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MIXED BOUNDARY VALUE PROBLEMS OF THE THIRD KIND IN A THEORY OF BENDING OF ELASTIC PLATES

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ABSTRACT. A modified boundary integral equation method is used to solve a specific type of mixed boundary value problem in a theory of bending of elastic plates with transverse shear deformation. This type of problem (often referred to as a mixed problem of the third kind) is characterized by the fact that a combination of transverse displacement and bending and twisting moments is prescribed on the curve which bounds the middle surface of the plate. Both interior and exterior problems are formulated and the corresponding existence and uniqueness results derived.

1. Introduction. Dirichlet (displacement) and Neumann (traction) problems for the bending of thin elastic plates with transverse shear deformation have been studied extensively in [**1**]. A class of corresponding mixed problems, when both displacement and traction are prescribed separately on different parts of the boundary of the plate, is discussed in [**9**]. The corresponding Robin problems are solved in [**10**]. Unfortunately, these results do not accommodate a specific type of mixed problem for plates defined by the requirement that a combination of transverse displacement and bending and twisting moments be prescribed on the curve bounding the middle surface of the plate. This problem is commonly referred to as a mixed boundary value problem of the third kind [**7**] for elastic plates. Despite its practical significance, to the authors' knowledge, a rigorous treatment of this problem remains absent from the literature. This can be attributed, in part, to the non-standard boundary condition and, in the case of the exterior problem, to the rapid growth at infinity of the corresponding matrix of fundamental solutions. Each of these features presents difficulties which are not accommodated by classical boundary integral techniques [**6**], [**7**].

In this paper we use a modified boundary integral equation method to solve both interior and exterior mixed problems of the third kind

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in the theory of bending of elastic plates presented in [**1**]. Using certain modifications to the classical techniques used in [**6**] and [**7**], including the use of a far-field pattern which allows for the possibility of divergence at infinity, we establish uniqueness and existence results in the appropriate function spaces.

Our technique is equally applicable, with modifications only in detail, to similar problems arising from existing theories where the governing system of equilibrium equations is elliptic, for example, problems from asymmetric elasticity [**4**], [**5**].

2. Preliminaries. In what follows, Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively; we sum over repeated indices, $\mathcal{M}_{m\times n}$ is the space of $(m \times n)$-matrices, E_n is the identity element in $\mathcal{M}_{n\times n}$, a superscript T indicates matrix transposition and $(\cdots)_{,\alpha} = \partial(\cdots)/\partial x_\alpha$. We define also the matrices

$$O_{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad I_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, if X is a space of scalar functions and v a matrix, $v \in X$ means that every component of v belongs to X.

Consider a homogeneous and isotropic plate occupying the region $S \times [-h_0/2, h_0/2]$, where $S \subset \mathbf{R}^2$ is a domain bounded by a closed C^2-curve ∂S and constant $= h_0 \ll \operatorname{diam} S$ is the plate's thickness. The equilibrium equations for bending in the absence of body forces and moments and of forces and moments acting on the faces, can be written in the form [**1**]

$$L(\partial_x)u(x) = 0, \tag{1}$$

where $x = (x_1, x_2)$ is a generic point in S, $u = (u_1, u_2, u_3)^T$ a vector characterizing the displacement field and $L(\partial_x) = L(\partial/\partial x_\gamma)$ is the matrix partial differential operator defined by the matrix $L(\xi) = L(\xi_\gamma)$ given by:

$$\begin{aligned} &L(\xi_1, \xi_2) \\ &= \begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{pmatrix}. \end{aligned} \tag{2}$$

Here, λ and μ are the Lamé constants of the material, $h^2 = h_0^2/12$ and $\Delta = \xi_\alpha\xi_\alpha$.

Together with L we consider the boundary stress operator $T(\partial_x; n)$ defined by

$$T(\xi_1,\xi_2;n) = \begin{pmatrix} h^2(\lambda+2\mu)n_1\xi_1 + h^2\mu n_2\xi_2 & h^2\mu n_2\xi_1 + h^2\lambda n_1\xi_2 & 0 \\ h^2\lambda n_2\xi_1 + h^2\mu n_1\xi_2 & h^2\mu n_1\xi_1 + h^2(\lambda+2\mu)n_2\xi_2 & 0 \\ \mu n_1 & \mu n_2 & \mu n_\alpha\xi_\alpha \end{pmatrix},$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S. For brevity, we write $T(\partial_x; n) \equiv T(\partial_x) \equiv T$.

With the assumption that $\lambda + \mu > 0$, $\mu > 0$, it is clear that the operator L is elliptic and the internal energy density is positive [**1**]. Further, $E(u,u) = 0$ if and only if

$$u(x) = (k_1, k_2, -k_1x_1 - k_2x_2 + k_3)^T, \tag{3}$$

where k_i are arbitrary constants. This is the most general rigid displacement compatible with this plate theory. If we write

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_1 & -x_2 & 1 \end{pmatrix},$$

where the columns $F^{(i)}$ form a basis for (3), then any vector of the form (3) can be written in the form Fk where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary. Further, it is clear that $LF = 0$ in $\mathbf{R}^2$ and that $TF = 0$ on ∂S.

The matrix of fundamental solutions for the operator L is given by [**1**]

$$D(x,y) = L^*(\partial_x)t(x,y),$$

where

$$t(x,y) = c_1[(4h^2 + |x-y|^2)\ln|x-y| + 4h^2K_0(h^{-1}|x-y|)],$$

$L^*(\xi)$ is the adjoint of L, K_0 the modified Bessel function of order zero and $c_1 = [8\pi h^2\mu^2(\lambda+2\mu)]^{-1}$. This choice of matrix of fundamental solutions seems natural since $D(x,y)$ is computed by means of Galerkin's method [**2**]. The columns $D^{(i)}(x,y)$ satisfy (1) at all $x \in \mathbf{R}^2$, $x \neq y$. Further we note that [**1**]

$$D(x,y) \text{ is } O(|x|^2 \ln |x|), \quad \text{as } |x| \to \infty, \quad y \in \partial S, \tag{4}$$

which will be seen to be significant in the application of the boundary integral equation method to the exterior mixed problem.

Along with $D(x,y)$ we consider the matrix of singular solutions

$$P(x,y;n) = (T(\partial_y;n)D(y,x))^T,$$

writing, for simplicity, $P(x,y;n(y)) \equiv P(x,y)$. It is not difficult to show that the columns of $P(x,y)$ satisfy (1) at all $x \in \mathbf{R}^2$, $x \neq y$ and for any direction n independent of x.

Finally, let $\mathcal{A}$ be the class of vectors $u \in \mathcal{M}_{3\times 1}$ admitting an asymptotic expansion of the form [**1**]:

$$\begin{aligned} u_1(r,\theta) = {} & r^{-1}[a_0 \sin\theta + 2a_1 \cos\theta - a_0 \sin 3\theta + (a_2 - a_1)\cos 3\theta] \\ & + r^{-2}[(2a_3 + a_4)\sin 2\theta + a_5 \cos 2\theta - 3a_3 \sin 4\theta + 2a_6 \cos 4\theta] \\ & + r^{-3}[2a_7 \sin 3\theta + 2a_8 \cos 3\theta + 3(a_9 - a_7)\sin 5\theta \\ & \qquad + 3(a_{10} - a_8)\cos 5\theta] + O(r^{-4}), \end{aligned}$$

$$\begin{aligned} u_2(r,\theta) = {} & r^{-1}[2a_2 \sin\theta + a_0 \cos\theta + (a_2 - a_1)\sin 3\theta + a_0 \cos 3\theta] \\ & + r^{-2}[(2a_6 + a_5)\sin 2\theta - a_4 \cos 2\theta + 3a_6 \sin 4\theta + 2a_3 \cos 4\theta] \\ & + r^{-3}[2a_{10} \sin 3\theta - 2a_9 \cos 3\theta + 3(a_{10} - a_8)\sin 5\theta \\ & \qquad + 3(a_7 - a_9)\cos 5\theta] + O(r^{-4}), \end{aligned}$$

$$\begin{aligned} u_3(r,\theta) = {} & -(a_1 + a_2)\ln r - [a_1 + a_2 + a_0 \sin 2\theta + (a_1 - a_2)\cos 2\theta] \\ & + r^{-1}[(a_3 + a_4)\sin\theta + (a_5 + a_6)\cos\theta - a_3 \sin 3\theta + a_6 \cos 3\theta] \\ & + r^{-2}[a_{11} \sin 2\theta + a_{12} \cos 2\theta + (a_9 - a_7)\sin 4\theta \\ & \qquad + (a_{10} - a_8)\cos 4\theta] + O(r^{-3}), \end{aligned}$$

where $a_0, \dots, a_{12}$ are arbitrary constants. Consider also the class

$$\mathcal{A}^* = \{u : u = Fk + u_0\},$$

where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary and $u_0 \in \mathcal{M}_{3\times 1} \cap \mathcal{A}$. Both $\mathcal{A}$ and $\mathcal{A}^*$ are classes of finite energy functions.

3. Boundary value problems. Let S^+ be the bounded domain enclosed by ∂S and $S^- = \mathbf{R}^2 \setminus (S^+ \cup \partial S)$.

We consider the following interior mixed boundary value problem of the third kind.

Find $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ satisfying (1) in S^+ and such that

$$O_{33}Tu(x) + I_{33}u(x) = f(x), \quad x \in \partial S, \tag{I}$$

where $f \in \mathcal{M}_{3\times 1}$ is prescribed on ∂S.

Similarly, we consider the corresponding exterior problem:

Find $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathcal{A}^*$ satisfying (1) in S^- and such that

$$O_{33}Tu(x) + I_{33}u(x) = g(x), \quad x \in \partial S, \tag{E}$$

where $g \in \mathcal{M}_{3\times 1}$ is prescribed on ∂S.

In view of (4), we pose the exterior problem (E) in $\mathcal{A}^*$ to allow as large a set of admissible functions as possible.

Remark 1 (Betti Formulae). The following subsidiary formulae are proved in [**1**].

(i) If $u \in C^2(S^+) \cap C^(\bar{S}^+)$ is a solution of (1) in S^+,

$$2\int_{S^+} E(u,u)\,d\sigma = \int_{\partial S} u^T Tu\,ds. \tag{5}$$

(ii) If $u \in C^2(S^-) \cap C^{-1}(\bar{S}^-) \cap \mathcal{A}^*$ is a solution of (1) in S^-,

$$2\int_{S^-} E(u,u)\,d\sigma = -\int_{\partial S} u^T Tu\,ds. \tag{6}$$

Using (5) in the case of the interior problem (I) and (6) in the case of the exterior problem (E), standard arguments [**1**] lead to the following uniqueness result for problems (I) and (E):

Theorem 1. Problems (I) and (E) have at most one solution.

4. Existence theorems. In [**1**], the elastic single layer potential

$$(V(\varphi))(x) = \int_{\partial S} D(x,y)\varphi(y)\,ds(y) \tag{7}$$

and the elastic double layer potential

$$(W(\varphi))(x) = \int_{\partial S} P(x,y)\varphi(y)\,ds(y) \tag{8}$$

where φ is a density (3×1)-matrix, are used to establish existence results for Dirichlet and Neumann type problems in the present theory of bending of elastic plates. Existence theorems for the corresponding mixed problems where Dirichlet and Neumann-type data are prescribed separately on different parts of the boundary are examined in [**9**]. Another type of mixed problem known as the Robin problem for this theory of plates is considered in [**10**]. To the authors' knowledge, a rigorous analysis of the corresponding mixed boundary value problems of the third kind does not appear in the literature.

In [**7**], mixed boundary value problems of the third kind are posed for a general three-dimensional elastic body. However, existence theorems are proved using specially modified potentials constructed specifically to deal with the nonstandard boundary condition. This requires certain modifications to the conventional boundary integral equation method.

In this paper, in contrast to the results presented in [**7**], we show how conventional elastic potentials of the type (7) and (8) can indeed be used to solve the corresponding mixed boundary value problems of the third kind from this theory of bending of elastic plates.

4.1 Interior mixed problem. Consider first the interior mixed problem (I). We seek the solution in the form

$$\begin{aligned}(u(\varphi))(x) = &\int_{\partial S}[P(x,y) - \mathcal{P}(x,y)]\begin{bmatrix}0\\0\\\varphi_3\end{bmatrix}(y)\,ds(y)\\ &+ \int_{\partial S}[D(x,y) - \mathcal{D}(x,y)]\begin{bmatrix}\varphi_1\\\varphi_2\\0\end{bmatrix}(y)\,ds(y)\end{aligned} \tag{9}$$

where $\varphi = [\varphi_1\ \varphi_2\ \varphi_3]^T \in \mathcal{M}_{3\times1}$ is some unknown vector-density and $\mathcal{P}, \mathcal{D} \in \mathcal{M}_{3\times3}$ are known matrices constructed as follows.

Let Ω_1 be a bounded domain with C^2-boundary $\partial\Omega_1$ such that $S^+ \cup \partial S \subset \Omega_1$.

We denote the columns of the matrix $\mathcal{P}$ by $\mathcal{P}^{(j)}$ each of which satisfies the boundary value problem:

$$\begin{aligned} L(\partial x)\mathcal{P}^{(j)}(x,y) &= 0, & x &\in \Omega_1, \\ \mathcal{P}^{(j)}(x,y) &= P^{(j)}(x,y), & x &\in \partial\Omega_1. \end{aligned}$$

Similarly, the columns of the matrix, $\mathcal{D}$ denoted by $\mathcal{D}^{(j)}$, satisfy the boundary value problem:

$$\begin{aligned} L(\partial x)\mathcal{D}^{(j)}(x,y) &= 0, & x &\in \Omega_1, \\ \mathcal{D}^{(j)}(x,y) &= D^{(j)}(x,y), & x &\in \partial\Omega_1. \end{aligned}$$

Noting that for $y \in \partial S$ and $x \in \partial\Omega_1$, $P^{(j)}(x,y)$ and $D^{(j)}(x,y)$ are of the class $C^{1,\alpha}(\partial\Omega_1)$, $0 < \alpha < 1$, (recall that $\partial S \subset \Omega_1$), it follows from the existence result for the interior Dirichlet problem of plates [**1**], that $\mathcal{P}^{(j)}(x,y)$ and $\mathcal{D}^{(j)}(x,y)$ exist uniquely for each $y \in \partial S$ in the class $C^2(\Omega_1) \cap C^1(\bar{\Omega}_1)$. In fact, for each y, $\mathcal{P}^{(j)}(x,y)$ and $\mathcal{D}^{(j)}(x,y)$ take the form of elastic double layer potentials (8) [**1**].

Suppose that the unknown density φ in (9) is of the class $C^{1,\alpha}(\partial S)$, $0 < \alpha < 1$.

Using the regularity properties of the elastic single and double layer potentials (7) and (8) with $C^{1,\alpha}$-density [**1**] and the properties of the matrix $\mathcal{P}$ and $\mathcal{D}$, it is clear that u from (9) satisfies the continuity and differentiability conditions of the problem (I) and the governing equations (1) in S^+.

Consider next the remaining boundary condition

$$O_{33}Tu(x) + I_{33}u(x) = f(x), \quad x \in \partial S. \tag{10}$$

The boundary integral equation method seeks to reduce the boundary condition (10) to a system of Cauchy singular integral equations on the curve ∂S. At first glance, this would not seem possible since

(10) requires the application of the boundary stress operator T to the elastic double layer potential appearing in (9). Since this potential is already Cauchy singular on the boundary ∂S [**1**], a further application of T would lead to a system of hypersingular integral equations. This is the main reason why, in [**7**], the authors found it necessary to construct specially modified potentials to deal with a similar nonstandard boundary condition of the type (10). For the present boundary value problems, however, we overcome this difficulty with the use of the following lemma.

We denote by $\{E_{ij}\}$ the standard ordered basis for the set of (3×3)-matrices and by $\varepsilon_{\gamma\beta}$ the alternating tensor.

Lemma 1. *Using the results established in* [**1**], *it is not difficult to show that*:

$$\varepsilon_{\gamma\beta}E_{\gamma\beta}\int_{\partial S}\left[\frac{\partial}{\partial s(y)}\ln|x-y|\right]\begin{bmatrix}0\\0\\\varphi_3\end{bmatrix}(y)\,ds(y)=0,\quad x\in\partial S,$$

where the integral exists as principal value uniformly for all $x\in\partial S$ and is of the class $C^{1,\alpha}(\partial S)$, $0<\alpha<1$.

In view of Lemma 1 and the properties of the elastic single and double layer potentials (7) and (8) [**1**], it follows that using (9) in the remaining boundary condition (10) leads to the following system of Cauchy singular integral equations over the boundary ∂S for the unknown density $\varphi=\left[\begin{smallmatrix}\varphi_1 & \varphi_2 & \varphi_3\end{smallmatrix}\right]^T$:

(11)

$$A(z)\varphi(z)+\frac{B(z)}{\pi i}\int_{\partial S}\frac{\varphi(\zeta)}{\zeta-z}\,d\zeta+\int_{\partial S}\mathcal{H}(z,\zeta)\varphi(\zeta)\,d\zeta=f(z),\quad z\in\partial S.$$

Here $z=x_1+ix_2$, $\zeta=\zeta_1+i\zeta_2$,

$$A(z)=\begin{bmatrix}\frac12 & 0 & *\\ 0 & \frac12 & *\\ 0 & 0 & -\frac12\end{bmatrix},\qquad B(z)=-\frac{ic_2}{2}\begin{bmatrix}0 & 1 & *\\ -1 & 0 & *\\ 0 & 0 & 0\end{bmatrix},$$

c_2 is the material constant defined by $c_2=(\mu/(\lambda+2\mu))$, '$*$' denotes a (possibly) nonzero entry whose specific form is not required and $\mathcal{H}$ is a weakly singular (Fredholm) kernel.

Clearly, with the conditions $\lambda + \mu > 0$ and $\mu > 0$ assumed above,

$$\det(A \pm B) = -\frac{1}{8}(1 - c_2^2) \neq 0 \quad \text{everywhere on } \partial S.$$

This means that the Cauchy singular integral equation (11) is of the regular type and is accommodated by the results presented in [**8**]. Furthermore, the index [**8**] of the singular integral operator from (11) is given by:

$$\kappa = \frac{1}{2\pi}\left[\arg \frac{\det(A - B)}{\det(A + B)}\right] = \frac{1}{2\pi} \arg(1) = 0,$$

which means that the Fredholm alternative holds for the system (11) and its associated system in the (real) dual system $(C^{0,\alpha}(\partial S), C^{0,\alpha}(\partial S))$, $\alpha \in (0,1)$, with the bilinear form

$$(\varphi, \psi) = \int_{\partial S} \varphi^T(y)\psi(y)\, ds(y).$$

Before we proceed with the main existence results, we note the following lemmas.

Lemma 2. *As in* [**1**], *we can prove that if* $f \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, *then any solution* $\varphi \in C^{0,\alpha}(\partial S)$ *of* (11) *belongs to* $C^{1,\alpha}(\partial S)$.

Lemma 3. *Let* $\Upsilon \subset \Omega_1$ *be the domain lying between* ∂S *and* $\partial\Omega_1$. *Using* (5) *it is not difficult to show that the following boundary value problem has only the trivial solution in* Υ:

Find $u \in C^2(\Upsilon) \cap C^1(\bar{\Upsilon})$ satisfying (1) in Υ and such that

$$\begin{aligned} O_{33}Tu(x) + I_{33}u(x) &= 0, \quad x \in \partial S, \\ u(x) &= 0, \quad x \in \partial\Omega_1. \end{aligned}$$

Theorem 2. *The homogeneous system* $(11)^0$ ((11) *with* $f \equiv 0$) *has only the trivial solution in the space* $C^{0,\alpha}(\partial S)$.

Proof. In view of Lemma 2, it suffices to prove the assertion in $C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$. Let $\varphi_0 \in C^{1,\alpha}(\partial S)$ be a solution of $(11)^0$. Then $(u(\varphi_0))(x)$ from (9) solves the homogeneous mixed problem $(I)^0$. Theorem 1 now yields $(u(\varphi_0))(x) = 0$, $x \in S^+$. Furthermore, since by construction u from (9) satisfies (1) in Υ (as defined in Lemma 3) and (using the properties of the matrices $\mathcal{P}$ and $\mathcal{D}$) $(u(\varphi_0))(x) = 0$, $x \in \partial\Omega_1$, we have, from Lemma 3, that $(u(\varphi_0))(x) = 0$, $x \in \Upsilon$. Thus $(u(\varphi_0))(x)$ vanishes on both sides of the boundary ∂S. From (9), using the continuity of the elastic single layer potential and the jump relations arising from the elastic double layer potential (8) as we cross the boundary ∂S [**1**], we obtain:

$$\varphi_3(x) = 0, \quad x \in \partial S.$$

Similarly, applying the T-operator to (9), noting that the quantity TW is continuous across the boundary ∂S [**1**] and the jump relations arising from the application of the T operator to the elastic single layer potential (7) as we cross the boundary ∂S, we obtain:

$$\varphi_\alpha(x) = 0, \quad x \in \partial S,$$

which completes the proof.

Theorem 3. *The system* (11) *is uniquely solvable for any* $f \in C^{1,\alpha}(\partial S)$, $0 < \alpha < 1$.

Proof. Using Theorem 2, since the Fredholm alternative applies to (11) and its associated system, the latter also has only the trivial solution in the space $C^{0,\alpha}(\partial S)$. This means that (11) is uniquely solvable for any $f \in C^{0,\alpha}(\partial S)$ and hence any $f \in C^{1,\alpha}(\partial S)$, $0 < \alpha < 1$.

From Theorems 1–3, we have the main existence result for the interior mixed problem (I):

Theorem 4. *The problem* (I) *has a unique solution for any* $f \in C^{1,\alpha}(\partial S)$, $0 < \alpha < 1$. *This solution is given by* (9) *with* $\varphi \in C^{1,\alpha}(\partial S)$, *the unique solution of the system* (11).

Remark 2. The condition $f \in C^{1,\alpha}(\partial S)$, $0 < \alpha < 1$, is sufficient but not necessary for the solvability of the problem (I). In fact, from the properties of the elastic double and single layer potentials (7) and (8), the solution (9) of the problem (I) requires only that

$$\varphi_\alpha \in C^{0,\alpha}(\partial S) \quad \text{and} \quad \varphi_3 \in C^{1,\alpha}(\partial S), \quad 0 < \alpha < 1,$$

which, in turn, requires that the conditions on the prescribed data can be weakened slightly to:

$$f_\alpha \in C^{0,\alpha}(\partial S) \quad \text{and} \quad f_3 \in C^{1,\alpha}(\partial S), \quad 0 < \alpha < 1.$$

4.2 Exterior mixed problem. In the case of the exterior mixed problem (E), the asymptotic behavior (4) of the matrix $D(x, y)$ suggests that we seek the solution in the form

$$\begin{aligned}(u(\varphi))(x) & \\ &= \int_{\partial S} [P(x,y) - \mathcal{P}_1(x,y)] \begin{bmatrix} 0 \\ 0 \\ \varphi_3 \end{bmatrix} (y)\, ds(y) \\ &\quad + \int_{\partial S} [D(x,y) - M^\infty(x) F^T(y) - \mathcal{D}_1(x,y)] \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ 0 \end{bmatrix} (y)\, ds(y)\end{aligned} \tag{12}$$

where $\varphi = [\varphi_1\ \varphi_2\ \varphi_3]^T \in \mathcal{M}_{3\times 1}$ is again some unknown vector-density and $M^\infty \in \mathcal{M}_{3\times 3}$ has columns [**3**]:

$$\begin{aligned} M^{\infty(1)}(r,\theta) = &- a_1\mu(\mu(2\ln r + 2 + \cos 2\theta),\ \mu \sin 2\theta, \\ &- (\mu r(2\ln r + 1) - 4h^2(\lambda + 2\mu) r^{-1}) \cos\theta)^T \end{aligned}$$

$$\begin{aligned} M^{\infty(2)}(r,\theta) = &- a_1\mu(\mu \sin 2\theta, \mu(2\ln r + 2 - \cos 2\theta), \\ &- (\mu r(2\ln r + 1) - 4h^2(\lambda + 2\mu) r^{-1}) \sin\theta)^T, \end{aligned}$$

$$\begin{aligned} M^{\infty(3)}(r,\theta) = &- a_1\mu(2\ln r + 1)\cos\theta, \mu r(2\ln r + 1)\sin\theta, \\ &- \mu r^2 \ln r + 4h^2(\lambda + 2\mu)\ln r + 4h^2(\lambda + 3\mu))^T, \end{aligned}$$

with polar coordinates (r, θ) given by $r = |x|$ and $\theta = \tan^{-1}(x_2/x_1)$. It is easily verified that $LM^\infty = 0$ in $\mathbf{R}^2 \setminus \{0\}$. The matrices

$\mathcal{P}_1, \mathcal{D}_1 \in \mathcal{M}_{3\times3}$ are known matrices constructed using a procedure similar to that used to construct the matrices $\mathcal{P}$ and $\mathcal{D}$ for problem (I). That is, let Ω_2 be an infinite domain with closed C^2-boundary $\partial\Omega_2$ such that

(i) $S^- \subset \Omega_2$,

(ii) $\partial S \subset \Omega_2$,

(iii) $\{0\} \notin \bar{\Omega}_2$.

We denote the columns of the matrix $\mathcal{P}_1$ by $\mathcal{P}_1^{(j)}$, each of which satisfies the boundary value problem:

$$\begin{aligned} L(\partial x)\mathcal{P}_1^{(j)}(x,y) &= 0, && x \in \Omega_2, \\ \mathcal{P}_1^{(j)}(x,y) &= P^{(j)}(x,y), && x \in \partial\Omega_2. \end{aligned}$$

Similarly, the columns of the matrix, $\mathcal{D}_1$, denoted by $\mathcal{D}_1^{(j)}$, satisfy the boundary value problem:

$$\begin{aligned} L(\partial x)\mathcal{D}_1^{(j)}(x,y) &= 0, && x \in \Omega_2, \\ \mathcal{D}_1^{(j)}(x,y) &= G^{(j)}(x,y), && x \in \partial\Omega_2, \end{aligned}$$

where $G \in \mathcal{M}_{3\times3}$ is given by $G(x,y) = D(x,y) - M^{\infty}(x)F^T(y)$. Noting that for $y \in \partial S$ and $x \in \partial\Omega_2$, $P^{(j)}(x,y)$ and $G^{(j)}(x,y)$ are of the class $C^{1,\alpha}(\partial\Omega_2)$, $0 < \alpha < 1$, (recall that $\partial S \subset \Omega_2$) it again follows from the existence result for the exterior Dirichlet problem in this theory of plates [**1**] that $\mathcal{P}_1^{(j)}(x,y)$ and $\mathcal{D}_1^{(j)}(x,y)$ exist uniquely for each $y \in \partial S$ in the class $C^2(\Omega_2) \cap C^1(\bar{\Omega}_2) \cap \mathcal{A}^*$. In fact, for each y, $\mathcal{P}_1^{(j)}(x,y)$ and $\mathcal{D}_1^{(j)}(x,y)$ take the form of the sum of a double layer potential and a matrix of the form (3).

Suppose that the unknown density φ in (12) is of the class $C^{1,\alpha}(\partial S)$, $0 < \alpha < 1$.

As in the case of problem (I), using the regularity properties of the elastic single and double layer potentials (7) and (8) with $C^{1,\alpha}$-density [**1**] and the properties of the matrices $\mathcal{P}_1, \mathcal{D}_1$ and M^{∞}, we clearly see that u from (12) satisfies the continuity and differentiability conditions of the problem (E) and the governing equations (1) in S^-. The fact that $u \in \mathcal{A}^*$ is proved as follows. If we define $\phi = [\varphi_1\ \varphi_2\ 0]$, as in [**3**], letting $|x| \to \infty$,

$$\begin{aligned}\int_{\partial S} [D(x,y) - M^\infty(x)F^T(y)]\phi(y)\,ds(y) \\ &= M^\infty(x)\int_{\partial S} F^T(y)\phi(y)\,ds(y) + u_0 \\ &\quad - M^\infty(x)\int_{\partial S} F^T(y)\phi(y)\,ds(y) \\ &= u_0 \in \mathcal{A}.\end{aligned}$$

Also,

$$\int_{\partial S} \mathcal{D}_1(x,y)\phi(y)\,ds(y) = \int_{\partial S} \mathcal{D}_1^{(j)}(x,y)\phi_j(y)\,ds(y) \in \mathcal{A}^*$$

since, as noted above, $\mathcal{D}_1^{(j)}(x,y) \in \mathcal{A}^*$ for each y. Finally, the term

$$\int_{\partial S} [P(x,y) - \mathcal{P}_1(x,y)] \begin{bmatrix} 0 \\ 0 \\ \varphi_3 \end{bmatrix} (y)\,ds(y)$$

is also in $\mathcal{A}^*$ by the properties of the elastic double layer potential (8) [**1**] and the fact that $\mathcal{P}_1^{(j)}(x,y) \in \mathcal{A}^*$ for each y as discussed above. Hence, $u(x) \in \mathcal{A}^*$.

The remaining boundary condition

$$O_{33}Tu(x) + I_{33}u(x) = g(x), \quad x \in \partial S$$

again leads to a system of Cauchy singular integral equations with zero index similar in type and form to that in (11) for the interior problem. Consequently, using the same arguments used above for problem (I), we can prove the following existence theorem for problem (E).

Theorem 5. *The problem* (E) *has a unique solution for any* $g \in C^{1,\alpha}(\partial S)$*,* $0 < \alpha < 1$*. This solution is given by* (12) *with* $\varphi \in C^{1,\alpha}(\partial S)$ *the unique solution of the corresponding system of Cauchy singular integral equations.*

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REFERENCES

1. C. Constanda, *A mathematical analysis of bending of plates with transverse shear deformation*, Longman, Harlow, 1990.

2. ———, *Some comments on the integration of certain systems of partial differential equations in continuum mechanics*, J. Math. Phys. **29** (1978), 835–839.

3. ———, *On integral solutions of the equations of thin plates*, Proc. Roy. Soc. London Ser. A **444** (1994), 317–323.

4. A.C. Eringen, *Linear theory of micropolar elasticity*, J. Appl. Math. Mech. **15** (1966), 909–923.

5. ———, *Theory of micropolar plates*, J. Math. Phys. **18** (1967), 12–30.

6. V.D. Kupradze, G. Gegelia, M.D. Basheleishvill and T.V. Burchuladze, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*, North-Holland, Amsterdam, 1979.

7. V.D. Kupradze, *Potential methods in the theory of elasticity*, Israel Program for Scientific Translations, Jerusalem, 1965.

8. N.I. Muskhelishvili, *Singular integral equations*, P. Noordhoff, Groningen, 1953.

9. P. Schiavone, *Mixed problems in the theory of bending of elastic plates with transverse shear deformation*, Quart. J. Mech. Appl. Math. **50** (1997), 239–249.

10. ———, *On the Robin problem for the equations of thin plates*, J. Integral Equations Appl. **8** (1996), 231–238.

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