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STABILITY OF THE SPLINE COLLOCATION METHOD FOR VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. Numerical stability of the spline collocation method for Volterra integral equations is investigated. General stability conditions are obtained and applied to the most practical types of splines. Results of several numerical tests are presented.

1. Introduction. We study the stability of the polynomial spline collocation method applied to Volterra integral equations of the second kind. Stability means here the boundedness of approximate solutions in uniform norm when the number of knots increases. A stability condition for a test type equation is found. Some general results are established and applied to several particular cases including linear, quadratic, and cubic splines. In many practical cases we get explicit formulae showing the dependence of the stability on collocation parameters. A series of numerical tests is given, and they support well the theoretical results.

2. The spline collocation method. We start with the description of the spline collocation method for the Volterra integral equation

$$y(t) = \int_0^t \mathcal{K}(t,s,y(s))\,ds + f(t), \quad t \in [0,T], \tag{2.1}$$

with given functions $f : [0,T] \to \mathbf{R}$, $\mathcal{K} : S \times \mathbf{R} \to \mathbf{R}$, and set $S = \{(t,s) : 0 \le s \le t \le T\}$.

A mesh $0 = t_0 < t_1 < \dots < t_N = T$ will be used and the process $N \to \infty$ is allowed, thus the t_n are dependent on N. Denote $h_n = t_n - t_{n-1}$ and $\sigma_n = (t_{n-1}, t_n]$, $n = 1, \dots, N$, $\Delta_N = \{t_1, \dots, t_{N-1}\}$.

For given integers $m \ge 1$ and $d \ge -1$, define the space of splines

$$S_{m+d}^d(\Delta_N) = \{u \in C^d[0,T] : u\,|_{\sigma_n} \in \mathcal{P}_{m+d},\ n = 1, \dots, N\},$$

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where $\mathcal{P}_k$ means the set of all polynomials with degree not exceeding k. The restriction u_n of a spline $u \in S^d_{m+d}$ on σ_n could be represented

$$u_n(t) = \sum_{k=0}^{m+d} b_{nk}(t - t_{n-1})^k, \quad t \in \sigma_n. \tag{2.2}$$

From $u \in C^d$ we get certain linear restrictions for the coefficients b_{nk}, and we give them in explicit form in the next section.

Suppose that there is a fixed selection of collocation parameters $0 < c_1 < \cdots < c_m \le 1$. Then define collocation points $t_{nj} = t_{n-1} + c_j h_n$, $j = 1, \dots, m$, $n = 1, \dots, N$, forming a set $X(N)$. In order to determine the approximate solution $u \in S^d_{m+d}(\Delta_N)$ of the equation (2.1), we impose the following collocation conditions

$$u(t) = \int_0^t \mathcal{K}(t, s, u(s))\, ds + f(t), \quad t \in X(N). \tag{2.3}$$

Starting the calculations by this method, we assume also that we can use the initial values $u_1^{(j)}(0) = y^{(j)}(0)$, $j = 0, \dots, d$ (or at least some approximations of them) which is justified by the requirement $u \in C^d[0,T]$. Another possible approach is to use only $u_1(0) = y(0) = f(0)$ (if $d \ge 0$) and more collocation points (if $d \ge 1$) to determine u_1. Thus, on every interval σ_n we have $d+1$ conditions of smoothness and m collocation conditions to determine $m+d+1$ parameters b_{nk}. This allows us to realize the method step-by-step going from an interval σ_n to the next one.

3. The method in the case of a test equation. Consider the test equation

$$y(t) = \lambda \int_0^t y(s)\, ds + f(t), \quad t \in [0,T], \tag{3.1}$$

where, in general, λ may be any complex number. Let in the sequel Δ_N be uniform, i.e., $h_n = h = T/N$ for all n. Representing $t \in \sigma_n$ as $t = t_{n-1} + \tau h$, $\tau \in (0,1]$, we have on σ_n

$$u_n(t_{n-1} + \tau h) = \sum_{k=0}^{m+d} a_{nk} \tau^k, \quad \tau \in (0,1],$$

where we passed to the parameters $a_{nk} = b_{nk}h^k$.

First, write down the smoothness conditions (for any $u \in S^d_{m+d}(\Delta_N)$)

$$u_n^{(j)}(t_n - 0) = u_{n+1}^{(j)}(t_n + 0),$$
$$j = 0, \dots, d, \quad n = 1, \dots, N-1.$$

We have

$$u_n^{(j)}(t_{n-1} + \tau h) = \frac{1}{h^j} \sum_{k=j}^{m+d} \frac{k!}{(k-j)!} a_{nk} \tau^{k-j}.$$

Hence

$$u_n^{(j)}(t_n - 0) = \frac{1}{h^j} \sum_{k=j}^{m+d} \frac{k!}{(k-j)!} a_{nk},$$
$$u_{n+1}^{(j)}(t_n + 0) = \frac{1}{h^j} j! a_{n+1,j}$$

and

$$a_{n+1,j} = \sum_{k=j}^{m+d} \frac{k!}{(k-j)!j!} a_{nk}, \tag{3.2}$$
$$j = 0, \dots, d, \quad n = 1, \dots, N-1.$$

The collocation conditions (2.3) applied to the test equation (3.1) give

$$u(t_{nj}) = \lambda \int_0^{t_{nj}} u(s)\, ds + f(t_{nj}),$$
$$j = 1, \dots, m, \quad n = 1, \dots, N.$$

Denote $\alpha_n = (a_{nk})_{k=0}^{m+d}$. Then we get

$$\begin{aligned}\sum_{k=0}^{m+d} a_{nk} c_j^k &= \sum_{r=1}^{n-1} \lambda \int_{t_{r-1}}^{t_r} u_r(s)\, ds + \lambda \int_{t_{n-1}}^{t_{nj}} u_n(s)\, ds + f(t_{nj}) \\ &= \sum_{r=1}^{n-1} \lambda h \int_0^1 \left(\sum_{k=0}^{m+d} a_{rk} \tau^k \right) d\tau + \lambda h \int_0^{c_j} \left(\sum_{k=0}^{m+d} a_{nk} \tau^k \right) d\tau \\ &\quad + f(t_{nj}) = \sum_{r=1}^{n-1} \lambda h \langle \alpha_r, q \rangle + \lambda h \sum_{k=0}^{m+d} a_{nk} \frac{c_j^{k+1}}{k+1} + f(t_{nj})\end{aligned}$$

or

$$\sum_{k=0}^{m+d} a_{nk}\left(1-\frac{\lambda h c_j}{k+1}\right)c_j^k = \lambda h\left\langle q, \sum_{r=1}^{n-1}\alpha_r\right\rangle + f(t_{nj}) \tag{3.3}$$

where $q = (1, 1/2, \ldots, 1/(m+d+1))$ and $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbf{R}^{m+d+1}$. The difference of the equations (3.3) with n and $n+1$ yields

$$\begin{aligned}\sum_{k=0}^{m+d} a_{n+1,k}&\left(1-\frac{\lambda h c_j}{k+1}\right)c_j^k\\ &= \sum_{k=0}^{m+d} a_{nk}\left(1-\frac{\lambda h c_j}{k+1}\right)c_j^k + \lambda h\langle q, \alpha_n\rangle + f(t_{n+1,j}) - f(t_{nj}),\\ &\qquad j = 1,\ldots,m, \quad n = 1,\ldots,N-1.\end{aligned} \tag{3.4}$$

Now we may write together the equations (3.2) and (3.4) in matrix form,

$$\begin{gathered}(V-\lambda h V_1)\alpha_{n+1} = (V_0 - \lambda h(V_1 - V_2))\alpha_n + g_n,\\ n = 1,\ldots,N-1,\end{gathered} \tag{3.5}$$

with $(m+d+1)\times(m+d+1)$ matrices V, V_0, V_1, V_2 as follows:

$$V = \left(\begin{array}{c|c} I & 0 \\ \hline \multicolumn{2}{c}{C}\end{array}\right), \qquad V_0 = \left(\begin{array}{c|c} A & B \\ \hline \multicolumn{2}{c}{C}\end{array}\right),$$

I being the $(d+1)\times(d+1)$ unit matrix,

$$C = \begin{pmatrix} 1 & c_1 & \cdots & c_1^{m+d} \\ \hdashline 1 & c_m & \cdots & c_m^{m+d}\end{pmatrix},$$

A being a $(d+1)\times(d+1)$ triangular matrix with ones on the main diagonal and zeros below,

$$V_1 = \begin{pmatrix} & & 0 & \\ c_1 & c_1^2/2 & \cdots & c_1^{m+d+1}/(m+d+1) \\ \hdashline c_m & c_m^2/2 & \cdots & c_m^{m+d+1}/(m+d+1)\end{pmatrix},$$

V_2 having first $d+1$ rows 0 and last m rows the vector q, and, finally, the $m+d+1$-dimensional vector $g_n = (0, \dots, 0, f(t_{n+1,1}) - f(t_{n1}), \dots, f(t_{n+1,m}) - f(t_{nm}))$. Thus, $g_n = O(h)$ for $f \in C^1$.

As V is invertible, so also is $V - \lambda h V_1$ for small h, and (3.5) can be written in the form

$$\alpha_{n+1} = (V^{-1}V_0 + W)\alpha_n + r_n$$

where $W = O(h)$, $r_n = O(h)$. Note that $W = 0$ if $\lambda = 0$. Set $M = V^{-1}V_0$.

4. Stability of the method. We have seen that the spline collocation method (2.3) for the test equation (3.1) leads to the iteration process

$$\alpha_{n+1} = (M+W)\alpha_n + r_n, \quad n = 1, \dots, N-1, \tag{4.1}$$

with $W = O(h)$, $r_n = O(h)$.

We distinguish the method with initial values $u_1^{(j)}(0) = y^{(j)}(0)$, $j = 0, \dots, d$, and another method which uses only $u_1(0) = y(0)$ and additional collocation points $t_{0j} = t_0 + c_{0j}h$, $j = 1, \dots, d$, with fixed $c_{0j} \in (0,1] \setminus \{c_1, \dots, c_m\}$ on the first interval σ_1. Denote $d_0 = \max\{d, 0\}$, $d_1 = \max\{d, 1\}$ for the method with initial values and $d_1 = 1$ for the method with additional initial collocation.

Definition 1. We say that the spline collocation method is stable if, for any $\lambda \in \mathbf{C}$ and any $f \in C^{d_1}[0,T]$, the approximate solution u remains bounded in $L_\infty(0,T)$ as $h \to 0$.

Let us notice that the boundedness of $\|u\|_{L_\infty(0,T)}$ is equivalent to the boundedness of $\|\alpha_n\|$ in n and h in any fixed norm of $\mathbf{R}^{m+d+1}$.

Proposition 2. *The spline collocation method is stable if and only if*

$$\|u\|_{L_\infty(0,T)} \le \text{const}\, \|f\|_{C^{d_1}[0,T]} \quad \forall\, f \in C^{d_1}[0,T], \tag{4.2}$$

where the constant may depend only on T, λ and on parameters c_j and c_{0j}.

Proof. We have to prove the "only if" part. But this follows from the principle of uniform boundedness.

Proposition 3. *If there exists a norm of* $\mathbf{R}^{m+d+1}$ *such that the corresponding matrix norm yields* $\|M\| \leq 1$, *then the spline collocation method is stable.*

Proof. If we use the initial values $a_{1j} = h^j y^{(j)}(0)/j!$, $j = 0, \dots, d$, then by (3.3) for $n = 1$ we have

$$(V - \lambda h V_1)\alpha_1 = g_0 \tag{4.3}$$

where $g_0 = (\alpha_{10}, \dots, \alpha_{1d}, f(t_{11}), \dots, f(t_{1n}))$. This enables us to estimate $\|g_0\|_\infty$ and thus $\|\alpha_1\|$ by const $\|f\|_{C^{d_0}[0,T]}$ because of

$$y^{(j)}(0) = \sum_{k=0}^{j} \lambda^{j-k} f^{(k)}(0). \tag{4.4}$$

The other choice of initial values $a_{10} = y(0) = f(0)$ and $a_{1j} = f(t_{0j})$, $j = 1, \dots, d$, gives $\|\alpha_1\| \leq \text{const}\, \|f\|_{C[0,T]}$. Since we have $\|\alpha_1\| \leq L_0$, $\|W\| \leq Kh$ and $\|r_n\| \leq Lh$, induction yields

$$\|\alpha_n\| \leq (1 + Kh)^{n-1} L_0 + \sum_{k=0}^{n-2} (1 + Kh)^k Lh.$$

Hence, taking into account $Nh = T$,

$$\|\alpha_n\| \leq (1 + Kh)^N L_0 + \sum_{k=0}^{N-1} (1 + Kh)^k Lh \leq e^{KT}(L_0 + LT)$$

which completes the proof.

Actually, for $n \geq 1$, we have $\|g_n\| \leq h$ const $\|f\|_{C^1[0,T]}$, and the proof gives once more the estimate $\|\alpha_n\| \leq$ const $\|f\|_{C^{d_1}[0,T]}$.

We will use the following fact: there exists a vector norm such that the corresponding matrix norm is equal to the spectral radius of the matrix if and only if all eigenvalues with maximal modulus have equal algebraic and geometric multiplicities.

Proposition 4. *The matrix M has eigenvalue $\mu = 1$ with geometric multiplicity m.*

Proof. It is clear that $\operatorname{Ker}(M - \mu I) = \operatorname{Ker}(V_0 - \mu V)$. The geometric multiplicity of $\mu = 1$ is $\dim \operatorname{Ker}(V_0 - V)$. But $\dim \operatorname{Ker}(V_0 - V) = m + d + 1 - \operatorname{rank}(V_0 - V)$. As $\operatorname{rank}(V_0 - V) = d + 1$, we get the assertion.

Proposition 5. *If all eigenvalues of M are in the closed unit disk and those which are on the unit circle have equal algebraic and geometric multiplicities, then the spline collocation method is stable. If M has an eigenvalue outside of the closed unit disk, then the spline collocation method is not stable.*

Proof. The first claim is proved in Proposition 3.

In order to prove the second one, suppose that M has an eigenvalue outside of the closed unit disk. Then $M + W$ also has an eigenvalue μ such that $|\mu| \geq 1 + \delta$ with some fixed $\delta > 0$ for any sufficiently small h. This property is based on the continuous dependence of the eigenvalues on the elements of the matrix. Fix for a moment a sufficiently small $h = T/N$. Then, for $\alpha_1 \neq 0$ (depending on h) such that $(M + W)\alpha_1 = \mu\alpha_1$ and $r_n = 0$, $n \geq 1$, we have $\|\alpha_{n+1}\| \geq (1 + \delta)^n \|\alpha_1\|$. In the case of the method of initial values the vector α_1 determines via (4.3) and (4.4) the values $f^{(j)}(0)$, $j = 0, \dots, d$, $f(t_{11}), \dots, f(t_{1m})$. Take f on $[0, h]$ as the polynomial interpolating the values $f^{(j)}(0)$, $j = 0, \dots, d$, $f(t_{1j})$, $j = 1, \dots, m$, and $f^{(j)}(h) = 0$, $j = 0, \dots, d_1$. In the case of the method of additional knots, let f be on $[0, h]$ the interpolating polynomial by the data $f(0)$, $f(t_{0j})$, $j = 0, \dots, d$, $f(t_{1j})$, $j = 1, \dots, m$, and $f^{(j)}(h) = 0$, $j = 0, \dots, d_1$ (here $d_1 = 1$). In both cases we ask f to be on $[nh, (n+1)h]$, $n \geq 1$, the interpolating polynomial by the values $f^{(j)}(nh) = 0$ and $f^{(j)}((n+1)h) = 0$, $j = 0, \dots, d_1$, $f(t_{n+1,j}) = f(t_{1j})$, $j = 1, \dots, m$. This guarantees that $f \in C^{d_1}[0, T]$, and we also get $r_n = 0$, $n \geq 1$. For example, the limit process in the classical Newton interpolation formula with single knots permits us to get the interpolant

f on $[t_n, t_{n+1}]$ as follows:

$$f(t) = f(t_n + \tau h) = \sum_{i=0}^{\kappa} \Bigg(\sum_{l=0}^{k_i} h^{s_l} p_{il} f^{(s_l)}(\xi_l) \Bigg) \prod_{r=0}^{i-1} (\tau - b_r) \tag{4.5}$$

with b_r being c_j or c_{0j}, ξ_l being t_{nj} or t_j, $0 \le s_l \le d_1$, $k_i \le i$, constants p_{il} depending on c_j and c_{0j}, $\kappa = m + d + d_1 + 1$ on the interval $[0, h]$ and $\kappa = m + 2d_1 + 1$ on the interval $[nh, (n+1)h]$, $n \ge 1$. We see that $s_l \ge 0$ holds only for $\xi_l = t_j$ and indeed it may be $f^{(s_l)}(\xi_l) \ne 0$ for $s_l > 0$ only in the method of initial values if $\xi_l = 0$.

Now replace h by h/k, $k = 1, 2, \ldots$, and keep $\|\alpha_1\| = 1$. Then $\|g_0\|_\infty$ remains bounded which means that $f(t_{1j})$, $j = 1, \ldots, m$, and $h^j y^{(j)}(0)/k^j$ or $h^j f^{(j)}(0)/k^j$, $j = 0, \ldots, d$, are bounded, too, in the process $k \to \infty$. Thus, (4.5) gives

$$\|f\|_{C^{d_1}[0,T]} \le \text{ const}\, k^{d_1}.$$

On the other hand

$$\|\alpha_{kN}\| \ge (1 + \delta)^{kN-1}$$

which yields that (4.2) cannot be satisfied. The proof is complete. $\square$

Let us now discuss the case where all eigenvalues of M are in the closed unit disk but one of them, say μ, belongs to the unit circle and has different algebraic and geometric multiplicities. Suppose $\lambda = 0$ and f is such that $r_n = 0$, $n \ge 1$. Then $\alpha_{n+1} = M^n \alpha_1$ and, from the Jordan decomposition $M = P^{-1} J P$ we see that $\alpha_{n+1} = P^{-1} J^n P \alpha_1$. The matrix J^n has at least one element $n\mu^{n-1}$. Thus, in the practical calculations, inevitable round-off errors in α_1 generate an increasing term which spoils possible stable behavior of the iteration process for some good function f. Actually, this instability is weak and cannot be observed (see the numerical tests, the case $d = 1$, $m = 2$, $c_1 = 0.5$, $c_2 = 1$). If there is a Jordan block of dimension $k > 2$ corresponding to μ, then J^n contains the elements of order n^{k-1} and the influence of round-off errors increases. But in practice only relatively low order splines are used and this phenomena does not take place (for example, in the case $d = 1$, $m = 2$ the eigenvalue $\mu = 1$ having geometric multiplicity 2 may have maximal algebraic multiplicity 3). Thus, the method is practically stable if and only if all eigenvalues of M are in the closed unit disk.

5. Convergence. For any $f \in C^{d_1}[0,T]$, the test equation (3.1) has the unique solution $y \in C^{d_1}[0,T]$. Let $u \in S^d_{m+d}(\Delta_n)$ be the approximate solution of (3.1). Here we are interested in the convergence of u to y in $L_\infty(0,T)$ in the process $h \to 0$.

Proposition 6. *The spline collocation method is convergent for the test equation* (3.1) *if and only if it is stable.*

Proof. The "only if" part can be proved by a standard argument based upon the principle of uniform boundedness.

Suppose that the method is stable. Take $y_h \in S^{d_1}_{m+d}(\Delta_N)$ such that $y_h \to y$ in $C^{d_1}[0,T]$. Let

$$g_h(t) = y_h(t) - \lambda \int_0^t y_h(s)\,ds, \quad t \in [0,T]. \tag{5.1}$$

Then

$$u(t) - y_h(t) = \lambda \int_0^t (u(s) - y_h(s))\,ds + f(t) - g_h(t), \\ t \in X(N),$$

(here the set $X(N)$ is supposed to contain also t_{0j}, $j = 1, \dots, d$, in the case of additional initial collocation) and with the help of (4.4)

$$\begin{aligned} u^{(j)}(0) - y_h^{(j)}(0) &= y^{(j)}(0) - y_h^{(j)}(0) \\ &= \sum_{k=0}^{j} \lambda^{j-k}(f^{(k)}(0) - g_h^{(k)}(0)), \quad j = 0, \dots, d_1, \end{aligned}$$

(the additional initial collocation needs only $j = 0$). By the stability

$$\|u - y_h\|_{L_\infty(0,T)} \le \text{const}\, \|f - g_h\|_{C^{d_1}[0,T]}.$$

But (3.1) and (5.1) yield $\|f - g_h\|_{C[0,T]} \to 0$. After that the differentiation of (3.1) and (5.1) give $\|f^{(j)} - g_h^{(j)}\|_{C[0,T]} \to 0$, $j = 0, \dots, d_1$, which completes the proof.

6. Examples. In this section we investigate special cases of d and m.

Case $d = -1$, $m \geq 1$. Then $V = V_0 = C$ and the eigenvalue $\mu = 1$ has algebraic and geometric multiplicity m. Thus, the method is stable. Let us point out here the special case $m = 1$. Then the method gives u as a piecewise constant function with the values

$$u_n = \frac{1}{1 - \lambda h c_1}\left(f(t_{n1}) + \frac{\lambda h}{1 - \lambda h c_1}\sum_{k=1}^{n-1}\left(\frac{1 + \lambda h(1 - c_1)}{1 - \lambda h c_1}\right)^{n-k-1} f(t_{k1})\right).$$

Using the inequalities $|1+\lambda h(1-c_1)| \leq 1+K_1 h$ and $|1-\lambda h c_1| \geq 1-K_2 h$ for some positive K_1 and K_2, we get

$$\left|\frac{1 + \lambda h(1 - c_1)}{1 - \lambda h c_1}\right|^{n-k+1} \leq \left(\frac{1 + K_1 h}{1 - K_2 h}\right)^N \longrightarrow e^{(K_1+K_2)T}.$$

Thus, u is bounded (uniformly in N) for any $f \in C[0,T]$.

It is well known, see, e.g., [**2**], that the collocation in $S_m^{-1}(\Delta_N)$ applied to the general equation (2.1) converges for any choice of parameters c_i.

Case $d = 0$, $m = 1$ (*space* $S_1^0(\Delta_N)$, *linear splines*). We have

$$V = \begin{pmatrix} 1 & 0 \\ 1 & c_1 \end{pmatrix}, \qquad V_0 = \begin{pmatrix} 1 & 1 \\ 1 & c_1 \end{pmatrix},$$

and the equation $\det(V_0 - \mu V) = 0$ besides $\mu = 1$ has the solution $\mu = 1 - 1/c_1$. The method is stable if and only if $1/2 \leq c_1 \leq 1$.

Case $d = 0$, $m = 2$ ($S_2^0(\Delta_N)$). By Proposition 4, $\mu = 1$ is a solution of $\det(V_0 - \mu V) = 0$ of geometric multiplicity 2. The third solution $\mu(c_1, c_2) = 1 - (c_1 + c_2 - 1)/c_1 c_2$ shows that if $c_1 + c_2 \leq 1$ the method is unstable. Suppose $c_1 + c_2 > 1$. Then $1/2 < c_2 \leq 1$. As $\mu(c_1, 1) = 0$, only the possibility $1/2 < c_2 < 1$ needs some analysis. Then $1 - c_2 < c_1 < c_2$. As $\mu(1 - c_2, c_2) = 1$, $0 < \mu(c_2, c_2) < 1$ and $\mu(c_1, c_2)$ is strictly decreasing in c_1, we conclude that $0 \leq \mu(c_1, c_2) < 1$ for $c_1 + c_2 > 1$ which yields the stability.

Case $d = 1$, $m = 1$ ($S_2^1(\Delta_N)$, *quadratic splines*). Here the geometric multiplicity of $\mu = 1$ as solution of the equation $\det(V_0 - \mu V) = 0$ is 1. We also get $c_1^2\nu^2 - (2c_1 + 1)\nu + 2 = 0$ with $\nu = 1 - \mu$. From $\nu = (1+2c_1 \pm (1+4c_1(1-c_1))^{1/2})/2c_1^2$, we see that $\nu > 0$ and thus $\mu < 1$. For $c_1 = 1$, there are eigenvalues $\mu = 0$ and $\mu = -1$ corresponding to $\nu = 1$ and $\nu = 2$. The function $\varphi(c_1) = (1+2c_1+(1+4c_1(1-c_1))^{1/2})/2c_1^2$ is decreasing ($\varphi'(c_1) < 0$) and hence for $c_1 < 1$, we get $\nu > 2$ and $\mu < -1$. Thus the method is stable if and only if $c_1 = 1$.

Case $d = 1$, $m = 2$ ($S_3^1(\Delta_N)$, *Hermite cubic splines*). Now the equation $\det(V_0 - \mu V) = 0$ has a root $\mu = 1$ of geometric multiplicity 2 and also gives

$$\nu^2 c_1^2 c_2^2 + \nu((c_1 + c_2)(1 - 2c_1c_2) - (c_2 - c_1)^2) + 2(c_1 + c_2)^2 - 3(c_1 + c_2) + 1 - 2c_1c_2 = 0, \tag{6.1}$$

with, as before, $\nu = 1 - \mu$. Taking here $c_1 = c_2 = 1$, we obtain $(\nu - 1)^2 = 0$. Consequently, if $c_1 < c_2$ and (c_1, c_2) is in a certain neighborhood of (1,1) then the method is stable. For example, taking $c_2 = 1$ and using (6.1), we get the eigenvalues $\mu = 0$ and $\mu = (1-c_1)/c_1$. Thus, the method with $c_2 = 1$ is stable if and only if $c_1 > 1/2$. Detailed analysis shows that the stability domain is strictly contained in the set $\{(c_1, c_2) : 0 < c_1 < c_2 \le 1, c_1 + c_2 > 3/2\}$. In the following table we list some eigenvalues μ corresponding to c_1 and c_2:

c_1	0.75	0.7	0.6	0.55
c_2	0.82	0.8	0.9	0.95
μ	1.027	1.386	1.276	1.165

Case $d = 2$, $m = 1$ ($S_3^2(\Delta_N)$, *cubic splines*). The equation $\det(V_0 - \mu V) = 0$ here gives, besides the root $\mu = 1$ of geometric multiplicity 1, the equation

$$\varphi(c_1, \nu) \equiv \nu(c_1^3\nu^2 - 3c_1^2\nu + c_1(6 - 3\nu)) - (\nu^2 - 6\nu + 6) = 0 \tag{6.2}$$

with $\nu = 1 - \mu$. For $c_1 = 1$, we get the roots $\mu = 0$, $\mu = -2 + \sqrt{3}$ and $\mu = -2 - \sqrt{3}$. Let $\nu_0 = 3 + \sqrt{3}$ (it corresponds to $\mu = -2 - \sqrt{3}$). We know that $\nu_0^2 - 6\nu_0 + 6 = 0$. By direct calculation we get

$$\varphi(c_1, \nu_0) = 3\nu_0 c_1(1 - c_1)(2c_1(1 - \nu_0) + 2 - \nu_0) < 0$$

for $c_1 < 1$. Then, as $\varphi(c_1, \nu) \to \infty$ when $\nu \to \infty$, we see that there is always a root ν of (6.2) such that $\nu > \nu_0$. For example, if $c_1 = 1/2$, we have $\nu = 22.95$. Hence the method with cubic splines is always unstable.

Case $d = 0$, $m \geq 1$, $c_m = 1$. By Proposition 4, $\mu = 1$ is a root with geometric multiplicity m for the equation $\det(V_0 - \mu V) = 0$. In addition, the direct calculation gives a root $\mu = 0$. Thus, in this case the method is stable which is in accordance with the cases $m = 1$ and $m = 2$.

7. Numerical tests. We chose the initial function $f(t) = \cos t$ and $\lambda = 1$ in the equation (3.1) on the interval $[0, 1]$. This equation has the exact solution $y(t) = (\sin t + \cos t + e^t)/2$ and was already used in [**1**] (see also [**2**]) as a test equation. As an approximate value of $\|u\|_\infty$ we actually calculated $\max_{1 \leq n \leq N} \max_{0 \leq k \leq 10} |u_n(t_{n-1} + (k/10)h)|$. The results are presented in the following tables.

Case $d = 0$, $m = 1$ (*linear splines*). The values of $\|u\|$ corresponding to different c_1 are given, and for $c_1 = 0.4$, we added in the last row $|\mu_{\max}(M + W)|^{N-1}$

N	4	8	16	32	64
$c_1 = 1.0$	2.05146	2.05038	2.05012	2.05005	2.05003
$c_1 = 0.5$	2.04432	2.04856	2.04966	2.04994	2.05000

N	4	16	64	256
$c_1 = 0.4$	2.04	2.05	76.5	$1.97 \cdot 10^{33}$
$\lvert\mu\rvert^{N-1}$	2.31	$2.74 \cdot 10^2$	$7.58 \cdot 10^{10}$	$4.86 \cdot 10^{44}$

Case $d = 0$, $m = 2$

N	4	8	16	64	4096
$c_1 = 0.7$ $c_2 = 1.0$	2.049939	2.050017	2.050026	2.050028	2.050028
$c_1 = 0.4$ $c_2 = 0.6$	2.048333	2.049603	2.049921	2.050021	2.050028

N	4	8	16	32	64
$c_1 = 0.2$ $c_2 = 0.4$	2.004	1.987	$3.18 \cdot 10^3$	$5.41 \cdot 10^{16}$	$2.64 \cdot 10^{40}$
$\|\mu\|^{N-1}$	$1.67 \cdot 10^2$	$2.09 \cdot 10^5$	$3.44 \cdot 10^{11}$	$9.60 \cdot 10^{23}$	$7.60 \cdot 10^{48}$

Case $d = 1$, $m = 1$ (*quadratic splines*)

N	4	8	16	32	64
$c_1 = 1.0$	2.050080	2.050031	2.050028	2.050028	2.050028
$c_1 = 0.7$	2.050162	2.052011	2.597786	$6.02 \cdot 10^5$	$1.13 \cdot 10^{19}$
$c_1 = 0.1$	$2.43 \cdot 10^2$	$2.64 \cdot 10^9$	$5.70 \cdot 10^{24}$	$4.50 \cdot 10^{56}$	$4.61 \cdot 10^{121}$

Case $d = 1$, $m = 2$ (*Hermite cubic splines*)

N	4	8	16	64	4096
$c_1 = 0.7$ $c_2 = 1.0$	2.050020	2.050026	2.050028	2.050028	2.050028

N	4	32	256	2048	16384
$c_1 = 0.5$ $c_2 = 1.0$	2.050029	2.050028	2.050028	2.050028	2.050028
$\|\mu\|^{N-1}$	2.117	2.635	2.708	2.717	2.718

N	4	8	16	32	64
$c_1 = 0.3$ $c_2 = 0.7$	1.99	54.0	$9.79 \cdot 10^8$	$5.12 \cdot 10^{24}$	$2.32 \cdot 10^{37}$
$c_1 = 0.1$ $c_2 = 0.2$	$3.70 \cdot 10^4$	$5.05 \cdot 10^{14}$	$1.66 \cdot 10^{36}$	$3.04 \cdot 10^{80}$	$1.67 \cdot 10^{170}$

N	4	8	16	32	64	128	256	512	1024	2048	4096
$c_1 = 0.75$ $c_2 = 0.82$	2.049960	2.050018	2.050026	2.050027	2.050028	2.050028	2.050028	2.050027	2.243082	$3.85 \cdot 10^{11}$	$5.59 \cdot 10^{35}$
$\lvert\mu\rvert^{N-1}$	2.12	2.40	2.55	2.63	4.22	23.3	$7.15 \cdot 10^{2}$	$6.73 \cdot 10^{5}$	$5.96 \cdot 10^{11}$	$4.96 \cdot 10^{23}$	$2.90 \cdot 10^{47}$

Case $d = 2$, $m = 1$ (*cubic splines*)

N	4	8	16	32	64
$c_1 = 1.0$	2.0498	2.0491	33.56	$3.10 \cdot 10^{19}$	$3.85 \cdot 10^{26}$
$c_1 = 0.5$	3.089	$1.42 \cdot 10^{4}$	$4.65 \cdot 10^{13}$	$8.33 \cdot 10^{33}$	$4.39 \cdot 10^{75}$
$c_1 = 0.1$	$3.33 \cdot 10^{5}$	$5.99 \cdot 10^{16}$	$3.42 \cdot 10^{40}$	$1.88 \cdot 10^{89}$	$9.26 \cdot 10^{187}$

8. Notes. A thorough treatment of the numerical solution of Volterra integral equations is given in [**2**]. The numerical stability of the polynomial spline collocation method is investigated in [**5**] with equidistant collocation points (i.e., $c_j = j/m$, $j = 1, \dots, m$) and in the general setting in [**3**]. Unfortunately, the proof of the main result (Theorem 3.3 of [**3**]) is not correct. In [**3**], this Theorem 3.3 is also applied to the particular cases, and stability conditions are obtained which are disproved by our results (Theorems 4.1, 4.2 (i) and 4.3 (i), (ii) are not valid).

The collocation with linear, quadratic and cubic splines in the knots (i.e., $c_1 = 1$) is already treated in [**6**]. For the test equation (3.1) in the case of cubic splines the divergence is established.

The special case $d = -1$, $m \geq 1$ is well investigated (see, e.g., [**2**], [**4**]) and the convergence is established for large classes of equations (including those with weakly singular kernels). Let us mention that the choice of collocation parameters $0 = c_1 < \cdots < c_m = 1$ (hence $m \geq 2$) in these investigations is of special interest. This case corresponds to our formulation of the problem with $d = 0$ and the parameters $0 < \tilde{c}_1 < \cdots < \tilde{c}_{m-1} = 1$ (i.e., $\tilde{c}_i = c_{i+1}$, $i = 1, \dots, m-1$) which is always stable (see Section 6).

Let us note that in [**7**] for the Cauchy problem $y' = f(x, y), y(0) = y_0$, the collocation on equidistant partition is considered and it is required that the spline satisfies the differential equation and its derivatives up to the order $m - 1$ at the knots (multiple collocation knots for $m \geq 2$). In particular, it is proved that such a method is divergent for $d \geq m + 2$ and convergent for $d \leq m + 1$. We have already made some efforts to give a similar completely determined partition of the set of m and d into stability and instability regions and to find out their dependence on collocation parameters c_i. Even for fixed choice of c_i, $i = 1, \dots, m$, and for some important particular cases as, for example, $c_m = 1$, we encountered difficulties in giving an answer because this needs the solution of a generalized eigenvalue problem.

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