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STABILITY IN DISTRIBUTION OF FORWARD STOCHASTIC FLOWS

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ABSTRACT. We discuss here the stability in distribution of a forward stochastic flow governed by its $B_b^{0,1}$-local characteristic. We first study the special case when the local characteristic belongs to the class $B_{ub}^{0,1}$ and is absolutely continuous with respect to the Lebesgue measure. The stability for the more general case is obtained via the time change procedure.

1. Introduction. Several stability properties of a Brownian flow have received reasonable attention in the past few years (see [**7**]); for example, a stability in large or asymptotic flatness property of the Brownian flow has been studied more recently by Basak and Kannan [**3**]. Our purpose in writing this article is to establish the stability in distribution of the stochastic forward flow $\varphi_{s,t}$, $0 \leq s \leq t < \infty$, governed by the stochastic equation

$$\varphi_{s,t}(x) = x + \int_s^T F(\varphi_{s,u}, du), \tag{1.1}$$

where F is a semi-martingale. Whereas Kunita [**7**] discusses the weak convergence of a stochastic flow by considering the joint law of the $\varphi_{s,t}$ and F, we instead consider the distribution of the flow itself. We shall show, under suitable conditions on the local characteristic of F, that the forward flow is stable in distribution. We will observe that the limiting probability is independent of the initial position, the initial moment, and the local characteristic of F. We begin by setting up some basic notations and recalling from [**7**] necessary definitions.

Notations, Definitions and Remarks.

• $(\Omega, \mathcal{F}, P)$ is a complete probability space supporting all our random variables.

• An $\mathbf{R}^d$-valued continuous random field $\varphi_{s,t}(x,\omega)$, with $x \in \mathbf{R}^d$, $\omega \in \Omega$, is called a *forward stochastic flow* provided there is a null set

$N \subset \Omega$ such that for all $\omega \in N^c$ and all s, t, u with $0 \leq s \leq t \leq u < \infty$, we have

– $\varphi_{s,s}(\omega)$ is the identity map for all $0 \leq s$,

– $\varphi_{s,u}(\omega) = \varphi_{t,u}(\omega) \circ \varphi_{s,t}(\omega)$, $s \leq t \leq u$,

– $\varphi_{s,t}(\omega) : \mathbf{R}^d \to \mathbf{R}^d$ is a continuous map for all s, t.

We shall suppress the argument ω in the above and in what follows, as is customary. Since the flow is continuous, we treat it as a separable and measurable version.

• $t \to F(x,t) = (F^1(x,t), \dots, F^d(x,t))$, $x \in \mathbf{R}^d$, will denote a continuous semi-martingale with values in $C = C(\mathbf{R}^d : \mathbf{R}^d) = \{f : \mathbf{R}^d \to \mathbf{R}^d \text{ is continuous}\}$. The semi-norms inducing a Frechet space topology on C is described *infra* (in a more general set up).

• Let $F^i(x,t) = M^i(x,t) + B^i(x,t)$ be a decomposition such that $M^i(x,t)$ is a continuous local martingale and $B^i(x,t)$ is a continuous process of bounded variation, $1 \leq i \leq d$.

• Set
$$\overline{\mathbf{A}}^{ij}(x,y,t) = \langle M^i(x,t), M^j(y,t)\rangle,$$
where the righthand side is the joint (quadratic) variation of the local martingales $M^i(x,t)$ and $M^j(y,t)$, $1 \leq i,\, j \leq d$.

• If A_t is a continuous strictly increasing process such that both $\overline{\mathbf{A}}^{ij}(x,y,t)$ and $B^i(x,t)$ are absolutely continuous with respect to A_t, a.s., for any $x,\, y \in \mathbf{R}^d$, then there exist predictable processes $a^{ij}(x,y,t)$ and $b^i(x,t)$ with parameters x, y such that, for all $t \in [0,T]$,

$$\overline{\mathbf{A}}^{ij}(x,y,t) = \int_0^t a^{ij}(x,y,s)\, dA_s,$$
$$B^i(x,t) = \int_0^t b^i(x,s)\, dA_s \text{ a.s.}$$

• Define a measure μ on $([0,T]\times\Omega, \mathcal{P})$ by $\mu(B) = E[\int_0^t \mathbf{1}(B)(s,w)\, dA_s]$, for all $B \in \mathcal{P}$, where $\mathcal{P}$ is the predictable σ-field and $\mathbf{1}(\cdot)$ is the indicator function.

• $a(x,y,t) = (a^{ij}(x,y,t))$, $i,j = 1, \dots, d$ is a $d \times d$-matrix valued function with the following properties:

(a) Symmetry: $a^{ij}(x,y,t) = a^{ji}(y,x,t)$ holds μ-a.e., for all x, y, i, j.

(b) Nonnegative definiteness: $\sum_{i,j,p,q} a^{ij}(x_p, x_q, t)\xi_p^i \xi_q^j \geq 0$ holds μ-a.e., for all $x_p = (\xi_p^1, \ldots, \xi_p^d)$, and $x_q = (\xi_q^1, \ldots, \xi_q^d)$, $p, q = 1, 2, \ldots, n$.

• $b(x,t) = (b^1(x,t), \ldots, b^d(x,t))'$.

• The triple $(a(x,y,t), b(x,t), A_t)$ is called the *local characteristic* of $F(x,t)$.

• Let $\mathbf{D}$ be a domain in $\mathbf{R}^d$, and let m be a nonnegative integer. Denote by $C^m(\mathbf{D}, \mathbf{R}^d) \equiv C^m = \{f : \mathbf{D} \mapsto \mathbf{R}^d$ is m-times continuously differentiable$\}$. When $m = 0$, C^0 is denoted by $C(\mathbf{D}, \mathbf{R}^d)$. For the multi-index of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, the differential operator D_x^α is defined by

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},$$

where $|\alpha| = \sum \alpha_i$. Let $K \subset \mathbf{D}$, and set

$$\|f\|_{m:K} = \sup_{x \in K} \frac{|f(x)|}{1 + |f(x)|} + \sum_{1 \leq |\alpha| \leq m} \sup_{x \in K} |D^\alpha f(x)|. \tag{1.2}$$

$C^m(\mathbf{D} : \mathbf{R}^d)$ is a Frechet space induced by the semi-norms $\| \; \|_{m,K}$.

• **Definition 1.1.** The local characteristic (a, b, A_t) of the semimartingale F *belongs to the class* $B_b^{0,1}$ if $a(x,y,t)$ and $b(x,t)$ have predictable modifications satisfying the following

Assumptions A: (A.1) For $t \geq 0$,

$$\int_0^\infty \|a(t)\|_1^\sim \, dA_t < \infty, \quad \text{a.s.},$$

where

$$\begin{aligned}\|a(t)\|_1^\sim &= \sup_{x,y \in \mathbf{R}^d} \frac{|a(x,y,t)|}{(1+|x|)(1+|y|)} \\ &\quad + \sup_{\substack{x,y,x',y' \in \mathbf{R}^d \\ x \neq x', y \neq y'}} \frac{|a(x,y,t) - a(x',y,t) - a(x,y',t) + a(x',y',t)|}{|x-x'||y-y'|} \\ &< \infty, \quad \forall\, \omega \in \Omega;\end{aligned}$$

(A.2) for $t \geq 0$,

$$\int_0^\infty \|b(t)\|_1 \, dA_t < \infty, \quad \text{a.s.,} \tag{1.3}$$

where

$$\|b(t)\|_1 = \sup_{x \in \mathbf{R}^d} \frac{|b(x,t)|}{1+|x|} + \sup_{\substack{x,y \in \mathbf{R}^d \\ x \neq y}} \frac{|b(x,t)-b(y,t)|}{|x-y|} < \infty, \quad \forall\, \omega \in \Omega. \tag{1.4}$$

The local characteristic (a, b, A_t) of F *belongs to the class* $B_{ub}^{0,1}$ provided there exists a constant L_1 such that

$$\|a(t)\|_1^\sim \leq L_1, \quad \text{for all } t \in [0,T] \quad \text{and for all } \omega \in \Omega, \tag{1.5}$$

and

$$\|b(t)\|_1 \leq L_1, \quad \text{for all } t \in [0,T], \quad \text{and for all } \omega \in \Omega; \tag{1.6}$$

in other words, (A.3)

$$\begin{aligned} |b(x,t)-b(y,t)| &\leq L_1|x-y|, \\ \|a(x,y,t)-2a(x,y,t)+a(y,y,t)\| &\leq L_1|x-y|^2, \\ |b(x,t)| &\leq L_1(1+|x|), \\ \|a(x,y,t)\| &\leq L_1(1+|x|)(1+|y|), \end{aligned}$$

hold for all $t \in [0,T]$, $x, y \in \mathbf{R}^d$ and $\omega \in \Omega$ where $|\ |$ and $\|\ \|$ denote norms of vectors and matrices, respectively.

• Let $t_0 \in [0,T]$ and $x_0 \in \mathbf{R}^d$. A continuous $\mathbf{R}^d$-valued process $\varphi_{t_0,t}$, $0 \leq t_0 \leq t \leq T$ adapted to $(\mathcal{F}_{t_0,t})$ is called a *solution* of Ito's stochastic differential equation based on $F(x,t)$ starting at x_0 at time t_0 if it satisfies

$$\varphi_{t_0,t} = x_0 + \int_{t_0}^t F(\varphi_{t_0,s}, ds). \tag{1.7}$$

Also $\varphi_{t_0,t}$ is said to be governed by Ito's stochastic differential equation based on $F(x,t)$.

• It is known from [**7**] that, if the local characteristic of F belongs to the class $B_b^{0,1}$, then for each t_0 and x_0, the equation (1.7) has a unique solution.

The following form of Ito's formula is central to our analysis. A proof of the formula can be found in [**7**]. (A discrete-time version of this formula, due to Kannan and Bo Zhang, will appear elsewhere.)

Theorem 1.2 (Generalized Ito formula). *Assume that $G(x,t)$, $x \in \mathbf{R}^d$, is a continuous C^2-process and also a continuous C^1-semi-martingale with local characteristic belonging to the class $B_b^{1,0}$. Let g_t be a continuous semi-martingale with values in $\mathbf{R}^d$. Then $G(g_t,t)$ is a continuous semi-martingale satisfying the formula*

$$\begin{aligned} G(g_t,t) - G(g_0,0) &= \int_0^t G(g_s, ds) + \sum_{i=1}^d \int_0^t \frac{\partial G}{\partial x^i}(g_s,s)\, dg_s^i \\ &\quad + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 G}{\partial x^i \partial x^j}(g_s,s)\, d\langle g_s^i, g_s^j\rangle \\ &\quad + \sum_{i=1}^d \Big\langle \int_0^t \frac{\partial G}{\partial x^i}(g_s, ds), g_t^i \Big\rangle. \end{aligned} \tag{1.8}$$

Comparing this formula with the standard Ito formula, we note that the last term here is a correction term. When the function G is deterministic, the generalized Ito formula reduces to the standard Ito formula. Now, since the solution of equation (1.7) is a continuous semi-martingale, we apply the generalized Ito formula to a deterministic C^2-function $G : \mathbf{R}^d \mapsto \mathbf{R}$ so that

$$\begin{aligned} G(\varphi_{t_0,t}) - G(x) &= \sum_{i=1}^d \int_{t_0}^t \frac{\partial G}{\partial x_i}(\varphi_{t_0,s})\, d\varphi_{t_0,s}^i \\ &\quad + \frac{1}{2}\sum_{i,j=1}^d \int_{t_0}^t \frac{\partial^2 G}{\partial x_i \partial x_j}(\varphi_{t_0,s})\, d\langle \varphi_{t_0,s}^i, \varphi_{t_0,s}^j\rangle. \end{aligned} \tag{1.9}$$

Since $\varphi_{t_0,t}$ is a solution of the SDE (1.7), viz. $d\varphi_{t_0,t} = F(\varphi_{t_0,t}, dt)$, we have from the semi-martingale decomposition of F that

$$\begin{aligned}\sum_{i=1}^{d}\int_{t_0}^{t}\frac{\partial G}{\partial x_i}(\varphi_{t_0,s},s)F^i(\varphi_{t_0,s},ds)\\ &=\sum_{i=1}^{d}\int_{t_0}^{t}\frac{\partial G}{\partial x_i}(\varphi_{t_0,s},s)M^i(\varphi_{t_0,s},ds)\\ &\quad+\sum_{i=1}^{d}\int_{t_0}^{t}\frac{\partial G}{\partial x_i}(\varphi_{t_0,s},s)B^i(\varphi_{t_0,s},ds).\end{aligned}$$

It follows from Theorem 3.2.4 in [**7**] that

$$\begin{aligned}\langle\varphi^i_{t_0,t},\varphi^j_{t_0,t}\rangle &=\Big\langle x+\int_{t_0}^{t}F^i(\varphi_{t_0,s},ds),x+\int_{x_0}^{t}F^j(\varphi_{t_0,s},ds)\Big\rangle\\ &=\Big\langle\int_{t_0}^{t}M^i(\varphi_{t_0,s},ds),\int_{t_0}^{t}M^j(\varphi_{t_0,s},ds)\Big\rangle\\ &=\int_{t_0}^{t}a^{ij}(\varphi_{t_0,s},s)\,dA_s.\end{aligned}\tag{1.11}$$

Hence we write $d\langle\varphi_{t_0,t},\varphi_{t_0,t}\rangle=a^{ij}(\varphi_{t_0,t},t)\,dA_t$. Also, since $B(x,t)=\int_{x_0}^{t}b(x,s)\,dA_s$, we shall write this as $B(x,dt)=b(x,t)\,dA_t$. We thus have

$$\begin{aligned}G(\varphi_{t_0-t})-G(x)&=\sum_{i=1}^{d}\int_{t_0}^{t}\frac{\partial G}{\partial x_i}(\varphi_{t_0,s})M^i(\varphi_{t_0,s},ds)\\ &\quad+\sum_{i=1}^{d}\int_{t_0}^{t}\frac{\partial G}{\partial x_i}(\varphi_{t_0,s})b^i(\varphi_{t_0,s},s)\,dA_s\\ &\quad+\frac{1}{2}\sum_{i,j=1}^{d}\int_{t_0}^{t}\frac{\partial^2 G}{\partial x_i\partial x_j}(\varphi_{t_0,s})a^{ij}(\varphi_{t_0,s},s)\,dA_s.\end{aligned}\tag{1.12}$$

In particular, when we take $G(x):=[\sum_{i,j=1}^{d}q_{ij}x_ix_j]^{1-\delta}=(x,Qx)^{1-\delta}$ for a $d\times d$ positive definite matrix $Q=(q_{ij})$, we then obtain, by the stopping time argument,

$$
\begin{aligned}
&E(\varphi_{t_0,t}, Q\varphi_{t_0,t})^{1-\delta} - (x, Qx)^{1-\delta} \\
&\quad = E\int_{t_0}^{t}\Big\{(1-\delta)(\varphi_{t_0,s}, Q\varphi_{t_0,s})^{-\delta}\Big[2(\varphi_{t_0,s}, Qb(\varphi_{t_0,s}, s)) \\
&\qquad + \mathrm{Tr}\,(Qa(\varphi_{t_0,s}, s)) + 2\delta\frac{(Q\varphi_{t_0,s}, a(\varphi_{t_0,s}, s)Q\varphi_{t_0,s})}{(\varphi_{t_0,s}, Q\varphi_{t_0,s})}\Big]\Big\}\, dA_s.
\end{aligned}
\tag{1.13}
$$

In this paper we discuss the stability in distribution of forward stochastic flow governed by equation (1.1). We first prove in Section 2 that the distribution of $\varphi_{s,t}$ converges to a probability measure when the local characteristic $(a, b, A_t) \in B_{ub}^{0,1}$. We later extend this main result to the more general case $(a, b, A_t) \in B_b^{0,1}$. In the case of $B_{ub}^{0,1}$-local characteristic, we first consider the flow when $A_t = t$ to obtain the main stability result. By changing the time scale we get the stability in distribution of the flow in the general case. Finally, for a flow governed by a $B_b^{0,1}$-local characteristic, we appeal to a fact which renders it equivalent to a $B_{ub}^{0,1}$-local characteristic and obtain the stability in distribution via conditions in terms of the $B_b^{0,1}$-local characteristic (a, b, A_t).

2. Stability in distribution of the flow. As pointed out in the introduction, we first establish the stability in distribution of the flow $\varphi_{s,t}$ when the local characteristic belongs to $B_{ub}^{0,1}$. We then address the more general case where the local characteristic belongs to $B_b^{0,1}$.

We begin by noticing that Assumptions (A.1)–(A.3) assure us of the existence and uniqueness of a solution of equation (1.1). We also need the following assumptions to prove the main stability results.

Assumptions B. *Put*

$$
\begin{aligned}
b(x, y, t) &:= b(x, t) - b(y, t),\\
\mathbf{A}(x, y, t) &:= a(x, x, t) - 2a(x, y, t) + a(y, y, t).
\end{aligned}
$$

(B.1) There is a symmetric positive definite matrix Q and a constant

$\gamma > 0$ satisfying

$$2((x-y), Qb(x,y,t)) - 2\frac{(x-y, Q\mathbf{A}(x,y,t)Q(x-y))}{((x-y), Q(x-y))} + \mathrm{Tr}\,(\mathbf{A}(x,y,t)Q) \le -\gamma(Q(x-y), x-y),$$

for all $t \in [0,T]$ and $x, y \in \mathbf{R}^d$, $x \neq y$, where $\mathrm{Tr}\,(\mathbf{A}) = \text{trace of } \mathbf{A}$.

(B.2) For a symmetric positive definite matrix C we have

$$2(x, Cb(x,t)) - 2\,\frac{(x, Ca(x,x,t)Cx)}{(x, Cx)} + \mathrm{Tr}\,(a(x,x,t)C) \le -\alpha(Cx, x),$$

for all sufficiently large $|x|$, $t \in [0,T]$ and for some $\alpha > 0$.

Remark 2.1. The assumptions (A.3) and (B.1) imply (B.2) for $C = Q$ and every $\alpha \in (0, \gamma)$.

The main results of this article present the weak convergence of the flow distribution $p_{s,t}(x, dy) = P(\varphi^x_{s,t} \in dy)$, $0 \le s \le t$, to a probability measure, as $t \to \infty$, and the limit being independent of s. Recall that the weak convergence of probability measures on a metric space S depends on the topology of S and not on how we metrize the topology. Dudley observe, in [**6**], that a nice class of functions to work with is the space **BL** of all bounded Lipschitz functions metrized as follows. Define, for a real-valued function f on a metric space (S, d),

$$\|f\|_L := \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)}, \qquad \|f\|_\infty := \sup_{x\in S} |f(x)|, \tag{2.1}$$

and

$$\|f\|_{\mathbf{BL}} := \|f\|_L + \|f\|_\infty.$$

The space **BL** is now defined by $\mathbf{BL}\,(S,d) := \{f : \|f\|_{BL} < \infty\}$. Dudley showed that the weak convergence of probability measures can be obtained as a metric convergence with respect to the **BL**-distance. This **BL**-distance between two probability measures μ and ν on the Borel σ-field of $\mathbf{R}^d$ is given by

$$\|\mu - \nu\|_{\mathbf{BL}} := \sup_{f\in BL} \left| \int f\,d\mu - \int f\,d\nu \right|. \tag{2.2}$$

To establish our first stability result, we need three auxiliary results. We start with the following estimate.

Lemma 2.2. *Assume that $A_t \equiv t$, that the local characteristics $(a,b) \in B^{0,1}_{ub}$, and that* (B.1) *holds. Then, for $t \geq t_0$, there exist an ε with $0 < \varepsilon < 1$ and a $\beta > 0$ such that*

$$E(Q\varphi^x_{t_0,t}, \varphi^x_{t_0,t})^\varepsilon \leq (Qx,x)^\varepsilon \exp(-\beta(t-t_0)). \tag{2.3}$$

Proof. Let $G(x) := (Qx,x)^{1-\delta_0}$, for some positive definite matrix Q and some $0 < \delta_0 < 1$. Then

$$\begin{aligned}
\mathcal{L}(Qx,x)^{1-\delta_0} &= (1-\delta_0)(Qx,x)^{-\delta_0}(Qx, 2b(x,t)) \\
&\quad + 2(1-\delta_0)(-\delta_0)(Qx,x)^{-\delta_0-1}(Qx, a(x,x,t)Qx) \\
&\quad + (1-\delta_0)(Qx,x)^{-\delta_0} Tr(Qa(x,x,t)) \\
&= (1-\delta_0)(Qx,x)^{-\delta_0}\bigg\{ 2(x, Qb(x,t)) + 2(1-\delta_0) \\
&\qquad \cdot \frac{(x, Qa(x,x,t)Qx)}{(Qx,x)} - 2\frac{(x, Qa(x,x,t)Qx)}{(Qx,x)} \\
&\qquad + Tr(a(x,x,t)Q)\bigg\}.
\end{aligned}$$

Since we have $K_Q(Qx,x) \leq (x,x) \leq K^Q(Qx,x)$ for some positive constants K_Q and K^Q, we obtain from (A.3) and (B.1) that

$$\begin{aligned}
\frac{(x, Qa(x,x,t)Qx)}{(Qx,x)} &\leq \frac{\|a(x,x,t)\|(Qx,Qx)}{(Qx,x)} \\
&\leq \|Q\|^2 (K^Q)^2 L_1 (Qx,x) \\
&= \lambda_a (Qx,x),
\end{aligned}$$

where $\lambda_a = \|Q\|^2(K^Q)^2 L_1$. Thus,

$$\begin{aligned}
\mathcal{L}(Qx,x)^{1-\delta_0} &\leq (1-\delta_0)(Qx,x)^{-\delta_0}[-\alpha(Qx,x) + 2(1-\delta_0)\lambda_a(Qx,x)] \\
&= \beta(Qx,x)^{1-\delta_0},
\end{aligned}$$

for all sufficiently large x, where $\beta = (1-\delta_0)\{\alpha - 2(1-\delta_0)\beta_1\lambda_\alpha\} > 0$ for δ_0 sufficiently close to 1. Therefore,

$$\sup_{|x|>M} \mathcal{L}(Qx,x)^{1-\delta_0} \leq -\beta_Q M^{2(1-\delta_0)}$$

for M large enough, where $\beta_Q = \beta\times$ (minimum eigenvalue of Q). It is easy to deduce from this that for some $\delta \in (\delta_0, 1)$, $(Q\varphi^x_{t_0,t}, \varphi^x_{t_0,t})^{1-\delta}$ is uniformly integrable. We now have by using the standard stopping time argument that

$$\begin{aligned} EG(\varphi^x_{t_0,t}) - G(x) &= E\int_{t_0}^t \mathcal{L}G(\varphi^x_{t_0,u})\,du \\ &\leq -\beta E\int_{t_0}^t G(\varphi^x_{t_0,u})\,du, \end{aligned}$$

so that

$$\frac{d}{dt}E(G(\varphi^x_{t_0,t})) \leq -\beta EG(\varphi^x_{t_0,t}).$$

This yields

$$E[G(\varphi^x_{t_0,t})\exp(\beta(t-t_0))] \leq G(x).$$

Therefore, taking $\varepsilon = 1-\delta$,

$$E(Q\varphi^x_{t_0,t}, \varphi^x_{t_0,t})^\varepsilon \leq (Qx,x)^\varepsilon \exp(-\beta(t-t_0)).$$

This completes the proof. □

Lemma 2.3. *Let the assumptions of Lemma* 2.2 *hold. Then, for* t_0, $r \geq 0$ *and all compact* $K \subset \mathbf{R}^d$, *there exists an* ε, $0 < \varepsilon < 1$, *such that*

$$\lim_{t\to\infty}\sup_{x\in K} E|\varphi_{t_0,t+r}(x) - \varphi_{t_0,t}(x)|^\varepsilon = 0. \tag{2.4}$$

Proof. Let $u = \varphi_{t_0,t+r}(x)$ and $v = \varphi_{t_0,t}(x)$ with $t \geq t_0$ and $r \geq 0$.

$$\begin{aligned} (Q(u-v),(u-v))^{1-\delta} &= ((Qu,u) - 2(Qu,v) + (Qu,v))^{1-\delta} \\ &\leq [2(Qu,u) + 2(Qv,v)]^{1-\delta} \\ &\leq 2^{1-\delta}\max\{[2(Qu,u)]^{1-\delta}, [2(Qv,v)]^{1-\delta}\} \\ &\leq 4^{1-\delta}[(Qu,u)^{1-\delta} + (Qv,v)^{1-\delta}]. \end{aligned} \tag{2.5}$$

Thus

$$\begin{aligned}
E(Q(\varphi_{t_0,t+r}(x)-\varphi_{t_0,t}(x)),\varphi_{t_0,t+r}(x)-\varphi_{t_0,t}(x))^{1-\delta}& \\
\le 4^{1-\delta}[E(Q\varphi_{t_0,t+r}(x),\varphi_{t_0,t+r}(x))^{1-\delta}& \\
+E(Q\varphi_{t_0,t}(x),\varphi_{t_0,t}(x))^{1-\delta}]& \\
\le 4^{1-\delta}[(Qx,x)^{1-\delta}\exp(-\beta(t+r-t_0))& \\
+(Qx,x)^{1-\delta}\exp(-\beta(t-t_0))],&
\end{aligned}$$

by (2.3),

$$\begin{aligned}
&= 4^{1-\delta}(Qx,x)^{1-\delta}\exp(-\beta(t-t_0))[\exp(-\beta r)+1] \\
&\le 2^{3-2\delta}(Qx,x)^{1-\delta}\exp(-\beta(t-t_0)) \\
&\to 0, \quad \text{(exponentially fast)}, \quad \text{as } t\to\infty.
\end{aligned} \tag{2.6}$$

Taking into account the quadratic nature of (Qx,x), this implies for $\varepsilon=1-\delta$ and for every compact $K\subset \mathbf{R}^d$ that

$$\lim_{t\to\infty}\sup_{x\in K} E|\varphi_{t_0,t+r}(x)-\varphi_{t_0,t}(x)|^{\varepsilon}=0,$$

and this completes the proof. □

Lemma 2.4. *Let the assumptions of Lemma 2.2 hold. Then, for* $t_0\ge 0$, *there exists an* ε, $0<\varepsilon<1$, *such that, for every compact* $K\subset\mathbf{R}^d$,

$$\lim_{t\to\infty}\sup_{x,y\in K} E|\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y)|^{\varepsilon}=0. \tag{2.7}$$

Proof. For $x,y\in\mathbf{R}^d$, $x\ne y$ and $r\ge 0$, define

$$\begin{aligned}
\lambda &:= (1-\delta)[\gamma-2(1-\delta)(K^Q)^2L_1\|Q\|^2], \\
w(t_0,t,x) &:= \exp(\lambda(t-t_0))(Qx,x)^{1-\delta}, \\
Z_{t_0,t}(x,y) &:= \varphi_{t_0,t}(x)-\varphi_{t_0,t}(y), \\
\bar\tau_r &:= \inf\{t\ge t_0 : |Z_{t_0,t}(x,y)|\le r\}, \\
\bar\tau_J^x &:= \inf\{t\ge t_0, |\varphi_{t_0,t}(x)-x|\ge J/3\}, \\
\tau_J' &:= \inf\{t\ge t_0 : |Z_{t_0,t}(x,y)|\ge J\}, \\
\tau_J &:= \tilde\tau_J^x\wedge\tilde\tau_J^y\wedge\tau_J', \\
\tau_r^J &:= t_r\wedge\tau_J, \quad \text{where } t_r:=t\wedge\tilde\tau_r, \quad \text{for } t\ge t_0.
\end{aligned}$$

Let $|x|, |y| < J/3$, $0 < r \le r_1 \le |x-y| < J/3$, and $t_0 \le s < t$. It follows from the Assumptions (A.3) and (B.1) and from Ito's lemma applied for the random time t_r^J, that

$$
\begin{aligned}
&w(t_0, t_r^J, Z_{t_0,t_r^J}(x,y)) \\
&= w(t_0, s_{r_1}^J, Z_{t_0,t_r^J}(x,y)) + \int_{s_{r_1}^J}^{t_r^J} \frac{\partial w}{\partial u}(t_0, u, Z_{t_0,u}(x,y))\, du \\
&\quad + \int_{s_{r_1}^J}^{t_r^J} \exp(\lambda(u-t_0)) \Big\{ (1-\delta)(Z_{t_0,u}(x,y))^{-\delta} \\
&\qquad\qquad \cdot \Big[2(Z_{t_0,u}(x,y), Qb(\varphi_{t_0,u}(x), \varphi_{t_0,u}(y), u)) \\
&\qquad\qquad + Tr(Q\mathbf{A}) - 2\delta \frac{(QZ_{t_0,u}(x,y), \mathbf{A}QZ_{t_0,u}(x,y))}{(Z_{t_0,u}(x,y), QZ_{t_0,u}(x,y))} \Big] \Big\}\, du \\
&\le w(t_0, s_{r_1}^J, Z_{t_0,t_r^J}(x,y)) \\
&\quad + \int_{s_{r_1}^J}^{t_r^J} \lambda \exp(\lambda(u-t_0))(QZ_{t_0,u}(x,y), Z_{t_0,u}(x,y))^{1-\delta}\, du \\
&\quad + \int_{s_{r_1}^J}^{t_r^J} \exp(\lambda(u-t_0))(1-\delta)[-\gamma + 2(1-\delta)L_1(K^Q\|Q\|)^2] \\
&\qquad\qquad \cdot (QZ_{t_0,u}(x,y), Z_{t_0,u}(x,y))^{1-\delta}\, du.
\end{aligned}
$$

Using the martingale convergence theorem, as $J \to \infty$, one has

$$
E[w(t_0, t_r, Z_{t_0,t_r}(x,y)) | \mathcal{F}_{t_0,s_{r_1}}] \le w(t_0, s_{r_1}, Z_{t_0,s_{r_1}}(x,y)) \text{ a.s.} \tag{2.9}
$$

Letting $r \downarrow 0$ and using Fatou's lemma, we get

$$
E[w(t_0, t\wedge\bar{\tau}_0, Z_{t_0,t\wedge\bar{\tau}_0}(x,y)) | \mathcal{F}_{t_0,s_{r_1}}] \le w(t_0, s_{r_1}, Z_{s_{r_1}}(x,y)) \text{ a.s.} \tag{2.10}
$$

Next letting $r_1 \downarrow 0$ and using martingale convergence theorem we observe that

$$
\begin{aligned}
&E[w(t_0, t\wedge\bar{\tau}_0, Z_{t_0,t\wedge\bar{\tau}_0}(x,y)) | \mathcal{F}_{t_0,s\wedge\bar{\tau}_0}] \\
&\qquad\qquad \le w(t_0, s\wedge\bar{\tau}_0, Z_{t_0,s\wedge\bar{\tau}_0}(x,y)) \text{ a.s.}
\end{aligned}
$$

In other words, $\{w(t_0, t \wedge \bar{\tau}_0, \varphi_{t\wedge\bar{\tau}_0}(x) - \varphi_{t\wedge\bar{\tau}_0}(y)) : t \geq t_0\}$ is a positive super-martingale, for any $\delta \in (\delta_0, 1)$ where δ_0 is chosen to be

$$\delta_0 := 1 - \frac{\gamma}{2L_1(K^Q\|Q\|)^2}.$$

Therefore,

$$E(w(t_0, t, \varphi_{t_0,t}(x) - \varphi_{t_0,t}(y))I_{t_0\leq t\leq\bar{\tau}_0}) \leq w(t_0, t_0, x - y).$$

Since $\varphi_{t_0,t}(x) - \varphi_{t_0,t}(y) = 0$ a.s., for $t \geq \bar{\tau}_0$,

$$E(\exp(\lambda(t-t_0))(Q(\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y)), \varphi_{t_0,t}(x)-\varphi_{t_0,t}(y))^{1-\delta}I_{t\geq\bar{\tau}_0}) = 0.$$

So

$$\begin{aligned} E((Q(\varphi_{t_0,t}(x) - \varphi_{t_0,t}(y)), \varphi_{t_0,t}(x) - \varphi_{t_0,t}(y))^{\varepsilon}) \\ \leq \exp(-\lambda(t - t_0))(Q(x - y), x - y)^{\varepsilon}. \end{aligned} \tag{2.11}$$

Hence, from the quadratic nature of (Qx, x), it follows that for $\varepsilon = 1-\delta$ and every compact K,

$$\lim_{t\to\infty} \sup_{x,y\in K} E|\varphi_{t_0,t}(x) - \varphi_{t_0,t}(y)|^{\varepsilon} = 0. \qquad \square \tag{2.12}$$

We are now ready to state and prove our first stability result.

Theorem 2.5. *Assume $A_t \equiv t$, the local characteristic $(a, b) \in B^{0,1}_{ub}$, and* (A.3) *and* (B.1) *hold. Then there exists a unique probability measure π such that*

$$\sup\{\|p_{t_0}(x, dy) - \pi(dy)\|_{B_L} : x \in K\} \longrightarrow 0$$

exponentially fast, as $t \to \infty$, for every compact K. Here the limit measure π is independent of the initial moment t_0.

Proof. In proving this theorem, we first show that the flow distribution $\{p_{t_0,t}(x, dy)\} = \{P(\varphi^x_{t_0,t} \in dy)\}$ is Cauchy (with respect to t) in the $d_{\mathbf{BL}}$-metric and that it converges to a probability measure π_{t_0}. We next show that the distribution π_{t_0} is independent of t_0.

For all $t \geq t_0$, and all $s \geq 0$,

$$\begin{aligned}
&\|p_{t_0,t+s}(x,dy) - p_{t_0,t}(x,dy)\|_{BL} \\
&\qquad = \sup_{f\in BL} |Ef(\varphi_{t_0,t+s}(x)) - Ef(\varphi_{t_0,t}(x))| \\
&\qquad \leq \sup_{f\in BL} |Ef(\varphi_{t_0,t+s}(x)) - f(\varphi_{t_0,t}(x))| \\
&\qquad \leq E\{|\varphi_{t_0,t+s}(x) - \varphi_{t_0,t}(x)| \wedge 2\}.
\end{aligned}$$

Since the inequality (2.3) renders the family of probability measures $\{p_{t_0,t}(x,dy) : t \geq t_0\}$ tight, there exists for all $\varepsilon > 0$, a compact subset K_{ε_0} of $\mathbf{R}^d$ such that

$$\int_{\mathbf{R}^d - K_{\varepsilon_0}} p_{t_0,s}(x,dy) < \frac{\varepsilon_0}{4}, \quad \forall\, s \geq t_0.$$

It follows from (2.4) that, for any compact $K \subset \mathbf{R}^d$, for all $\delta > 0$, there exists $T_0 > 0$ such that, for all $t \geq T_0$,

$$\sup_{x\in K} P(|\varphi_{t_0,t+s}(x) - \varphi_{t_0,t}(x)| > \delta) < \frac{\delta}{2}.$$

Therefore,

$$\begin{aligned}
&E\{|\varphi_{t_0,t+s}(x) - \varphi_{t_0,t}(x)| \wedge 2\} \\
&\quad \leq E[|\varphi_{t_0,t+s}(x) - \varphi_{t_0,t}(x)| I_{\{|\varphi_{t_0,t+s}(x) - \varphi_{t_0,t}(x)| \leq (\varepsilon_0/2)\}}] \\
&\qquad + E[(|\varphi_{t_0,t+s}(x) - \varphi_{t_0,t}(x)| \wedge 2) I_{\{|\varphi_{t_0,t+s}(x) - \varphi_{t_0,t}(x)| > (\varepsilon_0/2)\}}] \\
&\quad < \frac{\varepsilon_0}{2} + 2\frac{\varepsilon_0}{4} \\
&\quad = \varepsilon_0.
\end{aligned}$$

Hence, for each initial moment s, $\{p_{s,t}(x,\cdot)\}$ is Cauchy. Also, $p_{s,t}(x,\cdot) \to \pi_s(\cdot)$, as $t \to \infty$, in the **BL** metric $d_{\mathbf{BL}}$.

Before proving that the limit probability π_s is independent of the initial moment s, we show that π_s does not depend on the initial position x of the flow. For $x, z \in \mathbf{R}^d$,

$$\begin{aligned}
(2.13)\quad &\|p_{s,t}(z,dy) - \pi_s(dy)\|_{BL} \\
&\quad \leq \|p_{s,t}(z,dy) - p_{s,t}(x,dy)\|_{BL} + \|p_{s,t}(x,dy) - \pi_s(dy)\|_{BL}.
\end{aligned}$$

The second term on the righthand side of the above inequality goes to zero, as $t \to \infty$. Regarding the first term,

$$\begin{aligned} \|p_{s,t}(z,\cdot) - p_{s,t}(x,\cdot)\|_{BL} &= \sup_{f \in \mathbf{BL}} |E(f(\varphi_{s,t}(z)) - f(\varphi_{s,t}(x))| \\ &\leq E\{|\varphi_{s,t}(z) - \varphi_{s,t}(x)| \wedge 2\}. \end{aligned} \tag{2.14}$$

It now follows from (2.7) that, for any compact $K \subset \mathbf{R}^d$ and for all $\delta > 0$, there exists $\varepsilon' > 0$ such that

$$\sup_{x,z \in K} P(|\varphi_{t_0,t}(z) - \varphi_{t_0,t}(x)| > \delta) < \varepsilon'.$$

Now a $T_0 > 0$ exists such that for all $t \geq T_0$,

$$E\{|\varphi_{t_0,t}(z) - \varphi_{t_0,t}(x)| \wedge 2\} < \frac{\varepsilon_0}{4}, \quad \forall\, x, z \in K. \tag{2.15}$$

Thus, $\|p_{s,t}(z,dy) - \pi_s(dy)\|_{BL} \to 0$ as $t \to \infty$.

Finally we prove that for $0 \leq s \leq r$, $\pi_s = \pi_r$. Fix an r with $s \leq r \leq t$ and an $A \in \mathcal{B}(\mathbf{R}^d)$. Keeping in mind that our forward flow is continuous in x, we notice that

$$\begin{aligned} \{\varphi_{s,t}(x) \in A\} &= \{\varphi_{r,t}(\varphi_{s,r}(x)) \in A\} \\ &= \bigcup_{y=\varphi_{s,r}(x)} \{\varphi_{r,t}(y) \in A\} \\ &\supseteq \{\varphi_{r,t}(y) \in A\}, \end{aligned}$$

where $y = \varphi_{s,r}(x)$. Thus $P(\varphi_{s,t}(x) \in A) \geq P(\varphi_{r,t}(y) \in A)$. Now letting $t \to \infty$, we notice

$$\pi_s(A) \geq \pi_r(A). \tag{2.16}$$

To see the reverse inequality, we first note from the separability of the process that

$$\begin{aligned} \{\varphi_{r,t}(x) \in A\} &= \bigcup_{0 \leq s \leq r} \{\varphi_{r,t}(\varphi_{s,r}(\varphi_{s,r}^{-1}(x))) \in A\} \\ &= \bigcup_{0 \leq s \leq r} \{\varphi_{s,t}(\varphi_{s,r}^{-1}(x)) \in A\} \\ &= \bigcup_{0 \leq s \leq r} \bigcup_{y=\varphi_{s,r}^{-1}(x)} \{\varphi_{s,t}(y) \in A\} \\ &\supseteq \{\varphi_{s,t}(y) \in A\}, \end{aligned} \tag{2.17}$$

i.e., $P(\varphi_{r,t}(x) \in A) \geq P(\varphi_{s,t}(y) \in A)$. Now letting $t \to \infty$, we have

$$\pi_r(A) \leq \pi_s(A). \tag{2.18}$$

Hence $\pi_s(A) = \pi_r(A)$. This implies that, for any simple function f on $\mathbf{R}^d$,

$$\int f(y)\pi_s(dy) = \int f(y)\pi_r(dy). \tag{2.19}$$

Let $f \in \mathbf{BL}$; there exists a sequence of simple functions $g_n \in L^\infty(\mathbf{R}^d)$ such that $g_n \to f$ as $n \to \infty$ in L^∞. This implies

$$\begin{aligned}
&\left| \int f(y)\pi_s(dy) - \int f(y)\pi_r(dy) \right| \\
&\quad \leq \int |f(y) - g_n(y)|\pi_s(dy) + \left| \int g_n(y)\pi_s(dy) - \int g_n(y)\pi_r(dy) \right| \\
&\qquad + \int |f(y) - g_n(y)|\pi_r(dy) \\
&\quad \leq \int |f(y) - g_n(y)|\pi_s(dy) + \int |f(y) - g_n(y)|\pi_r(dy) \\
&\quad \longrightarrow 0, \quad \text{as } t \to \infty.
\end{aligned}$$

So

$$\int_{\mathbf{R}^d} f(y)\pi_s(dy) = \int_{\mathbf{R}^d} f(y)\pi_r(dy), \quad \forall\, f \in BL.$$

Therefore $\pi_s = \pi_r = \pi$. Hence the proof. $\square$

The following result of Basak [**2**] can be deduced as a special case of our Theorem 2.5.

Corollary 2.6. *Assume* (A.1), (A.2) *and* ($\hat{\text{B}}$.1)*: there is a symmetric positive definite matrix Q and a constant $\gamma > 0$ such that*

(2.20)
$$2((x-y), Qb(x,y)) - 2\,\frac{(x-y, Q\mathbf{A}(x,y)Q(x-y))}{((x-y), Q(x-y))} + \mathrm{Tr}\,(\mathbf{A}(x,y)Q) \leq -\gamma\,(Q\,(x-y), x-y)$$

holds for all $x, y \in \mathbf{R}^d$, $x \neq y$.

Then there exists a unique invariant probability π *for the flow*

$$\phi_t(x) = x + \int_0^t b(\phi_u)\,du + \int_0^t \sum_{k=1}^n \sigma_k(\phi_u)\,dB^k(u)$$

such that the transition probability of the flow converges weakly to π.

Proof. The Markovian property of a Brownian flow ϕ_u implies the invariance of the limiting measure π. □

We next discuss the case where A_t is a more general strictly increasing continuous process that is not necessarily equal to t. We still continue to assume that the local characteristic $(a, b, A) \in B^{0,1}_{ub}$.

Theorem 2.7. *Assume that the local characteristic* (a, b, A) *of* F *belongs to* $B^{0,1}_{ub}$, *and let* (B.1) *hold. Then there exists a unique probability* π *for the process* $\varphi_{t_0,t}(x)$ *and*

$$\sup\{\|p_{t_0,t}(x, dy) - \pi(dy)\|_{BL} : x \in K\} \longrightarrow 0 \quad \text{as } t \to \infty,$$

for every compact K. *The limit probability* π *is independent of the initial moment and initial position.*

We need the next three lemmas to prove Theorem 2.7. First let us recall that the topology on $C^{m,\gamma} = \{f \in C^m : D^\alpha f, |\alpha| = m$, are γ-Holder continuous$\}$ is defined by the semi-norm

$$\|f\|_{m+\gamma:K} = \|f\|_{m,K} + \sum_{|\alpha|=m} \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\gamma}. \tag{2.21}$$

Lemma 2.8. *Assume that* $(a, b, A) \in B^{0,1}_{ub}$ *and* (B.1) *holds. Then the solution* $\varphi_{t_0,t}$ *to the stochastic equation*

$$\varphi_{t_0,t}(x) = x + \int_{t_0}^t F(\varphi_{t_0,s}(x), ds)$$

satisfies the inequality

$$E(Q\varphi_{t_0,t}(x),\varphi_{t_0,t}(x))^{(1-\delta)/2} \leq (Qx,x)^{(1-\delta)/2}[E\exp(-\beta(A_t-A_{t_0}))]^{1/2}, \tag{2.22}$$

for some $0<\delta<1$ *and* $\beta>0$.

Proof. Let τ_t be the inverse function of A_t. Set $\tilde{\mathcal{F}}_t = \mathcal{F}_{\tau_t}$ and $\tilde{F}(x,t) = F(x,\tau_t)$. $\tilde{F}(x,t)$ is a $\mathbf{C}^{0,\gamma}$-semi-martingale for any $0<\gamma<1$, and its local characteristic $(a(x,y,\tau_t), b(x,\tau_t), t)$ belongs to $B^{0,1}_{ub}$. Moreover, (B.1) holds in terms of $a(x,y,\tau_t)$ and $b(x,y,\tau_t)$. This implies that the solution of the stochastic differential equation based on $\tilde{F}(x,t)$ has a modification of a stochastic flow $\tilde{\varphi}_{t_0,t}$ of homeomorphisms generated by $\tilde{F}(x,t)$. Therefore, we now know from (2.3) that

$$E[(Q\tilde{\varphi}_{t_0,t}(x),\tilde{\varphi}_{t_0,t}(x))^{1-\delta}\exp(\beta(t-t_0))] \leq (Qx,x)^{1-\delta}.$$

Also, for $t\geq t_0$,

$$E[(Q\tilde{\varphi}_{A_{t_0},A_t}(x),\tilde{\varphi}_{A_{t_0},A_t}(x))^{1-\delta}\exp(\beta(A_t-A_{t_0}))] \leq (Qx,x)^{1-\delta}.$$

Noticing that $\tilde{\varphi}_{t_0,t}(x) = \varphi_{\tau_{t_0},\tau_1}(x)$, we now have

$$E[(Q\varphi_{t_0,t}(x),\varphi_{t_0,t}(x))^{1-\delta}\exp(\beta(A_t-A_{t_0}))] \leq (Qx,x)^{1-\delta}. \tag{2.23}$$

Therefore,

$$\begin{aligned} E(Q\varphi_{t_0,t}(x),\varphi_{t_0,t}(x))^{(1-\delta)/2} &\leq \{E[(Q\varphi_{t_0,t}(x),\varphi_{t_0,t}(x))^{1-\delta}\exp(\beta(A_t-A_{t_0}))]\}^{1/2} \\ &\quad \cdot [E\exp(-\beta(A_t-A_{t_0}))]^{1/2} \\ &\leq (Qx,x)^{(1-\delta)/2}\,[E\exp(-\beta(A_t-A_{t_0}))]^{1/2}. \end{aligned} \tag{2.24}$$

This establishes the result. □

Remark 2.9. From Lemma 2.8 and the dominated convergence theorem, we know that $\{p_{t_0,t}(x,dy), t\geq t_0\}$ is tight if $A_t\to\infty$, a.s., as $t\to\infty$. Hence there exists a subsequence of $\{p_{t_0,t}(x,dy)\}$ converging to a probability measure π_{t_0}.

Lemma 2.10. *Assume that* $(a, b, A_t) \in B^{0,1}_{ub}$, (B.1) *holds and that* $\lim_{t\to\infty} A_t = \infty$, *a.s. If* $\varphi_{t_0,t}$ *is a solution to the stochastic equation*

$$\varphi_{t_0,t} = x + \int_{t_0}^{t} F(\varphi_{t_0,s}, ds),$$

then for every compact $K \subset \mathbf{R}^d$ *and* $r \geq 0$

$$\lim_{t\to\infty} \sup_{x\in K} E|\varphi_{t_0,t+r}(x) - \varphi_{t_0,t}(x)|^{\varepsilon} = 0, \tag{2.25}$$

for some $0 < \varepsilon < 1$.

Proof. We will first reduce the proof to the case $A_t \equiv t$ by changing the time scale. Let τ_u be the inverse function of A_u and put $\tilde{F}(x, u) = F(x, \tau_u)$. $\tilde{F}$ is a continuous $C^{0,\gamma}$-semi-martingale adapted to $(\tilde{\mathcal{F}}_u) = (\mathcal{F}_{\tau_u})$ for any positive $\gamma < 1$. The local characteristic $(a(x, y, \tau_u), b(x, \tau_u), u)$ of $\tilde{F}$ belongs to the class $B^{0,1}_{ub}$. Therefore, the solution $\tilde{\varphi}_{t_0,t}(x) = \varphi_{\tau_{t_0},\tau_t}(x)$ to the stochastic equation governed by $\tilde{F}$ satisfies, by (2.6), the following inequality.

(2.26)
$$E[\exp(\beta(t-t_0))(Q(\tilde{\varphi}_{t_0,t+r}(x)-\tilde{\varphi}_{t_0,t}(x)), \tilde{\varphi}_{t_0,t+r}(x)-\tilde{\varphi}_{t_0,t}(x))^{1-\delta}] \leq 2^{3-2\delta}(Qx, x)^{1-\delta},$$

where $t \geq t_0$, $r \geq 0$. Therefore, for $\bar{r} = A_{t+r} - A_t \geq 0$,

(2.27)
$$E[\exp(\beta(t-t_0))(Q(\tilde{\varphi}_{t_0,t+\bar{r}}(x)-\tilde{\varphi}_{t_0,t}(x)), \tilde{\varphi}_{t_0,t+\bar{r}}(x)-\tilde{\varphi}_{t_0,t}(x))^{1-\delta}] \leq 2^{3-2\delta}(Qx, x)^{1-\delta}.$$

We thus have

(2.28)
$$E[\exp(\beta(At - A_{t_0}))(Q(\tilde{\varphi}_{A_{t_0},A_t+\bar{r}}(x) - \tilde{\varphi}_{A_{t_0},A_t}(x)), \tilde{\varphi}_{A_{t_0},A_t+\bar{r}}(x) - \tilde{\varphi}_{A_{t_0},A_t}(x))^{1-\delta}] \leq 2^{3-2\delta}(Qx, x)^{1-\delta}.$$

That is,

$$E[\exp(\beta(A_t - A_{t_0}))(Q(\varphi_{t_0,t+r}(x) - \varphi_{t_0,t}(x)), \varphi_{t_0,t+r}(x) - \varphi_{t_0,t}(x))^{1-\delta}] \le 2^{3-2\delta}(Qx,x)^{1-\delta}. \tag{2.29}$$

Also,

$$\begin{aligned} &E[(Q(\varphi_{t_0,t+r}(x) - \varphi_{t_0,t}(x)), \varphi_{t_0,t+r}(x) - \varphi_{t_0,t}(x))^{(1-\delta)/2}] \\ &\le \sqrt{E[\exp(\beta(A_t - A_{t_0}))(Q(\varphi_{t_0,t+r}(x)\varphi_{t_0,t}(x)), \varphi_{t_0,t+r}(x) - \varphi_{t_0,t}(x))^{1-\delta}]} \\ &\quad \times \sqrt{E\exp(-\beta(A_t - A_{t_0}))} \\ &\le \sqrt{2^{3-2\delta}}(Qx,x)^{(1-\delta)/2}\sqrt{E\exp(-\beta(A_t - A_{t_0}))}. \end{aligned}$$

By the dominated convergence theorem we now have

$$\lim_{t\to\infty} E\exp(-\beta(A_t - A_{t_0})) = 0.$$

Now for $\varepsilon = (1-\delta)/2$, $r \ge 0$, and every compact K, we have

$$\lim_{t\to\infty}\sup_{x\in K} E|\varphi_{t_0,t+r}(x) - \varphi_{t_0,t}(x)|^{\varepsilon} = 0. \tag{2.30}$$ $\square$

Lemma 2.11. *Assume that* $(a,b,A_t) \in B^{0,1}_{ub}$, (B.1) *holds, and that* $\lim_{t\to\infty} A_t = \infty$ *a.s. If* $\varphi_{t_0,t}$ *is a solution to the stochastic differential equation*

$$\varphi_{t_0,t} = x + \int_{t_0}^{t} F(\varphi_{t_0,s}, ds),$$

then, for every compact $K \subset \mathbf{R}^d$,

$$\lim_{t\to\infty}\sup_{x,y\in K} E|\varphi_{t_0,t}(x) - \varphi_{t_0,t}(y)|^{\varepsilon} = 0, \tag{2.31}$$

where $0 < \varepsilon < 1$.

Proof. We again go to the time change, as in the proof of the previous lemma. It now follows from (2.11), that the solution $\tilde{\varphi}_{t_0,t}(x) =$

$\varphi_{\tau_{t_0},\tau_t}(x)$ to the stochastic differential equation governed by $\tilde{F}$, satisfies the following inequality

$$E[\exp(\lambda(t-t_0))(Q(\tilde{\varphi}_{t_0,t}(x)-\tilde{\varphi}_{t_0,t}(y)),\tilde{\varphi}_{t_0,t}(x)-\tilde{\varphi}_{t_0,t}(y))^{1-\delta}] \le (Q(x-y),x-y)^{1-\delta}. \tag{2.32}$$

Thus, we have for $t > t_0$,

$$E[\exp(\lambda(A_t-A_{t_0}))(Q(\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y)),\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y))^{1-\delta}] \le (Q(x-y),x-y)^{1-\delta}. \tag{2.33}$$

Next

$$\begin{aligned} &E[(Q(\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y)),\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y))^{(1-\delta)/2}] \\ &\le \sqrt{E[\exp(\lambda(A_t-A_{t_0}))(Q(\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y)),\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y))^{1-\delta}]} \\ &\quad \times \sqrt{E\exp(-\lambda(A_t-A_{t_0}))} \\ &\le (Q(x-y),x-y)^{(1-\delta)/2}\sqrt{E\exp(-\lambda(A_t-A_{t_0}))}. \end{aligned}$$

The dominated convergence theorem now gives

$$\lim_{t\to\infty} E\exp(-\lambda(A_t-A_{t_0})) = 0.$$

Now for $\varepsilon(=(1-\delta)/2)$, and every compact K, we have

$$\lim_{t\to\infty}\sup_{x,y\in K} E|\varphi_{t_0,t}(x)-\varphi_{t_0,t}(y)|^{\varepsilon} = 0. \tag{2.34}$$ □

We are now ready to prove Theorem 2.7.

Proof. From Lemma 2.8, Lemma 2.10, Lemma 2.11 and following the steps similar to those in the proof of Theorem 2.5, we obtain that $\{p(t_0,t;x,dy) : t \ge 0\}$ is Cauchy in the metric $d_{\mathbf{BL}}$. Its limiting probability $\pi_{t_0} = \pi$ is independent of the initial point, initial moment t_0, and is unique. That is,

$$\sup\{\|p_{t_0,t}(x,y) - \pi(dy)\|_{BL} : x \in K\} \longrightarrow 0, \quad \text{as } t \to \infty, \tag{2.35}$$

for every compact K. $\square$

Remark 2.12. If there exist constants $\gamma_1 > 0$ and $\delta_1 > 0$ such that, for all $s \geq 0$,

$$A_t - A_s \geq \gamma_1 (t-s)^{\delta_1}, \quad \text{a.s. for sufficiently large } t,$$

then the convergence in Theorem 2.7 is exponentially fast.

Finally we discuss the stability in distribution of the flows governed by equation (1.1) when the local characteristic of F is in $B_b^{0,1}$. Toward this end, we use the fact that a $B_b^{0,1}$-local characteristic is equivalent to a $B_{ub}^{0,1}$ local characteristic.

Assumptions.

- Set $W(t) = \|a(t)\|_1^{\tilde{}} + \|b(t)\|_1$.
- (B.3) There is a symmetric positive definite matrix Q and a constant $\gamma > 0$ such that

$$2((x-y), Qb(x,y,t)) - 2\frac{(x-y, Q\mathbf{A}(x,y,t)Q(x-y))}{((x-y), Q(x-y))} + \mathrm{Tr}\,(\mathbf{A}(x,y,t)Q) \\ \leq -\gamma(Q(x-y), x-y)(1+W(t)),$$

holds for all $x, y \in \mathbf{R}^d$, $x \neq y$, $\omega \in \Omega$.

- (B.4) The following condition holds for some symmetric positive definite matrix C and all $\omega \in \Omega$.

$$2(x, Cb(x,t)) - 2\frac{(x, Ca(x,x,t)Cx)}{(x, Cx)} + \mathrm{Tr}\,(a(x,x,t)C) \\ \leq -\alpha(Cx, x)(1+W(t)),$$

for all sufficiently large $|x|$ and some $\alpha > 0$.

From Assumption (A.1) we know that, for each $t > 0$ and all $\omega \in \Omega$,

$$\begin{aligned} \|a(x,\,x,t)\, -2a(x,\,y,t)\, +a(y,\,y,t)\| &\le |x-y|^2 \|a(t)\|\tilde{}_1, \quad \forall\, x,y \in \mathbf{R}^d, \\ \|a(x,\,x,t)\| &\le L_1 |x|^2 \|a(t)\|\tilde{}_1, \quad \forall\, x \in \mathbf{R}^d, \end{aligned}$$

where $0 < \int_0^\infty \|a(t)\|\tilde{}_1 \, dA_t < \infty$, a.s., and $L_1 > 0$ is a constant.

Similarly, the assumption (A.2) is equivalent to

$$\begin{aligned} |b(x,t)-b(y,t)| &\le |x-y| \|b(t)\|_1, \quad \forall\, x,y \in \mathbf{R}^d, \omega \in \Omega; \\ |b(x,t)| &\le L_1 |x| \|b(t)\|_1, \qquad \forall\, x \in \mathbf{R}^d, \omega \in \Omega. \end{aligned}$$

Here $0 < \int_0^\infty \|b(t)\|_1 \, dA_t < \infty$, a.s., and $L_1 > 0$ is a constant.

Remark 2.13. Notice that the assumptions $(a,b,A) \in B_b^{0,1}$ and (B.3) imply (B.4) for $Q = C$ and every $\alpha \in (0,\gamma)$.

Theorem 2.14. *Assume that the local characteristic* (a,b,A) *of* F *belongs to* $B_b^{0,1}$ *and that* (B.3) *holds. Then there exists a unique probability* π *such that*

$$\sup\{\|p(t;x,y) - \pi(dy)\|_{BL} : x \in K\} \longrightarrow 0, \quad \textit{as } t \to \infty,$$

for every compact K. *The limit probability* π *is independent of the initial moment and initial position.*

Proof. Let $b'(x,t) = (b(x,t))/(1+W(t))$, $a'(x,y,t) = (a(x,y,t))/(1+W(t))$, and $A'_t = \int_0^t (1+W(s))\, dA_s$. From [**7**] we know that $(a',b',A') \in B_{ub}^{0,1}$, and it is equivalent to the local characteristic (a,b,A). It is easy to see that (a',b',A') satisfies (A.3). Also, (B.3) implies (B.1) in terms of (a',b',A'). By Theorem 2.7, we know that $\{p(t;x,dy) : t \ge t_0\}$ is Cauchy in the metric $d_{\mathbf{BL}}$. Its limiting probability π is independent of the initial position and initial moment, and

$$\sup\{\|p(t;x,y) - \pi(dy)\|_{BL} : x \in K\} \longrightarrow 0, \quad \text{as } t \to \infty,$$

for every compact K. $\square$

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