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ON THE COMBINATION OF ROTHE'S METHOD AND BOUNDARY INTEGRAL EQUATIONS FOR THE NONSTATIONARY STOKES EQUATION

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ABSTRACT. We consider the exterior initial boundary value problem for the Stokes equation with Dirichlet boundary condition in $\mathbf{R}^2$. Using Rothe's method, the nonstationary problem is reduced to a system of boundary value problems for the Stokes resolvent equations. By a special approach we obtain a system of boundary integral equations and use a trigonometric quadrature method for the numerical solution. Numerical examples are presented.

1. Introduction and Rothe's method. The boundary integral equation method for the solution of boundary value problems in various applied sciences has been successfully applied for a long time. In the case of nonstationary problems, the use of this method is possible in different variants [**2**]. In one approach the initial boundary value problem can be directly reduced to time-dependent boundary integral equations by potential theory or by Green's formula [**8**], [**9**], [**13**]. Another method consists of having a preliminary semi-discretization of the time-dependent problem and reducing it to boundary value problems for elliptic equations, for example by an integral transformation. Then the integral equation method can be used for the time-independent problems [**2**], [**3**], [**7**]. Sometimes the combination of Laplace transform and boundary integral equations is used. But in this case some essential difficulties arise during numerical calculation of the inverse Laplace transformation (see [**2**]).

One of the possibilities for the semi-discretization consists of using Rothe's method with respect to the time variable. This method is also known as backwards Euler procedure or horizontal line method and is applied both to parabolic and hyperbolic problems. As a result one obtains boundary value problems for the elliptic equation with a recursive righthand side which contains solutions on previous

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time levels. Then a full discretization can be realized by various numerical solution procedures for the boundary integral equations. The combination of Rothe's method and boundary integral equation methods for the parabolic problem has been used in [**2**], [**10**]. A disadvantage of the approach in [**2**], [**10**] stems from the necessity to compute volume integrals leading to considerable computational costs, in particular in the case of an unbounded domain.

One proposal to remedy this drawback is to construct a special potential representation for a solution of the elliptic boundary value problems obtained by Rothe's method which leads to a system of boundary integral equations without volume integrals. Clearly this method can be applied to time-dependent problems with homogeneous differential equations and homogeneous initial conditions. Using this approach, in [**4**], [**6**] the nonstationary problems for the heat and telegraph equations, respectively, have been solved numerically. We note that this method is closely related to the operational quadrature method from [**19**], [**20**] (for details see [**6**]). In this paper we will extend the results in [**4**], [**6**] for the case of the nonstationary Stokes equation. The classical Rothe's method applied to the initial boundary value problems for the Stokes equation in the bounded domains is investigated in [**24**], [**25**], including an error and stability analysis. For the solution of the boundary value problems for the stationary Stokes equations by integral equations method, we refer to [**11**], [**12**], [**21**], [**23**].

The plan of the paper is as follows. In Section 2 we determine a fundamental matrix for the system of stationary Stokes equations which is obtained by Rothe's semi-discretization and introduce the single- and double-layer potentials for this system. Further, we reduce the Dirichlet boundary value problem by a potential approach to systems of boundary integral equations of the first and of the second kind. The parametrization of the integral equations of the first kind is described in Section 3. Here we separate the logarithmic singularity in the kernels and represent them in a form which is convenient for the numerical solution by a trigonometric quadrature method. Finally in Section 4 we present some results of numerical experiments.

Let $D \subset \mathbf{R}^2$ be an unbounded domain such that its complement is bounded and simply connected, and let us assume that the boundary Γ of D is of the class C^2, and let $T > 0$, $Q_T = D \times (0, T]$ and

$\Sigma_T = \Gamma \times (0,T]$. We consider the following initial boundary value problem

$$\frac{1}{c_r}\Delta u - \frac{\partial u}{\partial t} - \nabla p = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$u = f \quad \text{on } \Sigma_T, \tag{1.3}$$

$$u(\cdot,0) = 0 \quad \text{in } D, \tag{1.4}$$

$$u, p \to 0, \quad |x| \to \infty \quad \text{on } (0,T]. \tag{1.5}$$

Here c_r is the Reynolds number, $u = (u_1, u_2)$ and p are the unknown functions, Δ is the Laplace operator, ∇ is the nabla operator, div denotes the divergence and $f = (f_1, f_2)$ is a given function that satisfies the compatibility condition $f(\cdot,0) = 0$. The hydrodynamical initial boundary value problem describes the motion of a viscous incompressible fluid in the domain D [**1**], [**16**]. The vector function u is the velocity field and the scalar function p is the kinematics pressure.

On the equidistant mesh $\{t_n = (n+1)h,\ n = -1, \dots, N-1,\ h = T/N, N \in \mathbf{N}\}$, we approximate the solution (u,p) by the sequence (u_n, p_n), $n = 0, \dots, N-1$ that solves the system of the boundary value problems

$$\frac{1}{c_r}\Delta u_n - \kappa^2 u_n - \nabla p_n = -\kappa^2 u_{n-1} \quad \text{in } D, \tag{1.6}$$

$$\operatorname{div} u_n = 0 \quad \text{in } D, \tag{1.7}$$

$$u_n = f_n \quad \text{on } \Gamma, \tag{1.8}$$

$$u_n, p_n \to 0, \quad |x| \to \infty, \tag{1.9}$$

where $\kappa^2 = 1/h$, $f_n = f(\cdot, t_n)$ and $u_{-1} = 0$. The system (1.6)–(1.9) is obtained from (1.1)–(1.5) by a backwards Euler difference approximation for the time derivative on the grid points t_n.

Theorem 1.1. *The system* (1.6)–(1.9) *has at most one solution.*

Proof. The uniqueness of the classical solution of (1.6)–(1.9) for $n = 0$ is shown in [**23**]. Then the statement of the theorem follows by induction. $\square$

2. Boundary integral equation method. In this section we will reduce the boundary value problem (1.6)–(1.9) to a system of boundary integral equations. First we determine a fundamental solution of the system (1.6), (1.7).

Definition 2.1. The pair (E_n, e) consisting of a 2×2 matrix $E_n(x,y) = (E_{n,1}(x,y), E_{n,2}(x,y))$ with columns $E_{n,1}, E_{n,2}$, $n = 0, 1, \dots, N-1$ and a vector $e(x,y) = (e_1(x,y), e_2(x,y))$ is called a fundamental solution for the system (1.6), (1.7) if

$$\begin{cases} \dfrac{1}{c_r}\Delta E_{n,l} - \kappa^2 E_{n,l} - \nabla e + \kappa^2 E_{n-1,l} = \delta(x-y) I_l, \\ \operatorname{div} E_{n,l} = 0, \quad l = 1, 2. \end{cases} \tag{2.1}$$

Here $I = (I_1, I_2)$ is the 2×2 identity matrix, δ denotes the Dirac function, and the differentiation in (2.1) is taken with respect to x.

Let's consider the polynomials which will be used for compact representation of E_n,

$$v_n(r) = \sum_{m=0}^{[n/2]} a_{n,2m} r^{2m}, \qquad w_n(r) = \sum_{m=0}^{[(n-1)/2]} a_{n,2m+1} r^{2m+1}$$

for $n = 0, 1, \dots, N-1 (w_0 = 0)$, where the coefficients $a_{n,m}$ satisfy the recurrence relations

$$\begin{aligned} a_{n,0} &= 1, \quad n = 0, 1, \dots, N-1, \\ a_{n,n} &= \frac{\kappa}{2n} a_{n-1,n-1}, \quad n = 1, 2, \dots, N-1, \\ a_{n,m} &= \frac{1}{2\kappa m}\left\{4\left[\frac{m+1}{2}\right]^2 a_{n,m+1} + \kappa^2 a_{n-1,m-1}\right\}, \\ &\qquad m = n-1, \dots, 1 \end{aligned}$$

and $[r]$ denotes the integer part of $r \geq 0$. Next we introduce the sequences of functions

$$\Phi_n(\kappa, r) = K_0(\kappa r) v_n(r) + K_1(\kappa r) w_n(r)$$

and

$$\Psi_n(\kappa, r) = \frac{n+1}{\kappa^2} \ln \frac{1}{r} - \sum_{m=0}^{n} b_{n-m} [\Phi_m(\kappa, r) - \Phi_{m-1}(\kappa, r)],$$

where K_0 and K_1 are the modified Hankel functions of order zero and one, respectively, and $b_n := (n+1)/\kappa^2$. Throughout this paper all functions and constants with a negative index number are set equal to zero.

Note that in [**6**] it has been shown that Φ_n are singular solutions for the recurrence sequence of Helmholtz equations which are obtained for parabolic equation by Rothe's method.

Theorem 2.2. *The pair* (E_n, e) *with*

$$E_n(x,y) = -c_\pi c_r[I\Phi_n(\gamma, |x-y|) + \mathrm{grad}_x \mathrm{grad}_x^\top \Psi_n(\gamma, |x-y|)] \tag{2.2}$$

and

$$e(x,y) = -\frac{c_\pi(x-y)}{|x-y|^2} \tag{2.3}$$

is a fundamental solution of (1.6), (1.7). *Here we have set* $c_\pi := (2\pi)^{-1}$ *and* $\gamma := \kappa\sqrt{c_r}$.

Proof. For a function $g(x)$, $x \in \mathbf{R}^2$, we define the standard direct and inverse Fourier transform by

$$\hat{g}(\xi) = F(g) = c_\pi \int_{\mathbf{R}^2} g(x) e^{-i\langle x,\xi\rangle}\, dx,$$
$$g(x) = F^{-1}(\hat{g}) = c_\pi \int_{\mathbf{R}^2} \hat{g}(\xi) e^{i\langle x,\xi\rangle}\, d\xi,$$

respectively. Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbf{R}^2$. By using the direct Fourier transformation to the system (2.1), we obtain that

$$\begin{cases} -\left(\dfrac{|\xi|^2}{c_r} + \kappa^2\right)\hat{E}_{n,j} - i\xi\hat{e}_j + \kappa^2\hat{E}_{n-1,j} = c_\pi I_j, \\ \langle \xi, \hat{E}_{n,j}\rangle = 0, \end{cases} \tag{2.4}$$

where $\hat{E}_n = F(E_n)$ and $\hat{e} = F(e)$. From (2.4) it follows, for $n = 0$,

$$\begin{cases} \hat{E}_{0,j} = -\dfrac{c_r}{|\xi|^2 + \gamma^2}(c_\pi I_j + i\xi\hat{e}_j), \\ \langle \xi, \hat{E}_{0,j}\rangle = 0. \end{cases}$$

Then

$$\hat{e}(\xi) = \frac{ic_\pi \xi}{|\xi|^2}$$

and

$$\hat{E}_{0,j} = -\frac{c_\pi c_r}{|\xi|^2 + \gamma^2}\left(I_j - \frac{\xi_j \xi}{|\xi|^2}\right).$$

From this, it follows that

$$e(x,y) = F^{-1}(\hat{e}) = c_\pi \nabla \ln \frac{1}{|x-y|}.$$

Now by induction from (2.4) we can deduce that

$$\hat{E}_n(\xi) = -\, c_\pi c_r [I - J(\xi)] \hat{\Phi}_n(\xi), \tag{2.5}$$

where the matrix J is defined by $J(w) := ww^\top/|w|^2$ for $w \in \mathbf{R}^2 \setminus \{0\}$ and

$$\hat{\Phi}_n(\xi) = \sum_{m=0}^{n} \frac{\gamma^{2m}}{(|\xi|^2 + \gamma^2)^{m+1}}.$$

By reduction to ordinary differential equations, for the system of the elliptic equations

$$\Delta g_n - \kappa^2 g_n + \kappa^2 g_{n-1} = 0, \quad n = 0, 1, \ldots, N-1,\ g_{-1} = 0 \tag{2.6}$$

in a plane, it is shown in [**6**] that the functions $c_\pi \Phi_n(\kappa, |x-y|)$ are fundamental solutions. This gives us

$$F(c_\pi \Phi_n) = c_\pi \sum_{m=0}^{n} \frac{\gamma^{2m}}{(|\xi|^2 + \gamma^2)^{m+1}}$$

and then for $\hat{\Phi}_n$ we have the inverse Fourier transform

$$F^{-1}(c_\pi \hat{\Phi}_n) = c_\pi \Phi_n(\gamma, |x-y|). \tag{2.7}$$

With the help of the identity

$$\frac{1}{r(r+a)^n} = \frac{1}{a^n r} - \sum_{m=1}^{n} \frac{a^{m-n-1}}{(r+a)^m}, \quad a_r(r+a) \neq 0$$

we can write

$$\hat{\Phi}_n(\xi)|\xi|^{-2} = \frac{n+1}{\gamma^2|\xi|^2} - \sum_{m=0}^{n} b_{n-m} \frac{\gamma^{2m}}{(|\xi|^2+\gamma^2)^{m+1}}.$$

Now, since

$$F^{-1}(J(\xi)\hat{\Phi}_n) = -\operatorname{grad}_x \operatorname{grad}_x^\top (F^{-1}(\hat{\Phi}_n|\xi|^{-2}))$$

and

$$F^{-1}\left(\frac{c_\pi \gamma^{2n}}{(|\xi|^2+\gamma^2)^{n+1}}\right) = c_\pi[\Phi_n(\gamma,|x-y|) - \Phi_{n-1}(\gamma,|x-y|)],$$
$$n = 0, 1, \dots, N-1,$$

we obtain

$$F^{-1}(J(\xi)\hat{\Phi}_n) = -\operatorname{grad}_x \operatorname{grad}_x^\top \Psi_n(\gamma,|x-y|). \tag{2.8}$$

Thus (2.2) follows by taking the inverse Fourier transform of (2.5) using (2.7) and (2.8). □

Remark 2.3. 1. We note that for $n = 0$ from (2.2) and (2.3), the fundamental solution of the resolvent Stokes equation (see [**23**]) can be obtained.

2. The double grad calculation gives us the following equivalent representation for E_n

$$E_n(x,y) = -c_\pi c_r \left[I \left\{ \Phi_n(\gamma,r) + \frac{1}{r}\frac{\partial \Psi_n(\gamma,r)}{\partial r} \right\} + J(x-y) \left\{ \frac{\partial^2 \Psi_n(\gamma,r)}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi_n(\gamma,r)}{\partial r} \right\} \right] \tag{2.9}$$

with $r = |x-y|$. The further calculations show that the terms in figure brackets only have logarithmic singularity for $x = y$ (for details see Section 3).

For the system (1.6), (1.7), we consider both the single- and double-layer potentials:

a) for the velocity field

$$V_n(x) = \sum_{m=0}^{n} \int_\Gamma E_{n-m}(x,y)\varphi_m(y)\,ds(y)$$

and

$$W_n(x) = \sum_{m=0}^{n} \int_\Gamma \tilde{E}_{n-m}(x,y)\tilde{\varphi}_m(y)\,ds(y),$$

(b) for the pressure

$$v_n(x) = \sum_{m=0}^{n} \int_\Gamma \langle e(x,y), \varphi_m(y)\rangle\,ds(y)$$

and

$$w_n(x) = \sum_{m=0}^{n} \int_\Gamma \langle \tilde{e}(x,y), \tilde{\varphi}_m(y)\rangle\,ds(y),$$

respectively.

Here φ_n and $\tilde{\varphi}_n$ are unknown density vectors, $\tilde{E}_n = (\tilde{E}_{n,1}, \tilde{E}_{n,2})$ is the 2×2 matrix, and $\tilde{e} = (\tilde{e}_1, \tilde{e}_2)$, the vector with

$$\tilde{E}_{n,l} = T_x(E_{n,l}, e_l)\nu(y), \qquad \tilde{e} = T_x(e,0)\nu(y),$$

where ν is the outward unit normal vector to the boundary Γ and $T(u,p) = pI - (\nabla u + \nabla^\top u)$ is the stress tensor.

Theorem 2.4. *The single-layer potential* (V_n, v_n) *solves the system of boundary value problems* (1.6)–(1.9) *provided the densities* φ_n *solve the system of integral equations of the first kind*

$$\int_\Gamma E_0(x,y)\varphi_n(y)\,ds(y) = f_n(x) - \sum_{m=0}^{n-1} \int_\Gamma E_{n-m}(x,y)\varphi_m(y)\,ds(y), \quad x \in \Gamma, \tag{2.10}$$

for $n = 0, 1, \dots, N-1$.

Proof. Since (E_n, e) is a fundamental solution of the system (1.6), (1.7), the potential (V_n, v) satisfies this system for $x \in D$. From the asymptotic properties $K_0(r) \sim \ln(1/r)$ and $K_1(r) \sim (1/r)$ for $r \to 0$, it follows that the functions $E_n(x, y)$ have a logarithmic singularity for $x = y$. Therefore, as in the case of the hydrodynamic potentials for the stationary Stokes equation [**12**], [**16**], the potentials V_n are continuous in $\mathbf{R}^2$. The system of integral equations (2.10) ensures that the boundary conditions (1.8) are fulfilled. From the asymptotics of the modified Hankel functions for large arguments (see [**18**]) it follows that the V_n and v_n satisfy the condition (1.9). $\square$

Analogously, we can prove the following result for the double-layer potential.

Theorem 2.5. *The double-layer potential* (W_n, w_n) *solves the system of boundary value problems* (1.6)–(1.9) *provided the densities* $\tilde{\varphi}_n$ *solve the system of integral equations of the second kind*

$$\begin{aligned}(2.11)\quad \frac{1}{2}\tilde{\varphi}_n(x) &+ \int_\Gamma \tilde{E}_0(x, y)\tilde{\varphi}_n(y)\, ds(y)\\ &= f_n(x) - \frac{1}{2}\sum_{m=0}^{n-1} \tilde{\varphi}_m(x) - \sum_{m=0}^{n-1} \int_\Gamma \tilde{E}_{n-m}(x, y)\tilde{\varphi}_m(y)\, ds(y),\end{aligned}$$

for $n = 0, 1, \dots, N-1$.

3. Parametrization of the integral equations. In this section we consider the parametrization and the numerical solution of the integral equations of the first kind (2.10). The case of the integral equations of the second kind does not have the principal difference, unless the transformation of the kernels in (2.11) needs more complicated manipulation. On the other hand, in the case of the Stokes problem with nonsmooth or open boundary Γ, use of the integral equations of the first kind is preferable. First we want to analyze the kernels in our integral equations and show the logarithmic singularity in them. For compact representation of the results of the derivation in (2.9), we need to introduce the polynomials

$$\begin{aligned}
v_n^{(1)}(r) &= 2\sum_{m=1}^{[n/2]} m a_{n,2m} r^{2m-1},\\
v_n^{(2)}(r) &= 2\sum_{m=1}^{[n/2]} m(2m-1) a_{n,2m} r^{2m-2},\\
w_n^{(1)}(r) &= \sum_{m=0}^{[(n-1)/2]} (2m+1) a_{n,2m+1} r^{2m},\\
w_n^{(2)}(r) &= 2\sum_{m=1}^{[(n-1)/2]} m(2m+1) a_{n,2m+1} r^{2m-1}.
\end{aligned}$$

Next we define

$$\begin{aligned}
\eta_n^{(1)}(r) &= \frac{v_n^{(1)}(r)}{r} - \frac{\gamma w_n(r)}{r},\\
\eta_n^{(2)}(r) &= v_n^{(2)}(r) + \gamma^2 v_n(r) - 2\gamma w_n^{(1)}(r) + \frac{\gamma w_n(r)}{r},\\
\xi_n^{(1)}(r) &= -\frac{\gamma v_n(r)}{r} + \frac{w_n^{(1)}(r)}{r} - \frac{w_n(r)}{r^2},\\
\xi_n^{(2)}(r) &= \frac{\gamma v_n(r)}{r} - 2\gamma v_n^{(1)}(r) + \gamma^2 w_n(r)\\
&\qquad + \frac{2w_n(r)}{r^2} - \frac{2w_n^{(1)}(r)}{r} + w_n^{(2)}(r).
\end{aligned}$$

Now, by using the properties of the modified Hankel functions and straightforward calculations from (2.9), we obtain the following representation for E_n,

$$E_n(x,y) = -c_\pi c_r[\Psi_n^{(1)}(\gamma,|x-y|)I + \Psi_n^{(2)}(\gamma,|x-y|)J(x-y)]. \tag{3.1}$$

Here, for $r>0$, the functions $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$ are given by

$$\Psi_n^{(1)}(\gamma,r) = \Phi_n(\gamma,r) - \frac{n+1}{\gamma^2 r^2} - \sum_{m=0}^{n} b_{n-m}\big[\Phi_m^{(1)}(\gamma,r) - \Phi_{m-1}^{(1)}(\gamma,r)\big]$$

and

$$\Psi_n^{(2)}(\gamma, r) = \frac{2(n+1)}{\gamma^2 r^2} - \sum_{i=1}^{2} (-1)^i \sum_{m=0}^{n} b_{n-m} \big[\Phi_m^{(i)}(\gamma, r) - \Phi_{m-1}^{(i)}(\gamma, r)\big],$$

where we have set

$$\Phi_n^{(i)}(\gamma, r) = K_0(\gamma r)\eta_n^{(i)}(r) + K_1(\gamma r)\xi_n^{(i)}(r), \quad i = 1, 2.$$

By the power series for the functions K_0 and K_1 (see [**18**]), the functions $\Psi_n^{(i)}$, $i = 1, 2$, can be written in the form

$$\Psi_n^{(i)}(\gamma, r) = \ln\left(\frac{\gamma r}{2}\right)\theta_n^{(i)}(\gamma, r) + \chi_n^{(i)}(\gamma, r), \quad i = 1, 2, \tag{3.2}$$

where

$$\begin{aligned}
\theta_n^{(1)}(\gamma, r) &= -I_0(\gamma r)v_n(r) + I_1(\gamma r)w_n(r) \\
&\qquad - \sum_{m=0}^{n} b_{n-m} \sum_{i=0}^{1} (-1)^i [-I_0(\gamma r)\eta_{m-i}^{(1)}(r) + I_1(\gamma r)\xi_{m-i}^{(1)}(r)], \\
\theta_n^{(2)}(\gamma, r) &= -\sum_{m=0}^{n} b_{n-m} \sum_{k=1}^{2} \sum_{i=0}^{1} (-1)^i [-I_0(\gamma r)\eta_{m-i}^{(k)}(r) + I_1(\gamma r)\xi_{m-i}^{(k)}(r)].
\end{aligned}$$

in terms of the modified Bessel functions I_0 and I_1. Now we use the power series for I_0 and I_1 [**18**] and the definition of v_n, w_n, $\eta_n^{(i)}$ and $\xi_n^{(i)}$ and arrive at the expansions for $r \to 0$

$$\theta_n^{(i)}(\gamma, r) = \alpha_n(2 - i) + O(r^2), \quad i = 1, 2 \tag{3.3}$$

and

$$\chi_n^{(i)}(\gamma, r) = \beta_n^{(i)} + O(r^2), \quad i = 1, 2, \tag{3.4}$$

with

$$\alpha_n = -a_{n,0} - \sum_{m=0}^{n} b_{n-m} \sum_{i=0}^{1} (-1)^i \Big[-2a_{m-i,2} + \gamma a_{m-i,1} - \frac{\gamma^2}{2} a_{m-i,0} \Big],$$

$$\beta_n^{(1)} = -C a_{n,0} + \frac{1}{\gamma} a_{n,1} - \sum_{m=0}^{n} b_{n-m} \sum_{i=0}^{1} (-1)^i \Big[\frac{2}{\gamma} a_{m-i,3} - (1+2C) a_{m-i,2} - \gamma C a_{m-i,1} + \frac{\gamma^2}{4}(1-2C) a_{m-i,0} \Big],$$

$$\beta_n^{(2)} = \sum_{m=0}^{n} b_{n-m} \sum_{i=0}^{1} (-1)^i \Big[2a_{m-i,2} - \gamma a_{m-i,1} + \frac{\gamma^2}{2} a_{m-i,0} \Big].$$

Here $C = 0.57721\ldots$ is Euler's constant.

We assume that the boundary curve Γ is given through a parametric representation

$$\Gamma = \{x(s) : 0 \le s \le 2\pi\},$$

where $x : \mathbf{R} \to \mathbf{R}^2$ is C^2 and 2π-periodic with $|x'(s)| > 0$ for all s. Then we transform the system (2.10) into the parametric form

$$(3.5) \qquad -c_\pi \int_0^{2\pi} H_0(s,\sigma)\psi_n(\sigma)\,d\sigma = g_n(s) + c_\pi \sum_{m=0}^{n-1} \int_0^{2\pi} H_{n-m}(s,\sigma)\psi_m(\sigma)\,d\sigma, \quad 0 \le s \le 2\pi,$$

where $\psi_n(s) := |x'(s)|\varphi_n(x(s))$, $g_n(s) := (1/c_r) f_n(x(s))$, and where the kernels are given by

$$H_n(s,\sigma) := -\frac{1}{c_\pi c_r} E_n(x(s), x(\sigma))$$

for $s \ne \sigma$ and $n = 0, 1, \ldots, N-1$. From (3.1) and (3.2) it follows that the kernels in (3.5) can be written in the form

$$H_n(s,\sigma) = \ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) H_n^1(s,\sigma) + H_n^2(s,\sigma)$$

for $n = 0, 1, 2, \dots, N-1$, where

$$H_n^1(s,\sigma) = \frac{1}{2}[\theta_u^{(1)}(\gamma, |x(s)-x(\sigma)|)I + \theta_n^{(2)}(\gamma, |x(s)-x(\sigma)|)J(x(s)-x(\sigma))]$$

and

$$H_n^2(s,\sigma) = H_n(s,\sigma) - \ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)H_n^1(s,\sigma).$$

The asymptotic expansions (3.3) and (3.4) show that these kernels are smooth with the diagonal terms given through

$$H_n^1(s,s) = \frac{\alpha_n}{2}I,$$

and

$$H_n^2(s,s) = \frac{1}{2}\left(\alpha_n \ln\left(\frac{\gamma^2|x'(s)|^2 e}{4}\right) + 2\beta_n^{(1)}\right)I + \beta_n^{(2)}\tilde{J}(s)$$

for $n = 0, 1, \dots, N-1$. Here the matrix $\tilde{J}$ is defined by $\tilde{J}(s) := \tilde{w}(s)\tilde{w}^\top(s)$, where $\tilde{w}$ denotes the unit tangential vector to Γ. Thus we arrive at the system of integral equations

(3.6)

$$c_\pi \int_0^{2\pi} \left\{ \ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)H_0^1(s,\sigma) + H_0^2(s,\sigma)\right\}\psi_n(\sigma)\,d\sigma = G_n(s),$$
$$0 \le s \le 2\pi$$

with righthand sides

$$G_n(s) = -\,g_n(s) - c_\pi \sum_{m=0}^{n-1} \int_0^{2\pi} \left\{H_{n-m}^1(s,\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) + H_{n-m}^2(s,\sigma)\right\}\psi_m(\sigma)\,d\sigma.$$

The system (3.6) can also be written in operator form

$$(\alpha_0 S + A)\psi_n = G_n, \quad n = 0, 1, \dots, N-1,$$

with the integral operators

$$(S\varphi)(s) = \frac{c_\pi}{2}\int_0^{2\pi} \ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)\varphi(\sigma)\,d\sigma$$

and

$$(A\varphi)(s) = c_\pi \int_0^{2\pi} \left\{ \tilde{H}_0^1(s,\sigma) \ln \left(\frac{4}{e} \sin^2 \frac{s-\sigma}{2} \right) + H_0^2(s,\sigma) \right\} \varphi(\sigma)\, d\sigma,$$

where $\tilde{H}_0^1(s,\sigma) := H_0^1(s,\sigma) - (\alpha_0/2)I$.

Let $C^{0,\alpha}[0,2\pi]$ and $C^{1,\alpha}[0,2\pi]$ for $0 < \alpha \le 1$ denote the classical Hölder spaces of vector-valued functions. It is obvious that the operator S is bounded from $C^{0,\alpha}[0,2\pi]$ to $C^{1,\alpha}[0,2\pi]$ and has a bounded inverse (see [**14**]). By analogous arguments as in [**15**], we can show that the operator A is compact from $C^{0,\alpha}[0,2\pi]$ to $C^{1,\alpha}[0,2\pi]$. Hence, the uniqueness Theorem 1.1 and the Riesz theory (see [**14**]) lead us to the following existence result.

Theorem 3.1. *For any sequence g_n in $C^{1,\alpha}[0,2\pi]$, the system* (3.6) *possesses a unique solution ψ_n in $C^{0,\alpha}[0,2\pi]$.*

For the numerical solution of the integral equations (3.6), we used the trigonometric quadrature method that is described in detail in [**5**], [**6**], including a convergence and error analysis in Hölder spaces. This analysis exhibits the dependence of the convergence order on the smoothness of the boundary function and the boundary curve, i.e., the proposed method belongs to algorithms without the "saturation effect." We note that the numerical method in [**5**] requires the implementation of the factor $\sin^2((s-\sigma)/2)$ in the kernel $\tilde{H}_0^1$. As is shown in [**22**], this decomposition is not necessary for performing the error analysis. Next we remark that the error estimate of our quadrature method also depends on the kernel of the integral equation. On the other hand, the kernels of (3.6) contain the modified Hankel function $K_0(\gamma r)$ which has a pronounced delta function-like behavior when the time step size $h \to 0$ ($\gamma \to \infty$). This calls for a balance between the time and spatial discretization parameters for the numerical solution of the system (3.6). As a variant to the procedure described in this paper, one can consider the special collocation method in [**17**] that has a convergence rate independent of h.

TABLE 1. Relative errors for the velocity.

		e_r^1			e_r^2		
t	M	$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.2$	$h = 0.1$	$h = 0.05$
0.2	16	0.189967	0.093211	0.045853	0.138331	0.050058	0.012179
	32	0.189966	0.093211	0.045853	0.138331	0.050058	0.012179
0.4	16	0.122854	0.058207	0.028345	0.165779	0.092721	0.047793
	32	0.122854	0.058207	0.028345	0.165778	0.092720	0.047793
0.6	16	0.084955	0.041575	0.021089	0.257851	0.125482	0.063654
	32	0.084955	0.041575	0.021089	0.257850	0.125482	0.063654
0.8	16	0.066643	0.034062	0.017825	0.295588	0.144856	0.075443
	32	0.066643	0.034062	0.017825	0.295588	0.144856	0.075443
1.0	16	0.056692	0.029962	0.016005	0.323229	0.162898	0.086774
	32	0.056692	0.029962	0.016005	0.323228	0.162897	0.086774

4. Numerical experiments. In this section we will demonstrate the feasibility of the proposed method by a test example. As a boundary we choose the curve

$$\Gamma = \{x(s) = (0.2\cos s, 0.4\sin s - 0.3\sin^2 s)\}, \quad 0 \le s \le 2\pi, \tag{4.1}$$

which is illustrated in Figure 1. The boundary function f is given by the restriction of the fundamental solution for the Stokes equation (1.1) (see [**1**])

$$f(x,t) = \left(\frac{\partial^2 g}{\partial x_2^2}, -\frac{\partial^2 g}{\partial x_1 \partial x_2}\right)^\top, \tag{4.2}$$

where

$$g(x,t) = \ln|x| + \frac{1}{2}\,E_1\left(\frac{c_r|x|^2}{4t}\right).$$

Here E_1 denotes the exponential integral function [**18**]. The Reynolds number is chosen as $c_r = 1$ and the time interval is $[0,1]$. In Table 1, the relative errors

$$e_r(x,t_n) := |u(x,t_n) - \tilde{u}(x,t_n)|/|u(x,t_n)|$$

at the spatial point $x = (0.7, -0.5)$ are presented, where u is the exact solution of the problem (1.1), (1.5). The absolute errors for the pressure

$$e_a(x,t_n) := |p(x,t_n) - \tilde{p}(x,t_n)|$$

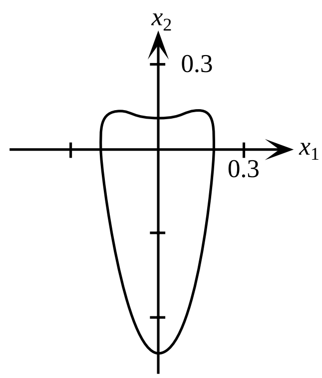

FIGURE 1. Boundary Γ.

at the same spatial point are illustrated in Figure 2. Note that the exact pressure in the case of the boundary function (4.2) has the representation

$$p(x,t) = \delta(t)\frac{x_1}{|x|^2}.$$

The exponential convergence with respect to the spatial discretization and linear convergence with respect to the time discretization are clearly exhibited.

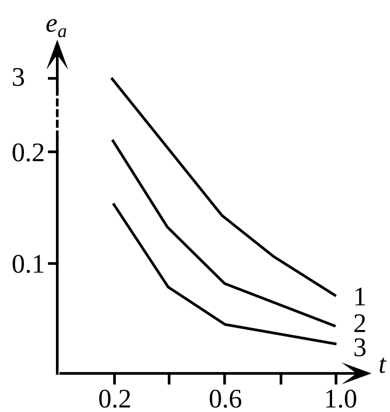

FIGURE 2. Absolute error for the pressure 1–$h = 0.2$, 2–$h = 0.1$, 3–$h = 0.05$.

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