.

approximant of type (2,2) is

$$\frac{G(x,1)}{H(1)} = 0.887x + x^2 - 0.293x^3 - 0.153x^4 + 0.023x^6. \tag{46}$$

Solution of (42) based on the squared Padé approximant of type (2,2) is

$$\frac{M(x,1)}{L(1)} = 0.730x + x^2 - 0.183x^3 - 0.166x^4 + 0.015x^6. \tag{47}$$

Solution of (42) using the integral Padé approximant of type (2,2) is

$$\frac{N(x,1)}{D(1)} = 0.888x + x^2 - 0.294x^3 - 0.153x^4 + 0.024x^6. \tag{48}$$

Solution of (42) based on the functional Padé approximant of type (2,4) is

$$\frac{p(x,1)}{q(1)} = 0.841x + 1.162x^2 - 0.607x^3 + 0.081x^4 + 0.049x^6. \tag{49}$$

Solution of (42) based on the modified Padé approximant of type (2,2) is

$$\frac{A(x,1)}{B(1)} = 0.889x + x^2 - 0.300x^3 - 0.149x^4 + 0.024x^6. \tag{50}$$

Solution of (42) based on the improved Padé approximant of type (2,2) is

$$\frac{P(x,1)}{Q(1)} = 0.876x + x^2 - 0.281x^3 - 0.157x^4 + 0.022x^6. \tag{51}$$

TABLE 7. Errors occurring in the solution of (42) using the six methods described.

x	**IPA** $\lambda = 1$	**MPA** $\lambda = 1$	**ISPA** $\lambda = 1$	**SPA** $\lambda = 1$	**IMPA** $\lambda = 1$	**FPA** $\lambda = 1$
0	0	0	0	0	0	0
0.25	0.107(-3)	-0.151(-3)	0.195(-3)	0.377(-1)	0.275(-2)	0.565(-2)
0.5	-0.509(-4)	-0.228(-3)	0.246(-3)	0.656(-1)	0.423(-2)	0.696(-2)
0.75	-0.111(-3)	0.129(-3)	-0.153(-3)	0.767(-1)	0.438(-2)	-0.260(-2)
1.0	0.226(-3)	0.733(-3)	0.102(-3)	0.687(-1)	0.388(-2)	-0.610(-1)

8.3 Numerical example 2. In this subsection we take another linear integral equation. We shall illustrate the convergence of the Padé-type methods for the Neumann series solution of the following linear integral equation

$$f(x,\lambda) = x + \lambda \int_0^1 [1 + \sin(x)\sin(y)] f(y,\lambda)\, dy. \tag{52}$$

This integral equation is a linear inhomogeneous Fredholm of the second kind, and the analytic solution of (52) is given as

$$f(x,\lambda) = \frac{2\pi\lambda + \pi^2/2(1 - (\pi\lambda/2)) + \sin(x)\pi}{1 - 3\pi\lambda/2 + (\pi^2/2 - 4)\lambda^2}. \tag{53}$$

It is easily found that the denominator of (53) has the following characteristic values $\lambda_1 = 0.2219814858\ldots$ and $\lambda_2 = 4.8190731647\ldots$.

The first few terms of the Neumann series are

(54)

$$
\begin{aligned}
f(x,\lambda) &= \sum_{i=0}^{\infty} C_i(x)\lambda^i \\
&= x + \left[\sin(x)\pi + \frac{1}{2}\pi\right]\lambda + \left[\frac{3}{2}\sin(x)\pi^2 + 2\pi + \frac{1}{2}\pi^3\right]\lambda^2 + \cdots .
\end{aligned}
$$

We repeat the procedure of the comparisons in the previous example. That is, we compare the estimates of the characteristic value for each of the methods. In Table 8 we list the estimates of the first characteristic value and the errors obtained by these methods in Table 9. As expected, the integral Padé approximant gives better approximations than the other similar methods. Moreover, the results in Tables 10a and 10b are the estimates of the first characteristic value but derived from the second row sequence, and, again, we list the errors obtained by these methods in Table 11. We see that the row sequence of the integral Padé approximants gives better approximation than the other Padé-type methods. We have found with this example that the estimates of the characteristic value based on the second row sequence for each of the Padé-type approximants to be exact after a few iterations, with the exception of the improved Padé approximants. Hence, we have omitted the tables of the second characteristic value.

TABLE 8. Estimates showing the precision of the characteristic value λ_1 derived using the six methods described. The exact value of $\lambda_1 = 0.22198148581854962\ldots$.

n	**IPA**$(n,1)$ =**FPA**$(n,2)$ λ_1	**ISPA**$(n,1)$ =**SPA**$(n,1)$ λ_1	**MPA**$(n,1)$ λ_1	**IMPA**(n) λ_1	**CPA**$(n,1)$ λ_1
1	0.22219	0.259	0.227	0.2265	0.1
2	0.2219819	0.221986	0.2222	0.2265	0.223
3	0.2219814868	0.221981496	0.221991	0.221996	0.22203
4	0.221981485821	0.22198148584	0.22198151	0.221996	0.221984

TABLE 9. Errors occurring in the estimates, shown in Table 8, of the six methods described.

n	**IPA=FPA** λ_c	**ISPA=SPA** λ_c	**MPA** λ_c	**IMPA** λ_c	**CPA** λ_c
1	-0.209(-3)	-0.371(-1)	-0.453(-2)	-0.453(-2)	0.144
2	-0.443(-6)	-0.503(-5)	-0.209(-3)	-0.453(-2)	-0.101(-2)
3	-0.940(-9)	-0.107(-7)	-0.962(-5)	-0.146(-4)	-0.467(-4)
4	-0.199(-11)	-0.226(-10)	-0.443(-6)	-0.146(-4)	-0.215(-5)

TABLE 10a. Estimates showing the precision of the characteristic value λ_1 derived using the six methods described. The exact value of $\lambda_1 = 0.22198148581854962\ldots$.

n	**ISPA**$(n,2)$ =**SPA**$(n,2)$ λ_1	**MPA**$(n,2)$ λ_1	**CPA**$(n,2)$ λ_1
1	0.2288	0.22227	–
2	0.221981496	0.22198148581854962...	0.222035
3	0.22198148581854962...	0.22198148581854962...	0.22198148581854962...
4	0.22198148581854962...	0.22198148581854962...	0.22198148581854962...

TABLE 10b. Estimates showing the precision of the characteristic value λ_1 derived using the six methods described. The exact value of $\lambda_1 = 0.22198148581854962\ldots$.

n	**IPA**$(n,2)$ λ_1	**FPA**$(n,4)$ λ_1	**IMPA**$(n+1)$ λ_1
1	0.2221981929	0.2288	0.2265
2	0.22198148581854962...	0.221981496	0.221996
3	0.22198148581854962...	0.22198148581854962...	0.221996
4	0.22198148581854962...	0.22198148581854962...	0.22198149

TABLE 11. Errors occurring in the estimates, shown in Tables 10a and 10b, of the six methods described.

	ISPA	**IPA**	**FPA**	**MPA**	**IMPA**	**CPA**
n	λ_1	λ_1	λ_1	λ_1	λ_1	λ_1
1	-0.679(-2)	0.443(-6)	0.185(-1)	-0.288(-3)	-0.453(-2)	–
2	-0.100(-7)	0	-0.857(-9)	0	-0.146(-4)	-0.530(-4)
3	0	0	0	0	-0.146(-4)	0
4	0	0	0	0	-0.669(-8)	0

8.4 Precision of the approximate solution. In Figure 2 we display the exact solution and its approximations obtained using the improved squared Padé approximant of type (2,1), the integral Padé approximant of type (2,1), the modified Padé approximant of type (2,1), the original squared Padé approximant of type (2,1), the improved Padé approximant of type (2,1) and the functional Padé approximant of type (2,2). Also, in Figure 2 we see a remarkable precision of the improved squared Padé approximant and other Padé-type approximants. Graphically there is no significant difference between the improved squared Padé approximant and the exact solution, whereas the original squared Padé approximant is visibly different from the exact solution. For the purpose of this paper, we have shown that the precision of the characteristic function for the improved squared Padé approximant has been improved when compared to the original squared Padé approximant. Therefore, in Table 7 we show the errors incurred by the improved squared Padé approximant and the original squared Padé approximant together with the other Padé-type approximants for $x = 0(0.25)1$ in the solution of (42). For a particular value of $\lambda = 0.5$ we list the appropriate rational functions displayed, but first we state the analytic solution

$$f(x, 0.5) = x - 1.871 - 1.399\sin(x). \tag{55}$$

Solution of (52) based on the improved squared Padé approximant of type (2,1) is

$$\frac{G(x, 0.5)}{H(0.5)} = x - 1.882 - 1.384\sin(x). \tag{56}$$

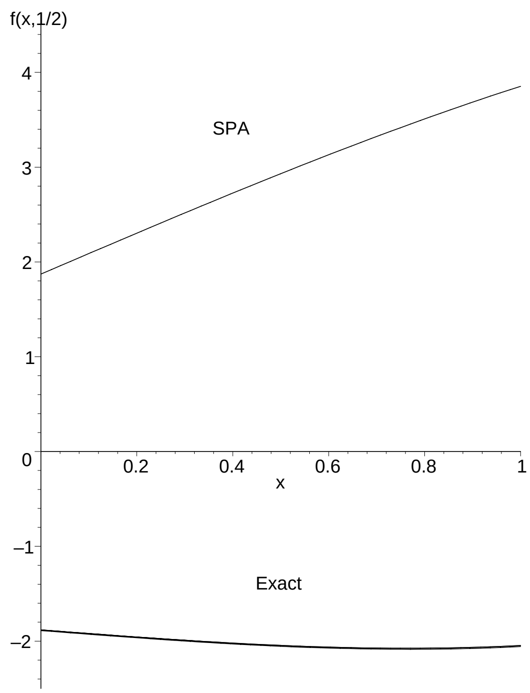

FIGURE 2. The analytic solution (exact) of (52) for $\lambda = 1/2$. All the approximant curves are indistinguishable from the exact curve whereas the curve SPA is perceptibly different.

Solution of (52) based on the square Padé approximant of type (2,1) is

$$\frac{M(x,0.5)}{L(0.5)} = x + 1.872 + 1.166\sin(x). \tag{57}$$

Solution of (52) based on the integral Padé approximant of type (2,1) is

$$\frac{N(x,0.5)}{D(0.5)} = x - 1.881 - 1.384\sin(x). \tag{58}$$

Solution of (52) based on the functional Padé approximant of type (2,2) is

$$\frac{p(x,0.5)}{q(0.5)} = x - 2.041 - 1.150\sin(x). \tag{59}$$

Solution of (52) based on the modified Padé approximant of type (2,1) is

$$\frac{A(x,0.5)}{B(0.5)} = x - 1.889 - 1.389\sin(x). \tag{60}$$

Solution of (52) based on the improved Padé approximant of type (2,1) is

$$\frac{P(x,0.5)}{Q(0.5)} = x - 2.044 - 1.301\sin(x). \tag{61}$$

TABLE 12. Errors occurring in the solution of (52) by the six Padé-type approximants.

x	**IPA** $\lambda = 0.5$	**MPA** $\lambda = 0.5$	**ISPA** $\lambda = 0.5$	**SPA** $\lambda = 0.5$	**IMPA** $\lambda = 0.5$	**FPA** $\lambda = 0.5$
0	0.103(-1)	0.176(-1)	0.104(-1)	-3.74	0.172	0.170
0.25	0.656(-2)	0.151(-1)	0.675(-2)	-4.38	0.196	0.108
0.50	0.307(-2)	0.128(-1)	0.328(-2)	-4.97	0.218	0.500(-1)
0.75	0.229(-4)	0.108(-1)	0.259(-3)	-5.49	0.237	-0.376(-3)
1.0	-0.238(-2)	0.916(-2)	0.212(-2)	-5.90	0.252	-0.402(-1)

10. Remarks and conclusion. The Padé-type methods are rational approximation solutions to linear integral equations, and these methods are essentially for accelerating the convergence of a sequence of functions. We have demonstrated the Padé-type approximants for two types of row sequence purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of each of the methods.

In this paper we show that simply expressing the denominator polynomial of the improved squared Padé approximant in terms of the zeros obtained by the original squared Padé approximant, we actually increase the precision of the characteristic function of a linear integral equation. The precision of the characteristic value of the improved squared Padé approximant is substantially more accurate when compared to the improved Padé approximant, the modified Padé approximant and the classical Padé approximant. Moreover, the precision of the characteristic function is substantially more accurate when compared to the original squared Padé approximant, the functional Padé approximant, the improved Padé approximant, the modified Padé approximant and the classical Padé approximant. Unfortunately, the drawback of the original squared Padé approximant is also common to the improved squared Padé approximant, which is the inadequacy of both of the methods when the generating function possesses an alternating or a negative power series. Hence, further investigation is needed to eliminate the drawback.

In all the numerical examples performed we have found that the integral Padé approximant is much more efficient and does not have the drawbacks of the other similar Padé-type approximants considered. We observe the remarkable precision of the estimates of the characteristic value and the characteristic function of the integral equation given in this paper and in the previous study [**20**]. We shall describe the drawback of the other Padé-type methods compared to the integral Padé approximants. The poor performance of the functional Padé approximant is due to the superfluous zeros in the denominator polynomial [**6**], [**8**], [**12**], [**14**], [**20**]. Also, it is well established that the drawback of the classical Padé approximant is the problem of assigning a particular value of x in the Neumann series to determine the estimate of the characteristic value [**6**], [**14**], [**20**]. The disadvantage with the improved Padé approximant and the modified Padé approximant is that they lack the remarkable precision of the integral Padé approximant. The results of this investigation are similar to the previous study. However, the purpose of this paper was to illustrate the improvement of the original squared Padé approximant and demonstrate the consistency of the Padé-type methods. Finally, an analytical investigation of the Padé-type approximants is a subject of further research.

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