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FINITE ELEMENT APPROXIMATION WITH QUADRATURE TO A TIME DEPENDENT PARABOLIC INTEGRO-DIFFERENTIAL EQUATION WITH NONSMOOTH INITIAL DATA

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ABSTRACT. In this paper we analyze the effect of numerical quadrature in the finite element analysis for a time dependent parabolic integro-differential equation with nonsmooth initial data. Both semi-discrete and fully discrete schemes are discussed and optimal order error estimates are derived in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms using energy method when the initial function is only in H_0^1. Further, quasi-optimal maximum norm estimate is shown to hold for rough initial data.

1. Introduction. In this paper we consider a finite element Galerkin method with spatial quadrature for the following time dependent parabaolic integro-differential equation

$$
\begin{aligned}
u_t + A(t)u &= \int_0^t B(t,s)u(s)\,ds \quad \text{in } \Omega\times J,\\
u &= 0 \quad \text{on } \partial\Omega\times J,\\
u(\cdot,0) &= u_0 \quad \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where Ω is a convex bounded domain in R^2 with boundary $\partial\Omega$, $u(x,t)$ is a real valued function in R^2 and J denotes the time interval $(0,T]$ with $T<\infty$. Here $A(t)$ is a time dependent, positive definite, self-adjoint and uniformly elliptic second order partial differential operator and $B(t,s)$ is a general second order partial differential operator, both with smooth coefficients.

Let $H_0^1(\Omega)=\{\phi\in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$, and let $A(t;\cdot,\cdot)$ and $B(t,s;\cdot,\cdot)$ be the bilinear forms associated with the operators A and B, respectively. The weak formulation of the problem (1.1) is defined

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as follows. Find $u : \bar{J} \to H_0^1(\Omega)$ such that

$$(1.2)\qquad \begin{aligned} (u_t,\phi) + A(t;u,\phi) &= \int_0^t B(t,s;u(s),\phi)\,ds \\ &\forall\,\phi \in H_0^1(\Omega), \quad t \in J, \\ u(0) &= u_0. \end{aligned}$$

For the purpose of numerical solution we shall assume that we are given a family of subspaces $\{S_h\}$, $0 < h < 1$, of $H_0^1(\Omega)$ consisting of continuous, piecewise linear functions on a quasi-uniform triangulation $\mathcal{F}_h = \{\tau_h\}$ of Ω with its boundary vertices on $\partial\Omega$ and which vanish outside the polygonal domain $\Omega_h \subset \Omega$ determined by $\mathcal{F}_h$. Further, the finite dimensional subspaces $\{S_h\}$ are such that for $r = \{1,2\}$,

$$(1.3)\qquad \begin{aligned} \inf_{\chi\in S_h}\{\|\phi-\chi\| + h\|\phi-\chi\|_1\} &\le Ch^r\|\phi\|_r, \\ \phi &\in H_0^1(\Omega)\cap H^r(\Omega). \end{aligned}$$

Throughout this paper we shall work on the standard Sobolev spaces $H_0^1(\Omega)$ and $H^s(\Omega)$ with norm $\|\cdot\|_1$ and $\|\cdot\|_s$, respectively. We shall use the notations $\|\cdot\|$ and $\|\cdot\|_{0,p}$ to denote the norm on $L^2(\Omega)$ and $L^p(\Omega)$, $p \neq 2$, respectively. The norm of the Sobolev space $W^{s,\infty}(\Omega)$ is denoted by $\|\cdot\|_{s,\infty}$, and that of $L^\infty(\Omega)$ by $\|\cdot\|_\infty$.

We recall that the standard semi-discrete finite element approximation $\bar{u}_h$ of u is a function $\bar{u}_h : \bar{J} \to S_h$ satisfying

$$(1.4)\qquad \begin{aligned} (\bar{u}_{ht},\chi) + A(t;\bar{u}_h,\chi) &= \int_0^t B(t,s;\bar{u}_h(s),\chi)\,ds \\ &\forall\,\chi \in S_h, \quad t \in J, \\ \bar{u}_h(0) &= \bar{u}_{h,0}, \end{aligned}$$

where $\bar{u}_{h,0} \in S_h$ is a suitable approximation to u_0.

In practice, the spatial integrals appearing in (1.4) are evaluated numerically by using some quadrature rules. In this paper we shall apply the following numerical quadrature to approximate the inner products and the bilinear forms in (1.4). Let $p_{\tau_h,j}$, $j = 1,2,3$, be the vertices of a triangle $\tau_h \in \mathcal{F}_h$. Define an approximation of the inner product in S_h and induced norm by

$$(1.5)\qquad (\phi,\psi)_h = \sum_{\tau_h\in\mathcal{F}_h} \frac{1}{3}\,\mathrm{area}\,(\tau_h) \sum_{j=1}^{3} \phi(p_{\tau_h,j})\psi(p_{\tau_h,j}),$$

and

$$\|\phi\|_h = (\phi, \phi)_h^{1/2},$$

respectively. For the purpose of applying quadrature rules to all the terms in (1.4), we set $A_h(t; \cdot, \cdot)$ and $B_h(t, s; \cdot, \cdot)$ as the bilinear forms with quadrature corresponding to the operators A and B, respectively. Therefore, instead of (1.4), we consider the following semi-discrete finite element method with quadrature: Find $u_h(t) \in S_h$ such that

$$(1.6)\qquad \begin{aligned} (u_{ht}, \chi)_h + A_h(t; u_h, \chi) &= \int_0^t B_h(t, s; u_h(s), \chi)\, ds \\ &\forall\, \chi \in S_h, \quad t \in J, \\ &u_h(0) = P_h u_0. \end{aligned}$$

Here $P_h u_0$ is the L^2-projection of u_0 onto S_h defined by

$$(1.7)\qquad (P_h u_0, \chi) = (u_0, \chi), \quad \chi \in S_h.$$

Along with the semi-discrete approximation we shall also discuss the time discretization of (1.6) based on the backward Euler method. Let $k > 0$ be the time step, $t_n = nk$ with $T = Nk$, and let $U^n = U(t_n)$ be an approximation of $u(t_n)$. Then the discrete time Euler scheme is to seek a function U^n, $n = 1, 2, \ldots, N$, satisfying

$$(1.8)\qquad \begin{aligned} (\bar{\partial}_t U^n, \chi)_h + A_h(t_n; U^n, \chi) &= k \sum_{j=0}^{n-1} B_h(t_n, t_j; U^j, \chi) \\ &\forall\, \chi \in S_h, \\ &U^0 = P_h u_0, \end{aligned}$$

where $\bar{\partial}_t U^n = k^{-1}(U^n - U^{n-1})$ and the integral term in (1.6) is approximated by the rectangle rule

$$\int_0^{t_n} \phi(s)\, ds \approx k \sum_{j=0}^{n-1} \phi(t_j), \quad 0 < t_n \le T.$$

Earlier, Raviart [**13**] first studied the effect of quadrature on the finite element solution to a parabolic differential equation, and the analysis for nonlinear parabolic equation with quadrature was discussed in Chou and Li [**4**] and Nie and Thomée [**9**]. For the heat equation in two space dimensions, Chen and Thomée [**3**] considered both semi-discrete and fully discrete lumped mass Galerkin methods and obtained error estimates of optimal order in L^2 and of almost optimal in L^∞ when the initial data is in $H^2(\Omega) \cap H_0^1(\Omega)$. Subsequently, Thomée et al. [**16**] have obtained improved estimates with initial data only in $H_0^1(\Omega)$ using semi-group theoretic arguments.

Recently, Pani and Peterson [**10**] extended the results of Chen and Thomée [**3**] to a time independent parabolic integro-differential equation, $A(t) \equiv A$, with quadrature formula applied to all the terms appearing in the Galerkin formulation and obtained optimal error estimates for both smooth and nonsmooth initial data. For the nonsmooth case, it is assumed that the initial function $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. In addition, almost optimal pointwise estimates are shown to hold under strong compatibility assumptions on u_0, i.e., $Au_0 = 0$ on $\partial\Omega$ (see [**10**], p. 1095, last paragraph).

In the present paper an attempt has been made to improve upon the results of Pani and Peterson [**10**] by requiring less regularity assumptions on the initial function u_0. More precisely, optimal estimates in $L^\infty(H^1)$, $L^\infty(L^2)$ norms and quasi-optimal estimate in $L^\infty(L^\infty)$ norm are derived using energy arguments, when the initial function $u_0 \in H_0^1(\Omega)$. However, for optimal estimates, the initial condition is implemented as an L^2-projection of u_0. Further, some results related to the Ritz-Volterra projection are also explored, which are subsequently used to obtain quasi-optimal L^∞-error estimates. Compared to Pani and Peterson [**10**], it is possible to avoid the stringent compatibility condition for maximum norm estimate using modified pointwise estimates of Ritz-Volterra projection and an auxiliary estimate $\| \int_0^t u(s)\, ds\|_3$ (see (4.8) in Theorem 4.2).

The finite element error analysis without quadrature (i.e., in the case of exact integration for the parabolic integro-differential equation with nonsmooth initial data can be found in Crouzeix and Thomée [**5**], when B is of lower order. For an arbitrary second order differential operator B and time dependent A, Pani and Sinha [**11**] proved an optimal L^2-estimate using energy arguments, when the initial data are in $L^2(\Omega)$. Earlier, Thomée and Zhang [**17**] discussed the error analysis for time independent A with $u_0 \in L^2$ using the semi-group theoretic approach. The numerical solution by means of finite element methods has been investigated by Le Roux and Thomée [**6**], Cannon and Lin [**1, 2**], Lin et al. [**7**], Pani et al. [**12**] and Zhang [**18**].

The plan of this paper is as follows. Section 2 contains some preliminary materials and a prior estimates for u and u_h. Ritz-Volterra projection and related estimates are derived in Section 3. Section 4 is devoted to semi-discrete error analysis for nonsmooth initial data. Finally, Section 5 deals with the discrete-in-time backward Euler method and optimal error estimates are obtained for rough data.

2. Preliminaries. In this section we state some basic results without proof. To begin with, we define the quadrature error

$$q(\phi)(\psi) = (\phi, \psi)_h - (\phi, \psi),$$

where $(\cdot,\cdot)_h$ is given by (1.5). Similarly, define the quadrature error associated with A_h and B_h by $q_A(\phi)(\psi)$ and $q_B(\phi)(\psi)$, respectively, in the obvious way. Sometimes we shall use $\|\phi\|_{1,h}$ as

$$\|\phi\|_{1,h} = (\|\phi\|_h^2 + \|\nabla\phi\|_h^2)^{1/2}.$$

In the rest of the paper C and c denote generic positive constants independent of h and k but may depend on T and are not necessarily the same at each occurrence.

The following lemmas will be frequently used in our analysis. For a proof, see [**3**] and [**10**].

Lemma 2.1. *For $\phi, \chi \in S_h$*

$$|q(\phi)(\chi)| \leq Ch^2\|\phi\|_1\|\chi\|_1.$$

The above inequality also holds if q is replaced by either q_A or q_B.

Lemma 2.2. *On S_h the norms $\|\cdot\|$ and $\|\cdot\|_h$ are equivalent. Likewise, for $\|\cdot\|_1$ and $\|\cdot\|_{1,h}$. For $\phi, \psi \in S_h$,*

$$|B_h(\phi,\psi)|, |A_h(t;\phi,\psi)| \leq C\|\phi\|_1\|\psi\|_1,$$

and

$$c\|\phi\|_1^2 \leq A_h(t;\phi,\phi),$$

provided that h is sufficiently small.

We shall also use the Ritz projection $R_h = R_h(t) : H_0^1(\Omega) \to S_h$ associated with the bilinear form $A(t;\cdot,\cdot)$ which is defined by

$$A(t; R_h\phi - \phi, \chi) = 0, \quad \chi \in S_h, \quad t \in \bar{J}. \tag{2.1}$$

It is now quite standard to prove that, for $2 \le p < \infty$ and $r = \{1, 2\}$, $R_h u$ satisfies

$$\begin{aligned} \|R_h\phi - \phi\|_{0,p} + h\|R_h\phi - \phi\|_{1,p} \le C_p h^r \|\phi\|_{r,p}, \\ \phi \in H_0^1(\Omega) \cap W^{r,p}(\Omega). \end{aligned} \tag{2.2}$$

Let $I_h v$ be the linear interpolant of v with respect to the vertices of $\mathcal{F}_h$. Then, for $v \in W^{r,p}(\Omega) \cap H_0^1(\Omega)$, $r = \{1, 2\}$, we have

$$\|I_h v - v\|_{0,p} + h\|I_h v - v\|_{1,p} \le Ch^r \|v\|_{r,p}, \quad 2 \le p \le \infty. \tag{2.3}$$

Further, for $\chi \in S_h$, the following inverse estimates

$$\begin{aligned} \|\chi\|_{i,p} \le Ch^{-2(q^{-1}-p^{-1})-(i-j)} \|\chi\|_{j,q}, \\ 1 \le q \le p \le \infty, \quad 0 \le j \le i \le 1, \end{aligned} \tag{2.4}$$

and Sobolev imbedding result

$$\|\chi\|_\infty \le C|\log h|^{1/2} \|\chi\|_1 \tag{2.5}$$

hold.

In the rest of this section we shall discuss some a priori estimates for u and its semi-discrete solution u_h. It is easy to modify Theorem 4 of Pani and Peterson [**10**] to obtain the following a priori bounds for u. Therefore, we state the result in terms of a theorem without proof.

Theorem 2.1. *Let u be the exact solution of* (1.1). *Further, let $0 \le i \le 4$, $0 \le j \le 3$ and $0 \le k \le 2$. For $t \in (0, T]$, the following estimates hold.*

(a) *If $0 \le k + 2j - i \le 1$, then*

$$t^i \|D_t^j u(t)\|_k^2 \le C(T)\|u_0\|_{k+2j-i}^2,$$

(b) *If* $0 \le k+2j-i-1 \le 1$, *then*

$$\int_0^t s^i \|D_t^j u(s)\|_k^2\,ds \le C(T)\|u_0\|_{k+2j-i-1}^2,$$

where $D_t^j = (d^j/dt^j)$.

Below we shall prove the stability estimates for u_h which will be used in Section 5.

Lemma 2.3. *Assume that* $u_h(0) = P_h u_0$. *Then, for* $u_0 \in H_0^1(\Omega)$, *we have*

$$\|u_h(0)\|_1 \le C\|u_0\|_1.$$

Proof. Note that

$$\begin{aligned} \|u_h(0)\|_1 &= \|P_h u_0\|_1 \\ &\le Ch^{-1}\|P_h u_0 - \tilde{I}_h u_0\| + \|\tilde{I}_h u_0 - u_0\|_1 + \|u_0\|_1, \end{aligned} \tag{2.6}$$

where $\tilde{I}_h u_0$ is a local averaging interpolation operator satisfying, [**15**, p. 491]

$$\|\tilde{I}_h u_0 - u_0\| + h\|\tilde{I}_h u_0 - u_0\|_1 \le Ch\|u_0\|_1.$$

Since $\|P_h u_0 - \tilde{I}_h u_0\| \le \|P_h u_0 - u_0\| + \|u_0 - \tilde{I}_h u_0\| \le Ch\|u_0\|_1$, the desired estimate follows from (2.6), and this completes the proof. □

Finally, using the standard energy arguments and Lemma 2.3, we obtain the following stability estimates.

Theorem 2.2. *Let* u_h *be the finite element solution of* (1.6). *Further, let* $0 \le i \le 3$, $0 \le j \le 2$ *and* $0 \le k \le 1$. *Then, for* $u_0 \in H_0^1(\Omega)$ *and* $t \in (0,T]$, *the following stability results hold*:

(a) *If* $0 \le k+2j-i \le 1$, *then*

$$t^i \|D_t^j u_h(t)\|_k^2 \le C(T)\|u_0\|_{k+2j-i}^2.$$

(b) *If* $0 \le k+2j-i-1 \le 1$, *then*

$$\int_0^t s^i \|D_t^j u_h(s)\|_k^2 \, ds \le C(T)\|u_0\|_{k+2j-i-1}^2.$$

3. Ritz-Volterra projection and related estimates. Following Lin, Thomée and Wahlbin [**7**] (see also [**1**]–[**2**]), we now define the Ritz-Volterra projection $W_h = W_h(t) : H_0^1(\Omega) \to S_h$ by

$$A(t;(W_h u - u)(t),\chi) = \int_0^t B(t,s;(W_h u - u)(s),\chi)\,ds \quad \forall\,\chi \in S_h, \quad t \in \bar{J}. \tag{3.1}$$

Let $\rho = W_h u - u$. In this section we shall modify the arguments of Lin et al. [**7**] to derive some estimates of ρ in L^p-norm. This, in return, will be used to prove pointwise error estimates when the initial function $u_0 \in H_0^1(\Omega)$. For the remaining part of this paper we shall use the following notation

$$\hat{\phi}(t) = \int_0^t \phi(s)\,ds.$$

Integrating (3.1) with respect to time t, we obtain

$$\begin{aligned} A(t;\hat{\rho}(t),\chi) &= \int_0^t A_s(s;\hat{\rho}(s),\chi)\,ds + \int_0^t B(s,s;\hat{\rho}(s),\chi)\,ds \\ &\quad - \int_0^t \int_0^s B_\tau(s,\tau;\hat{\rho}(\tau),\chi)\,d\tau\,ds, \quad \chi \in S_h. \end{aligned} \tag{3.2}$$

Based on the analysis of Lin et al. [**7**, Theorem 2.1], it is easy to obtain the following estimates for $\hat{\rho}$.

Lemma 3.1. *Let* $\hat{\rho}$ *satisfy* (3.2) *and* $2 \le p < \infty$. *Then a constant* C_p *exists such that, for* $r = \{1,2\}$, *we have*

$$\|\hat{\rho}(t)\|_{0,p} + h\|\hat{\rho}(t)\|_{1,p} \le C_p h^r \left[\|\hat{u}(t)\|_{r,p} + \int_0^t \|\hat{u}(s)\|_{r,p}\,ds \right].$$

Below we shall modify the arguments in Lin et al. [**7**] to obtain L^p-estimates for $\rho(t)$ and $\rho_t(t)$.

Lemma 3.2. *Let ρ satisfy* (3.1) *and $2 \le p < \infty$. Then a constant C_p exists such that, for $r = \{1, 2\}$, we have*

$$\|\rho(t)\|_{0,p} + h\|\rho(t)\|_{1,p} \le C_p h^r \left[\|u(t)\|_{r,p} + \|\hat{u}(t)\|_{r,p} + \int_0^t \|\hat{u}(s)\|_{r,p}\, ds\right],$$

and

$$\begin{aligned}\|\rho_t(t)\|_{0,p} &+ h\|\rho_t(t)\|_{1,p} \\ &\le C_p h^r \left[\|u(t)\|_{r,p} + \|u_t(t)\|_{r,p} + \|\hat{u}(t)\|_{r,p} + \int_0^t \|\hat{u}(s)\|_{r,p}\, ds\right].\end{aligned}$$

Proof. Let ρ_{x_j}, $1 \le j \le 2$, be the jth component of $\nabla\rho$ in $L^p(\Omega)$. With $p^{-1} + q^{-1} = 1$, we write

$$\|\rho_{x_j}\|_{0,p} = \sup\{(\rho_{x_j}, \phi^*); \phi^* \in C_0^\infty(\Omega), \|\phi^*\|_{0,q} = 1\}.$$

Let ψ^* be the solution of

$$\begin{aligned} A(t)\psi^* &= -\phi^*_{x_j} \quad \text{in } \Omega, \\ \psi^* &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The following elliptic regularity estimate

$$\|\psi^*\|_{1,q} \le C_p \|\phi^*\|_{0,q} \tag{3.3}$$

holds, see Schechter [**14**]. Using the definition of ρ, i.e., the equation (3.1), it follows that

$$\begin{aligned}(\rho_{x_j}, \phi^*) &= -(\rho, \phi^*_{x_j}) = A(t; \rho, \psi^*) \\ &= A(t; \rho, \psi^* - R_h\psi^*) + \int_0^t B(t, s; \rho(s), R_h\psi^*)\, ds \\ &= I_1 + I_2.\end{aligned}$$

For I_1, using (2.1) and (2.2), we obtain

$$|I_1| = |A(t; R_h u - u, \psi^*)| = |((R_h u - u)_{x_j}, \phi^*)| \le C_p h^{r-1} \|u\|_{r,p} \|\phi^*\|_{0,q}.$$

Integrating by parts with respect to s, we note that

$$\begin{aligned} I_2 &= B(t,t;\hat{\rho}(t), R_h\psi^*) - \int_0^t B_s(t,s;\hat{\rho}(s), R_h\psi^*)\,ds \\ &= B(t,t;\hat{\rho}(t), R_h\psi^* - \psi^*) + B(t,t;\hat{\rho}(t),\psi^*) \\ &\quad - \int_0^t B_s(t,s;\hat{\rho}(s), R_h\psi^* - \psi^*)\,ds \\ &\quad - \int_0^t B_s(t,s;\hat{\rho}(s),\psi^*)\,ds. \end{aligned}$$

Hence, using (2.2), (3.3) and Lemma 3.1, it follows that

$$\begin{aligned} |I_2| &\le C\left(\|\hat{\rho}(t)\|_{1,p} + \int_0^t \|\hat{\rho}(s)\|_{1,p}\,ds\right)(\|R_h\psi^* - \psi^*\|_{1,q} + \|\psi^*\|_{1,q}) \\ &\le C_p h^{r-1}\left(\|\hat{u}(t)\|_{r,p} + \int_0^t \|\hat{u}(s)\|_{r,p}\,ds\right)\|\phi^*\|_{0,q}. \end{aligned}$$

Altogether, we obtain

$$\|\rho(t)\|_{1,p} \le C_p h^{r-1}\left(\|u(t)\|_{r,p} + \|\hat{u}(t)\|_{r,p} + \int_0^t \|\hat{u}(s)\|_{r,p}\,ds\right).$$

For the estimation of ρ in the L^p-norm, we recall Aubin-Nitsche duality arguments. Let ψ^* be the solution of

$$\begin{aligned} A(t)\psi^* &= \phi^* \quad \text{in } \Omega, \\ \psi^* &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where ϕ^* is smooth and $\|\phi^*\|_{0,q} = 1$. Then the following regularity estimate

$$\|\psi^*\|_{2,q} \le C_p \|\phi^*\|_{0,q} \tag{3.4}$$

holds, see again [**14**]. Now

$$
\begin{aligned}
(\rho, \phi^*) &= A(t; \rho, \psi^*) = A(t; R_h u - u, \psi^*) \\
&\quad + \int_0^t B(t, s; \rho(s), R_h\psi^* - \psi^*)\, ds + \int_0^t B(t, s; \rho(s), \psi^*)\, ds \\
&= (R_h u - u, \phi^*) + B(t, t; \hat{\rho}(t), R_h\psi^* - \psi^*) + B(t, t; \hat{\rho}(t), \psi^*) \\
&\quad - \int_0^t B_s(t, s; \hat{\rho}(s), R_h\psi^* - \psi^*)\, ds - \int_0^t B_s(t, s; \hat{\rho}(s), \psi^*)\, ds \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

For I_1, we use (2.1) and (2.2) to obtain

$$|I_1| \le \|R_h u - u\|_{0,p} \|\phi^*\|_{0,q} \le C_p h^r \|u\|_{r,p} \|\phi^*\|_{0,q}.$$

For I_2 and I_4, use of Lemma 3.1 with estimate (2.2) yields

$$|I_2| + |I_4| \le C_p h^r \left(\|\hat{u}(t)\|_{r,p} + \int_0^t \|\hat{u}(s)\|_{r,p}\, ds \right) \|\phi^*\|_{0,q}.$$

To estimate I_3, we first note that

$$I_3 = B(t, t; \hat{\rho}(t), \psi^*) = (\hat{\rho}(t), B^*(t, t)\psi^*),$$

where B^* is the formal adjoint of B and then use Lemma 3.1 with (3.4) to obtain

$$
\begin{aligned}
|I_3| &\le \|\hat{\rho}(t)\|_{0,p} \|\psi^*\|_{2,q} \\
&\le C_p h^r \left(\|\hat{u}(t)\|_{r,p} + \int_0^t \|\hat{u}(s)\|_{r,p}\, ds \right) \|\phi^*\|_{0,q}.
\end{aligned}
$$

Similarly, $|I_5|$ can be estimated as

$$|I_5| \le C_p h^r \int_0^t \|\hat{u}(s)\|_{r,p}\, ds \|\phi^*\|_{0,q}.$$

Combining all these estimates, it follows that

$$\|\rho\|_{0,p} \le C_p h^r \left(\|u(t)\|_{r,p} + \|\hat{u}(t)\|_{r,p} + \int_0^t \|\hat{u}(s)\|_{r,p}\, ds \right),$$

and this proves the first inequality. To prove the second estimate, differentiate (3.1) with respect to time t to have

$$A(t;\rho_t,\chi) = -A_t(t;\rho,\chi) + B(t,t;\rho(t),\chi) + \int_0^t B_t(t,s;\rho(s),\chi)\,ds, \quad \chi \in S_h.$$

Proceeding as in the proof of ρ, it is easy to derive L^p-estimates for ρ_t in a similar way, and this completes the proof. $\square$

In the following lemma, we shall obtain an estimation of ρ in maximum norm.

Lemma 3.3. *For $\varepsilon > 0$ a constant C_ε exists depending on ε such that, for $r = \{1,2\}$,*

$$\|\rho(t)\|_\infty \le C_\varepsilon h^{r-\varepsilon}\bigg[\|u(t)\|_{r,\infty} + \|\hat{u}(t)\|_{r,\infty} + \int_0^t \|\hat{u}(s)\|_{r,\infty}\,ds\bigg].$$

Proof. Writing ρ as $\rho = (W_h u - I_h u) + (I_h u - u) = \rho_1 + \rho_2$, where $I_h u$ is the linear interpolant of u satisfying (2.3). From (2.3), it follows that

$$\|\rho_2(t)\|_\infty \le Ch^r\|u(t)\|_{r,\infty}, \quad r \in \{1,2\}.$$

Further, an application of (2.4) yields

$$\|\rho_1(t)\|_\infty \le Ch^{-2/p}\|\rho_1(t)\|_{0,p} \le Ch^{-2/p}(\|\rho(t)\|_{0,p} + \|\rho_2(t)\|_\infty).$$

Now use Lemma 3.2 to obtain

$$\begin{aligned}\|\rho(t)\|_\infty &\le \|\rho_1(t)\|_\infty + \|\rho_2(t)\|_\infty \\ &\le Ch^{-2/p}(\|\rho(t)\|_{0,p} + h^r\|u(t)\|_{r,\infty}) \\ &\le C_\varepsilon h^{r-\varepsilon}\bigg(\|u(t)\|_{r,\infty} + \|\hat{u}(t)\|_{r,\infty} + \int_0^t \|\hat{u}(s)\|_{r,\infty}\,ds\bigg),\end{aligned}$$

where $\varepsilon = 2/p$ for large p, and this completes the proof. $\square$

4. Semi-discrete error estimates for nonsmooth data. In this section we shall derive optimal error estimates when the initial function $u_0 \in H_0^1(\Omega)$. For optimal estimates, let us split the error as: $u_h(t) - u(t) = (u_h(t) - W_h u(t)) + (W_h u(t) - u(t)) = \theta(t) + \rho(t)$. From (1.2), (1.6) and (3.1), we write an equation in θ as

$$(4.1)\quad (\theta_t, \chi)_h + A_h(t;\theta,\chi) = \int_0^t B_h(t,s;\theta(s),\chi)\,ds - (\rho_t,\chi) + G(w)(\chi),$$

where $G(\phi)(\chi) = -q(\phi_t)(\chi) - q_A(\phi)(\chi) + \int_0^t q_B(t,s;\phi(s))(\chi)\,ds$ and $w = W_h u$. Integrating with respect to time (1.2) and (1.6), and using (3.2) with $u_h(0) = P_h u_0$, we obtain the following equation in $\hat\theta$

$$(4.2)\quad \begin{aligned}(\hat\theta_t(t),\chi)_h + A_h(t;\hat\theta(t),\chi) = &\int_0^t A_{hs}(s;\hat\theta(s),\chi)\,ds\\ &+ \int_0^t B_h(s,s;\hat\theta(s),\chi)\,ds\\ &- \int_0^t\int_0^s B_{h\tau}(s,\tau;\hat\theta(\tau),\chi)\,d\tau\,ds\\ &- (\rho,\chi) + \overline{G}(w)(\chi) + q(P_h u_0)(\chi),\end{aligned}$$

where

$$\begin{aligned}\overline{G}(w)(\chi) = -q(\hat w_t)(\chi) - q_A(\hat w)(\chi) &+ \int_0^t q_{A_s}(s;\hat w(s))(\chi)\,ds\\ + \int_0^t q_B(s,s;\hat w(s))(\chi)\,ds&\\ - \int_0^t\int_0^s q_{B_\tau}(s,\tau;\hat w(\tau))(\chi)\,d\tau\,ds.&\end{aligned}$$

Below we shall prove a series of lemmas which altogether lead to the desired result. For a linear functional F on S_h, define

$$\|F\|_{-1,h} = \sup_{\chi\in S_h}\frac{F(\chi)}{\|\chi\|_{1,h}}.$$

Lemma 4.1. *With $G(w)$ and $\overline{G}(w)$ as above, the following estimates*

$$\|\overline{G}(w)\|_{-1,h}^2 + \int_0^t s\|\overline{G}_s(w)\|_{-1,h}^2\,ds \le Ch^4\|u_0\|_1^2,$$

and

$$t^2\|G(w)\|^2_{-1,h} + t^4\|G_t(w)\|^2_{-1,h} \le Ch^4\|u_0\|_1^2$$

hold.

Proof. By Lemma 2.1 and Theorem 2.1, we have

$$\begin{aligned}\|\overline{G}(w)\|_{-1,h} &\le Ch^2\left[\|W_h u(t)\|_1 + \int_0^t \|W_h u(t)\|_1\,ds\right]\\ &\le Ch^2\left[\|u(t)\|_1 + \int_0^t \|u(s)\|_1\,ds\right]\\ &\le Ch^2\|u_0\|_1.\end{aligned}$$

Since

$$\|\overline{G}_t(w)\|_{-1,h} \le Ch^2\left[\|u_t(t)\|_1 + \|u(t)\|_1 + \int_0^t \|u(s)\|_1\,ds\right],$$

we obtain using Theorem 2.1,

$$\int_0^t s\|\overline{G}_s(w)\|^2_{-1,h}\,ds \le Ch^4\|u_0\|_1^2.$$

Similarly,

$$\|G(w)\|_{-1,h} \le Ch^2\left[\|u_t(t)\|_1 + \|u(t)\|_1 + \int_0^t \|u(t)\|_1\,ds\right],$$

and

$$\|G_t(w)\|_{-1,h} \le Ch^2\left[\|u_{tt}(t)\|_1 + \|u_t(t)\|_1 + \|u(t)\|_1 + \int_0^t \|u(t)\|_1\,ds\right].$$

Now the desired results follow from Theorem 2.1, and this completes the proof. □

Lemma 4.2. *With θ and $\hat{\theta}$ as above, there is a positive constant C such that*

$$t\|\hat{\theta}(t)\|_1^2 + \int_0^t s\|\theta(s)\|^2\,ds \le C(T)th^4\|u_0\|_1^2.$$

Proof. Take $\chi = t\theta(t)$ in (4.2), and write

$$A_h(t;\hat{\theta}, t\hat{\theta}_t) = \frac{1}{2}\,\frac{d}{dt}\,\{tA_h(t;\hat{\theta}(t),\hat{\theta}(t))\} - \frac{t}{2}A_{ht}(t;\hat{\theta}(t),\hat{\theta}(t)) - \frac{1}{2}A_h(t;\hat{\theta}(t),\hat{\theta}(t)).$$

Then an integration with respect to time from 0 to t leads to

$$\begin{aligned}
\frac{t}{2}A_h(t;\hat{\theta}(t),&\hat{\theta}(t)) + \int_0^t s\|\theta(s)\|^2\,ds \\
&= \frac{1}{2}\int_0^t [sA_{hs}(s;\hat{\theta}(s),\hat{\theta}(s))\,ds + A_h(s;\hat{\theta}(s),\hat{\theta}(s))]\,ds \\
&\quad + \int_0^t\int_0^s sA_{h\tau}(\tau;\hat{\theta}(\tau),\hat{\theta}_s(s))\,d\tau\,ds \\
&\quad + \int_0^t\int_0^s sB_h(\tau,\tau;\hat{\theta}(\tau),\hat{\theta}_s(s))\,d\tau\,ds \\
&\quad - \int_0^t\int_0^s\int_0^\tau sB_{h\tau'}(\tau,\tau';\hat{\theta}(\tau'),\hat{\theta}_s(s))\,d\tau'\,d\tau\,ds \\
&\quad - \int_0^t s(\rho(s),\theta(s))\,ds \\
&\quad + \int_0^t s\overline{G}(w)(\hat{\theta}_s(s))\,ds + \int_0^t q(P_h u_0)(s\hat{\theta}_s))\,ds \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

The terms I_1 and I_5 can be estimated as

$$\begin{aligned}
|I_1| + |I_5| \le C\int_0^t s[\|\hat{\theta}(s)\|_1^2 + \|\rho(s)\|^2]\,ds& \\
+ C\int_0^t \|\hat{\theta}(s)\|_1^2\,ds + \frac{1}{2}\int_0^t s\|\theta(s)\|^2\,ds&.
\end{aligned}$$

For I_2, integrate by parts with respect to s to have

$$\begin{aligned}
I_2 &= \int_0^t\int_\tau^t sA_{h\tau}(\tau;\hat{\theta}(\tau),\hat{\theta}_s(s))\,ds\,d\tau \\
&= t\int_0^t A_{h\tau}(\tau;\hat{\theta}(\tau),\hat{\theta}(t))\,d\tau - \int_0^t \tau A_{h\tau}(\tau;\hat{\theta}(\tau),\hat{\theta}(\tau))\,d\tau \\
&\quad - \int_0^t\int_0^s A_{h\tau}(\tau;\hat{\theta}(\tau),\hat{\theta}(s))\,d\tau\,ds.
\end{aligned}$$

Similarly, I_3 and I_4 can be written as

$$I_3 = t\int_0^t B_h(\tau,\tau;\hat{\theta}(\tau),\hat{\theta}(t))\,d\tau - \int_0^t \tau B_h(\tau,\tau;\hat{\theta}(\tau),\hat{\theta}(\tau))\,d\tau - \int_0^t\int_0^s B_h(\tau,\tau;\hat{\theta}(\tau),\hat{\theta}(s))\,d\tau\,ds,$$

and

$$I_4 = -\int_0^t\int_0^t tB_{h\tau'}(\tau,\tau';\hat{\theta}(\tau'),\hat{\theta}(t))\,d\tau'\,d\tau + \int_0^t\int_0^t \tau' B_{h\tau'}(\tau,\tau';\hat{\theta}(\tau'),\hat{\theta}(\tau'))\,d\tau'\,d\tau + \int_0^t\int_0^s\int_0^\tau B_{h\tau'}(\tau,\tau';\hat{\theta}(\tau'),\hat{\theta}(s))\,d\tau'\,d\tau\,ds.$$

Hence, we obtain

$$|I_2|+|I_3|+|I_4| \le \frac{1}{8}\,t\|\hat{\theta}(t)\|_1^2 + C\int_0^t [\|\hat{\theta}(s)\|_1^2 + s\|\hat{\theta}(s)\|_1^2]\,ds + C\int_0^t\int_0^s \|\hat{\theta}(\tau)\|_1^2\,d\tau\,ds.$$

For the estimation of I_6, we integrate by parts with respect to s to find that

$$I_6 = t\overline{G}(w)(\hat{\theta}(t)) - \int_0^t \overline{G}(w)(\hat{\theta}(s))\,ds - \int_0^t s\overline{G}_s(w)(\hat{\theta}(s))\,ds,$$

and then

$$\begin{aligned}|I_6| &\le t\|\overline{G}(w)\|_{-1,h}\|\hat{\theta}(t)\|_1 \\ &\quad + \int_0^t [\|\overline{G}(w)\|_{-1,h} + s\|\overline{G}_s(w)\|_{-1,h}]\|\hat{\theta}(s)\|_1\,ds \\ &\le \frac{1}{8}\,t\|\hat{\theta}(t)\|_1^2 + Ct\|\overline{G}(w)\|_{-1,h}^2 + C\int_0^t [\|\overline{G}(w)\|_{-1,h}^2 \\ &\quad + s^2\|\overline{G}_s(w)\|_{-1,h}^2 + \|\hat{\theta}(s)\|_1^2]\,ds.\end{aligned}$$

Finally, for I_7, an integration by parts yields

$$I_7 = q(P_h u_0)(t\hat{\theta}(t)) - \int_0^t q(P_h u_0)(\hat{\theta}(s))\,ds,$$

and hence, using Lemmas 2.1 and 2.3, it follows that

$$|I_7| \le Cth^4\|u_0\|_1^2 + \frac{1}{4}t\|\hat{\theta}(t)\|_1^2 + \int_0^t \|\hat{\theta}(s)\|_1^2\,ds.$$

Combining the estimates of $I_1, \dots, I_7$, use Lemmas 3.2, 4.1 and Theorem 2.1 to obtain

$$t\|\hat{\theta}(t)\|_1^2 + \int_0^t s\|\theta(s)\|^2\,ds \le Cth^4\|u_0\|_1^2 + C\int_0^t \|\hat{\theta}(s)\|_1^2\,ds + C\int_0^t s\|\hat{\theta}(s)\|_1^2\,ds.$$

Before we apply Gronwall's lemma, we need to estimate the last but one term on the righthand side of the above inequality. Now set $\chi = \hat{\theta}(t)$ in (4.2) and rewrite $(\rho, \hat{\theta}) = (d/dt)(\hat{\rho}, \hat{\theta}) - (\hat{\rho}, \theta)$. Then integrate the resulting equation from 0 to t to obtain

$$\begin{aligned}\|\hat{\theta}(t)\|_h^2 + \int_0^t \|\hat{\theta}(s)\|_1^2\,ds \le C\bigg[&\int_0^t\int_0^s \|\hat{\theta}(\tau)\|_1\|\hat{\theta}(s)\|_1\,d\tau\,ds \\ &+ \int_0^t\int_0^s\int_0^\tau \|\hat{\theta}(\tau')\|_1\|\hat{\theta}(s)\|_1\,d\tau'\,d\tau\,ds \\ &+ \|\hat{\rho}(t)\|\|\hat{\theta}(t)\| + \int_0^t \|\hat{\rho}(s)\|\|\theta(s)\|\,ds \\ &+ \int_0^t \|\overline{G}(w)\|_{-1,h}\|\hat{\theta}(s)\|_1\,ds \\ &+ h^2\int_0^t \|P_h u_0\|_1\|\hat{\theta}(s)\|_1\,ds\bigg].\end{aligned}$$

Here we have used Lemma 2.1 to estimate the last term. For the third and fourth terms on the righthand side, we note that

$$\begin{aligned}\|\hat{\rho}(t)\|\|\hat{\theta}(t)\| &= \left\|\int_0^t \rho(s)\,ds\right\|\|\hat{\theta}(t)\| \\ &\le t^{1/2}\left(\int_0^t \|\rho(s)\|^2\,ds\right)^{1/2}\|\hat{\theta}(t)\| \\ &\le \frac{1}{2}t\int_0^t \|\rho(s)\|^2\,ds + \frac{1}{2}\|\hat{\theta}(t)\|^2,\end{aligned}$$

and

$$\int_0^t \|\hat{\rho}(s)\|\|\theta(s)\|\,ds = \int_0^t \left\|\int_0^s \rho(\tau)\,d\tau\right\|\|\theta(s)\|\,ds$$
$$\leq \int_0^t s^{1/2}\left(\int_0^s \|\rho(\tau)\|^2\,d\tau\right)^{1/2}\|\theta(s)\|\,ds$$
$$\leq C(\varepsilon)\int_0^t\int_0^s \|\rho(\tau)\|^2\,d\tau\,ds + \frac{\varepsilon}{2}\int_0^t s\|\theta(s)\|^2\,ds.$$

For the remaining terms, we apply Cauchy-Schwarz inequality. Altogether, we obtain

$$\int_0^t \|\hat{\theta}(s)\|_1^2\,ds \leq Cth^4\|u_0\|_1^2 + C(\varepsilon)\bigg[t\int_0^t \|\rho(s)\|^2\,ds + \int_0^t\int_0^s \|\rho(\tau)\|^2\,d\tau\,ds + \int_0^t \|\overline{G}(w)\|_{-1,h}^2\,ds\bigg] + \frac{\varepsilon}{2}\int_0^t s\|\theta(s)\|^2\,ds + C\int_0^t\int_0^s \|\hat{\theta}(\tau)\|_1^2\,d\tau\,ds.$$

Applying Lemmas 3.2, 4.1, Theorem 2.1 and then Gronwall's lemma, it follows that

$$\int_0^t \|\hat{\theta}(s)\|_1^2\,ds \leq C(\varepsilon)th^4\|u_0\|_1^2 + C\varepsilon\int_0^t s\|\theta(s)\|^2\,ds.$$

On substitution of the above estimate, we choose ε appropriately so that $(1 - C\varepsilon) > 0$. Now an application of Gronwall's lemma completes the rest of the proof. □

Remark 4.1. In fact, we now have from Lemmas 4.2 and 4.3,

$$\|\hat{\theta}(t)\|_1^2 + \int_0^t \|\hat{\theta}(s)\|_1^2\,ds \leq Cth^4\|u_0\|_1^2. \tag{4.3}$$

Lemma 4.3. *With θ as above, the following estimates*

$$t^2\|\theta(t)\|^2 + \int_0^t s^2\|\theta(s)\|_1^2\,ds \leq Cth^4\|u_0\|_1^2,$$

and

$$t^3\|\theta(t)\|_1^2 + \int_0^t s^3\|\theta_s(s)\|^2\,ds \le Cth^4\|u_0\|_1^2$$

hold.

Proof. For the first estimate, set $\chi = t^2\theta$ and, for the second one, choose $\chi = t^3\theta_t$ in (4.1). With appropriate modifications of the proofs of Lemmas 6 and 7 in Pani et al. [**10**, pp. 1094–1095] and with straightforward estimates for the terms due to time dependence of A, we obtain the required estimates. Note that we have to use (4.3), Lemmas 3.2, 4.1, 4.2 and Theorem 2.1 to complete the proof. □

Theorem 4.1. *Let u be the exact solution of* (1.1)*, and let u_h be the corresponding semi-discrete approximation defined by* (1.6)*. Further, let $u_0 \in H_0^1(\Omega)$ and $u_h(0) = P_h u_0$. Then the following estimates*

$$\|u_h(t) - u(t)\| \le Ct^{-1/2}h^2\|u_0\|_1,$$

and

$$\|u_h(t) - u(t)\|_1 \le Ct^{-1}h\|u_0\|_1$$

hold.

Proof. Writing $u_h(t) - u(t)$ as $u_h(t) - u(t) = \theta(t) + \rho(t)$. The first estimate is an immediate consequence of Lemmas 3.2, 4.3, Theorem 2.1 and the triangle inequality. The second inequality follows from Lemmas 3.2, 4.3 and Theorem 2.1. □

Our final result in this section is an almost optimal pointwise estimate of $u_h - u$. using the modified estimates of ρ in L^∞-norm and a priori bounds for $\|\hat{u}(t)\|_3$, it is possible to relax the stringent compatibility conditions assumed in Pani and Peterson (see [**10**], p. 1095, the last paragraph).

Theorem 4.2. *Let u and u_h be the solutions of* (1.1) *and* (1.6)*, respectively. Further, let $u_0 \in H_0^1(\Omega)$ and $u_h(0) = P_h u_0$. Then, for any $\varepsilon > 0$, there is a constant C_ε such that*

$$\|u_h(t) - u(t)\|_\infty \le C_\varepsilon t^{-1}h^{2-\varepsilon}\|u_0\|_1.$$

Proof. Writing

$$\|u_h(t) - u(t)\|_\infty \le \|\theta(t)\|_\infty + \|\rho(t)\|_\infty, \tag{4.4}$$

we have by the Sobolev imbedding result and Lemma 4.3,

$$\|\theta(t)\|_{0,p} \le C_p \|\theta(t)\|_1 \le C_p h^2 t^{-1} \|u_0\|_1,$$

and by (2.4)

$$\|\theta(t)\|_\infty \le C_p h^{-2/p} \|\theta(t)\|_{0,p} \le C_\varepsilon h^{2-\varepsilon} t^{-1} \|u_0\|_1, \tag{4.5}$$

where $\varepsilon = 2/p$ for large p. From Lemma 3.3, we have

$$\|\rho(t)\|_\infty \le C_\varepsilon h^{2-2\varepsilon} \Big[\|u(t)\|_{2-\varepsilon,\infty} + \|\hat{u}(t)\|_{2-\varepsilon,\infty} + \int_0^t \|\hat{u}(s)\|_{2-\varepsilon,\infty}\, ds \Big].$$

Again, use the Sobolev imbedding theorem to obtain

$$\|u(t)\|_{2-\varepsilon,\infty} \le C_\varepsilon \|u(t)\|_3 \quad \text{and} \quad \|\hat{u}(t)\|_{2-\varepsilon,\infty} \le C_\varepsilon \|\hat{u}(t)\|_3.$$

Hence,

$$t\|\rho(t)\|_\infty \le C_\varepsilon h^{2-2\varepsilon} \Big[t\|u(t)\|_3 + t\|\hat{u}(t)\|_3 + t\int_0^t \|\hat{u}(s)\|_3\, ds \Big]. \tag{4.6}$$

Integrating the term on the right of (1.1) by parts with respect to s, we have

$$A(t)u(t) = -u_t + B(t,t)\hat{u}(t) - \int_0^t B_s(t,s)\hat{u}(s)\, ds.$$

By elliptic regularity

$$t\|u(t)\|_3 \le Ct\|A(t)u(t)\|_1 \le t\|u_t(t)\|_1 + Ct\|\hat{u}(t)\|_3 + C\int_0^t \|\hat{u}(s)\|_3\, ds. \tag{4.7}$$

To estimate $\|\hat{u}(t)\|_3$, integrate (1.1) from 0 to t to have

$$A(t)\hat{u}(t) = -u(t) + u_0 + \int_0^t B(s,s)\hat{u}(s)\, ds - \int_0^t \int_0^s B_\tau(s,\tau)\hat{u}(\tau)\, d\tau\, ds + \int_0^t A_s(s)\hat{u}(s)\, ds.$$

Again an application of elliptic regularity property yields

$$\begin{aligned}\|\hat{u}(t)\|_3 &\le C\|A(t)\hat{u}(t)\|_1\\ &\le C\bigg[\|u(t)\|_1+\|u_0\|_1+\int_0^t \|B(s,s)\hat{u}(s)\|_1\,ds\\ &\qquad +\int_0^t\int_0^s \|B(s,\tau)\hat{u}(\tau)\|_1\,d\tau\,ds+\int_0^t \|A_s(s)\hat{u}(s)\|_1\,ds\bigg],\end{aligned}$$

and this implies

$$\|\hat{u}(t)\|_3 \le C\bigg[\|u(t)\|_1+\|u_0\|_1+\int_0^t \|\hat{u}(s)\|_3\,ds\bigg].$$

Use Theorem 2.1 and then apply Gronwall's lemma to obtain

$$\|\hat{u}(t)\|_3 \le C\|u_0\|_1. \tag{4.8}$$

The desired estimate now follows from (4.4)–(4.8), and this completes the proof. □

5. Backward Euler scheme. In this section we shall discuss the backward Euler scheme (1.8) for the time discretization of the problem (1.6). For $\phi\in S_h$, define $\|\phi\|_{-j,h}$ as

$$\|\phi\|_{-j,h}=\sup_{g\in S_h}\frac{(\phi,g)}{\|g\|_j},\quad j=0,1.$$

The following discrete version of Gronwall's lemma will be used in our subsequent analysis. For a proof, we refer to Pani et al. [**12**].

Lemma 5.1. *If $\xi_n\ge 0$, $\alpha_n\ge\alpha_{n-1}$, $\beta_j\ge 0$ and $\xi_n\le\alpha_n+\sum_{j=0}^{n-1}\beta_j\xi_j$, then $\xi_n\le\alpha_n\exp(\sum_{j=0}^{n-1}\beta_j)$.*

Let $\eta^n=U^n-u_h^n$. Then, from (1.6) and (1.8), η^n satisfies

$$\begin{aligned}(\bar{\partial}_t\eta^n,\chi)_h&+A_h(t_n;\eta^n,\chi)\\ &=k\sum_{j=0}^{n-1}B_h(t_n,t_j;\eta^j,\chi)+Q_B^n(u_h)(\chi)+(\tau^n,\chi)_h,\\ \eta^0&=0,\end{aligned}\tag{5.1}$$

where $\tau^n = u_{ht}^n - \bar{\partial}_t u_h^n$, and $Q_B^n(u_h)(\chi) = k\sum_{j=0}^{n-1} B_h(t_n, t_j; u_h^j, \chi) - \int_0^{t_n} B_h(t_n, s; u_h(s), \chi)\, ds$. Define $\hat{\eta}^n = k\sum_{j=0}^{n} \eta^j$. Clearly, $\bar{\partial}_t \hat{\eta}^n = \eta^n$ and $\hat{\eta}^0 = 0$.

Multiply (1.8) by k and then sum with respect to n from 1 to m with $1 \leq n \leq m \leq N$ to have

$$(5.2)\qquad (U^m, \chi)_h + k\sum_{n=1}^{m} A_h(t_n; U^n, \chi) = k^2 \sum_{n=1}^{m}\sum_{j=0}^{n-1} B_h(t_n, t_j; U^j, \chi) + (P_h u_0, \chi)_h.$$

Integrate (1.6) from 0 to t to obtain

$$(5.3)\qquad (u_h(t), \chi)_h + \int_0^t A_h(s; u_h(s), \chi)\, ds = (P_h u_0, \chi)_h + \int_0^t \int_0^s B_h(s, \tau; u_h(\tau), \chi)\, d\tau\, ds.$$

Using (5.3) at $t = t_m$ and (5.2), we find that

$$(\bar{\partial}_t \hat{\eta}^m, \chi)_h + k\sum_{n=1}^{m} A_h(t_n; \eta^n, \chi) = k^2 \sum_{n=1}^{m}\sum_{j=0}^{n-1} B_h(t_n, t_j; \eta^j, \chi) + Q_A^m(u_h(\chi) + \bar{Q}_B^m(u_h)(\chi),$$

where

$$Q_A^m(u_h)(\chi) = -k\sum_{n=1}^{m} A_h(t_n; u_h^n, \chi) + \int_0^{t_m} + A_h(s; u_h(s), \chi)\, ds,$$

and

$$\overline{Q}_B^m(u_h)(\chi) = k^2 \sum_{n=1}^{m}\sum_{j=0}^{n-1} B_h(t_n, t_j; u_h^j, \chi) - \int_0^{t_m} \int_0^s B_h(s, \tau; u_h(\tau), \chi)\, d\tau\, ds.$$

Note that

$$k\sum_{n=1}^{m} A_h(t_n; \eta^n, \chi) = A_h(t_m; \hat{\eta}^m, \chi) - k\sum_{n=1}^{m} (\bar{\partial} A_h)(t_n; \hat{\eta}^{n-1}, \chi),$$

where $(\bar{\partial}A_h)(t_n;\cdot,\cdot) = k^{-1}[A_h(t_n;\cdot,\cdot) - A_h(t_{n-1};\cdot,\cdot)]$ is the backward difference quotient of $A_h(t,\cdot,\cdot)$ with respect to first variable at $t = t_n$. Hence, we obtain

$$
\begin{aligned}
(\bar{\partial}_t\hat{\eta}^m,\chi)_h + A_h(t_m;\hat{\eta}^m,\chi) &= k\sum_{n=1}^{m}(\bar{\partial}A_h)(t_n;\hat{\eta}^{n-1},\chi)\\
&\quad + k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1}B_h(t_n,t_j;\eta^j,\chi)\\
&\quad + Q_A^m(u_h)(\chi) + \overline{Q}_B^m(u_h)(\chi).
\end{aligned} \tag{5.4}
$$

We shall prove the following lemmas for our subsequent use.

Lemma 5.2. *With Q_A^m, Q_B^m and $\overline{Q}_B^m$ as above, we have for $m = 1,\ldots,N$,*

$$
\begin{aligned}
&\|Q_A^m(u_h)\|_{-1,h} + \|Q_B^m(u_h)\|_{-1,h} + \|\overline{Q}_B(u_h)\|_{-1,h}\\
&\qquad\qquad \le C(T)k\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)\|u_0\|_1.
\end{aligned}
$$

Proof. Using the rectangle quadrature rule, it follows that

$$
\begin{aligned}
|Q_A^m(u_h)(\chi)| &= \left|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}(t_{j-1}-s)\frac{\partial}{\partial s}[A_h(s;u_h(s),\chi)]\,ds\right|\\
&\le \int_0^{t_1} s|[A_h(s;u_{hs}(s),\chi) + A_{hs}(s;u_h(s),\chi)]|\,ds\\
&\quad + k\sum_{j=2}^{m}\int_{t_{j-1}}^{t_j}|[A_h(s;u_{hs}(s),\chi) + A_{hs}(s;u_h(s),\chi)]|\,ds\\
&\le C\int_0^{t_1} s[\|u_h\|_1 + \|u_{hs}\|_1]\,ds\|\chi\|_1\\
&\quad + Ck\sum_{j=2}^{m}\int_{t_{j-1}}^{t_j}[\|u_h\|_1 + \|u_{hs}\|_1]\,ds\|\chi\|_1\\
&\le Ck\left(\int_0^{t_1} s[\|u_h\|_1^2 + \|u_{hs}\|_1^2]\,ds\right)^{1/2}\|\chi\|_1
\end{aligned}
$$

$$+ Ck \sum_{j=2}^{m} \left(\int_{t_{j-1}}^{t_j} \frac{1}{s}\, ds \right)^{1/2} \cdot \left(\int_{t_{j-1}}^{t_j} s[\|u_h(s)\|_1^2 + \|u_{hs}(s)\|_1^2]\, ds \right)^{1/2} \|\chi\|_1.$$

Apply Theorem 2.2 to obtain

$$\begin{aligned}\|Q_A^m(u_h)\|_{-1,h} &\le Ck\Bigg[\|u_0\|_1 + \left(\sum_{j=2}^{m} \log \frac{t_j}{t_{j-1}} \right)^{1/2} \cdot \left(\sum_{j=2}^{m} \int_{t_{j-1}}^{t_j} s[\|u_h\|_1^2 + \|u_{hs}\|_1^2]\, ds \right)^{1/2}\Bigg] \\ &\le Ck\Bigg[\|u_0\|_1 + \left(\log \frac{1}{k} \right)^{1/2} \cdot \left(\int_0^{t_m} s[\|u_h\|_1^2 + \|u_{hs}\|_1^2]\, ds \right)^{1/2}\Bigg] \\ &\le Ck\left(1 + \left(\log \frac{1}{k} \right)^{1/2}\right) \|u_0\|_1.\end{aligned}$$

Similarly, using the left rectangle rule, we derive the required estimate for Q_B^m. Finally, we note that

$$\begin{aligned}\overline{Q}_B^m(u_h)(\chi) &= k \sum_{n=1}^{m} Q_B^n(u_h)(\chi) \\ &\quad + \Bigg[k \sum_{n=1}^{m} \int_0^{t_n} B_h(t_n, s; u_h(s), \chi)\, ds \\ &\quad - \int_0^{t_m} \int_0^s B_h(s, \tau; u_h(\tau), \chi)\, d\tau\, ds \Bigg].\end{aligned}$$

Again, using the estimate of $Q_B^n(u_h)$ and the right rectangle rule for the second term on the righthand side, we complete the rest of the proof. $\square$

Lemma 5.3. *There is a constant C independent of k such that, for $n = 1, \dots, N$,*

$$\|\hat{\eta}^n\|^2 + k\sum_{j=1}^{n}\|\hat{\eta}^j\|_1^2 \le C(T)t_n k^2\left(1 + \left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2.$$

Proof. Taking $\chi = \hat{\eta}^m$ in (5.4) and using the fact $(\bar{\partial}_t\hat{\eta}^m, \hat{\eta}^m)_h = (1/2)\bar{\partial}_t\|\hat{\eta}^m\|_h^2 + (k/2)\|\bar{\partial}_t\hat{\eta}^m\|_h^2$, we obtain

$$\begin{aligned}\frac{1}{2}\,&\bar{\partial}_t\|\hat{\eta}^m\|_h^2 + A_h(t_m;\hat{\eta}^m,\hat{\eta}^m) + \frac{k}{2}\,\|\bar{\partial}_t\hat{\eta}^m\|_h^2\\ &= k\sum_{n=1}^{m}(\bar{\partial}A_h)(t_n;\hat{\eta}^{n-1},\hat{\eta}^m) + k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1}B_h(t_n,t_j;\eta^j,\hat{\eta}^m)\\ &\quad + Q_A^m(u_h)(\hat{\eta}^m) + \overline{Q}_B^m(u_h)(\hat{\eta}^m).\end{aligned}$$

Since $\hat{\eta}^0 = 0$, we now have for $m = 1$,

$$\|\hat{\eta}^1\|^2 + \|ck\|\hat{\eta}^1\|_1^2 \le k(\|Q_A^1(u_h)\|_{-1,h} + \|\overline{Q}_B^1(u_h)\|_{-1,h})\|\hat{\eta}^1\|_1.$$

An application of Young's inequality yields

$$\|\hat{\eta}^1\|^2 + k\|\hat{\eta}^1\|_1^2 \le Ck[\|Q_A^1(u_h)\|_{-1,h}^2 + \|\overline{Q}_B^1(u_h)\|_{-1,h}^2]. \tag{5.5}$$

For $m \ge 2$, we note that

$$\begin{aligned}k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1}B_h(t_n,t_j;\eta^j,\chi) &= k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}B_h(t_n,t_j;\bar{\partial}_t\hat{\eta}^j,\chi)\\ &= k\sum_{n=2}^{m}B_h(t_n,t_{n-1};\hat{\eta}^{n-1},\chi)\\ &\quad - k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}(\bar{\partial}_2B_h)(t_n,t_j;\hat{\eta}^{j-1},\chi),\end{aligned}$$

where $(\bar{\partial}_2B_h)(t_n,t_j;\cdot,\cdot) = k^{-1}[B(t_n,t_j;\cdot,\cdot) - B(t_n,t_{j-1};\cdot,\cdot)]$ is the backward difference quotient with respect to the second variable at

$t = t_j$, and hence we obtain

$$\frac{1}{2}\,\bar{\partial}_t\|\hat{\eta}^m\|_h^2 + c\|\hat{\eta}^m\|_1^2 \leq C\bigg[k\sum_{n=1}^{m}\|\hat{\eta}^{n-1}\|_1\|\hat{\eta}^m\|_1 + k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}\|\hat{\eta}^{j-1}\|_1\|\hat{\eta}^m\|_1 + (\|Q_A^m(u_h)\|_{-1,h} + \|\overline{Q}_B^m(u_h)\|_{-1,h})\|\hat{\eta}^m\|_1\bigg].$$

Multiply by $2k$ and then sum m from 2 to l with $m \leq l \leq N$ to find that

$$\|\hat{\eta}^l\|^2 + k\sum_{m=2}^{l}\|\hat{\eta}^m\|_1^2 \leq C\bigg[\|\hat{\eta}^1\|^2 + k^2\sum_{m=2}^{l}\sum_{n=1}^{m}\|\hat{\eta}^{n-1}\|_1^2 + k^3\sum_{m=2}^{l}\sum_{n=2}^{m}\sum_{j=1}^{n-1}\|\hat{\eta}^{j-1}\|_1^2 + k\sum_{m=2}^{l}\|Q_A^m(u_h)\|_{-1,h}^2 + k\sum_{m=2}^{l}\|\overline{Q}_B^m(u_h)\|_{-1,h}^2\bigg].$$

Add $k\|\hat{\eta}^1\|_1^2$ to both sides of the above equation and use (5.5) to obtain

$$\|\hat{\eta}^l\|^2 + k\sum_{m=1}^{l}\|\hat{\eta}^m\|_1^2 \leq C\bigg[k\sum_{m=1}^{l}(\|Q_A^m(u_h)\|_{-1,h}^2 + \|\overline{Q}_B^m(u_h)\|_{-1,h}^2) + k^2\sum_{m=1}^{l-1}\sum_{n=1}^{m}\|\hat{\eta}^n\|_1^2\bigg].$$

Finally, use Lemma 5.2 and then apply the discrete version of Gronwall's lemma to complete the rest of the proof. ◻

Lemma 5.4. *For $\varepsilon > 0$, constants C_1 exist depending on ε and C_2 such that*

$$k\sum_{j=1}^{n}t_j\|\eta^j\|^2 + t_n\|\hat{\eta}^n\|_1^2 \leq C_1(\varepsilon)t_nk^2\left(1 + \left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2 + C_2\varepsilon k\sum_{j=1}^{n}t_j^2\|\eta^j\|_1^2.$$

Proof. Choose $\chi = t_m\eta^m$ in (5.4) and use identity

$$2t_mA_h(t_m;\hat\eta^m,\eta^m) = \bar\partial_t[t_mA_h(t_m;\hat\eta^m,\hat\eta^m)] + t_mkA_h(t_m;\bar\partial_t\hat\eta^m,\bar\partial_t\hat\eta^m) - t_{m-1}(\bar\partial A_h)(t_m;\hat\eta^{m-1},\hat\eta^{m-1}) - A_h(t_m;\hat\eta^{m-1},\hat\eta^{m-1}),$$

to obtain

(5.6)

$$\begin{aligned} t_m\|\eta^m\|_h^2 &+ \frac{1}{2}\,\bar\partial_t[t_mA_h(t_m;\hat\eta^m,\hat\eta^m)] + \frac{t_m}{2}\,kA_h(t_m;\bar\partial_t\hat\eta^m,\bar\partial_t\hat\eta^m)\\ &= k\sum_{n=1}^{m}t_m(\bar\partial A_h)(t_n;\hat\eta^{n-1},\eta^m) + k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1}t_mB_h(t_n,t_j;\eta^j,\eta^m)\\ &\quad + \frac{1}{2}\,[t_{m-1}(\bar\partial A_h)(t_m;\hat\eta^{m-1},\hat\eta^{m-1})A_h(t_m;\hat\eta^{m-1},\hat\eta^{m-1})]\\ &\quad + [Q_A^m(u_h)(t_m\eta^m) + \overline{Q}_B^m(u_h)(t_m\eta^m)]\\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For $m = 1$, use of Young's inequality yields

$$kt_1\|\eta^1\|^2 + t_1\|\hat\eta\|_1^2 \le C(\varepsilon)k[\|Q_A^1(u_h)\|_{-1,h}^2 + \|\overline{Q}_B^1(u_h)\|_{-1,h}^2] + \varepsilon k t_1^2\|\eta^1\|_1^2. \tag{5.7}$$

For $m \ge 2$, we note that

$$\begin{aligned} k^2\sum_{n=2}^{m}\sum_{j=0}^{n-1}t_mB_h(t_n,t_j;\eta^j,\eta^m) &= k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}t_mB_h(t_n,t_j;\bar\partial_t\hat\eta^j,\eta^m)\\ &= k\sum_{n=2}^{m}t_mB_h(t_n,t_{n-1};\hat\eta^{n-1},\eta^m)\\ &\quad - k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}t_m(\bar\partial_2B_h)(t_n,t_j;\hat\eta^{j-1},\eta^m). \end{aligned}$$

For I_1 and I_2, we have

$$|I_1| + |I_2| \le C(\varepsilon)k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}\|\hat\eta^n\|_1^2 + \frac{\varepsilon}{2}t_m^2\|\eta^m\|_1^2.$$

The term I_3 can be written as

$$|I_3| \leq C[\|\hat{\eta}^{m-1}\|_1^2 + t_{m-1}\|\hat{\eta}^{m-1}\|_1^2].$$

Further, for I_4, an application of Young's inequality yields

$$|I_4| \leq C(\varepsilon)[\|Q_A^m(u_h)\|_{-1,h}^2 + \|\overline{Q}_B^m(u_h)\|_{-1,h}^2] + \frac{\varepsilon}{2} t_m^2 \|\eta^m\|_1^2.$$

After substituting we now sum (5.6) with respect to m from 2 to l to obtain

$$\begin{aligned} k\sum_{m=2}^{l} t_m\|\eta^m\|^2 &+ t_l\|\hat{\eta}^l\|_1^2 \\ &\leq C(\varepsilon)\Bigg[t_1\|\hat{\eta}^1\|_1^2 + k^2\sum_{m=2}^{l-1}\sum_{n=1}^{m}\|\hat{\eta}^n\|_1^2 + k\sum_{m=1}^{l-1}\|\hat{\eta}^m\|_1^2 \\ &\qquad + k\sum_{m=2}^{l}(\|Q_A^m(u_h)\|_{-1,h}^2 + \|\overline{Q}_B^m(u_h)\|_{-1,h}^2) + k\sum_{m=1}^{l-1} t_m\|\hat{\eta}^m\|_1^2\Bigg] \\ &\quad + \varepsilon C_k \sum_{m=2}^{l} t_m^2\|\eta^m\|_1^2. \end{aligned}$$

Now the term $kt_1\|\eta^1\|^2$ may be added to both sides of the above inequality and then use (5.7), Lemmas 5.2–5.3 and the discrete Gronwall's lemma to complete the rest of the proof. ◻

Lemma 5.5. *With $\tau^n = u_{ht}(t_n) - \bar{\partial}_t u_h(t_n)$, the following estimates hold true.*

(a) $\sum_{j=1}^{n} \|\tau^j\|_{-1,h}^2 \leq C\|u_0\|_1^2$.

(b) $\sum_{j=1}^{n} t_j^2\|\tau^j\|_{-1,h}^2 + \sum_{j=1}^{n} t_j^3\|\tau^j\|^2 \leq Ckt_n\|u_0\|_1^2$.

Proof. We write τ^j as

$$\tau^j = \frac{1}{k}\int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{hss}(s)\, ds. \tag{5.8}$$

Differentiating (1.6) with respect to t, we find that

$$(u_{htt}(t),\chi)_h = -A_h(t;u_{ht}(t),\chi) - A_{ht}(t;u_h(t),\chi) + B_h(t,t;u_h(t),\chi) + \int_0^t B_{ht}(t,s;u_h(s),\chi)\,ds.$$

It now follows that

$$\|u_{htt}(t)\|_{-1,h} \le C\Big[\|u_h(t)\|_1 + \|u_{ht}(t)\|_1 + \int_0^t \|u_h(s)\|_1\,ds\Big].$$

Thus, using the above estimate, we obtain

$$\begin{aligned}\|\tau^j\|_{-1,h}^2 &\le \frac{1}{2}\int_{t_{j-1}}^{t_j} s\|u_{hss}(s)\|_{-1,h}^2\,ds\\ &\le C\int_{t_{j-1}}^{t_j} s\{\|u_h(s)\|_1^2 + \|u_{hs}(s)\|_1^2 + \int_0^s \|u_h(\tau)\|_1^2\,d\tau\}\,ds,\end{aligned}$$

and by Theorem 2.2,

$$\begin{aligned}\sum_{j=1}^n \|\tau^j\|_{-1,h}^2 &\le C\int_0^{t_n} s(\|u_h(s)\|_1^2 + \|u_{hs}(s)\|_1^2 + \int_0^s \|u_h(\tau)\|_1^2\,d\tau)\,ds\\ &\le C\|u_0\|_1^2,\end{aligned}$$

which yields (a). To estimate (b), since $(s-t_{j-1})t_j \le ks$ for $s \in [t_{j-1},t_j]$, we have

$$\begin{aligned}\sum_{j=1}^n t_j^2\|\tau^j\|_{-1,h}^2 &\le \sum_{j=1}^n \frac{1}{k}\int_{t_{j-1}}^{t_j} (s-t_{j-1})^2 t_j^2 \|u_{hss}(s)\|_{-1,h}^2\,ds\\ &\le Ck\int_0^{t_n} s^2\{\|u_h(s)\|_1^2 + \|u_{hs}(s)\|_1^2 + \int_0^s \|u_h(\tau)\|_1^2\,d\tau\}\,ds\\ &\le Ct_nk\|u_0\|_1^2.\end{aligned}$$

Further, we note that

$$\begin{aligned}\sum_{j=1}^n t_j^3\|\tau^j\|^2 &= \sum_{j=1}^n (t_{j-1}^3 + 3kt_{j-1}^2 + 3k^2t_{j-1} + k^3)\|\tau^j\|^2\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

From (5.8), we have for I_1

$$I_1 \leq \frac{k}{3}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} s^3\|u_{hss}(s)\|^2\,ds$$
$$\leq \frac{k}{3}\int_0^{t_n} s^3\|u_{hss}(s)\|^2\,ds \leq Ckt_n\|u_0\|_1^2.$$

Similarly, we obtain

$$I_2 + I_3 \leq Ckt_n\|u_0\|_1^2.$$

Finally to estimate I_4, we have

$$I_4 \leq k^2\int_0^{t_n} s^2\|u_{hss}(s)\|^2\,ds \leq Ck^2\|u_0\|_1^2.$$

Altogether the above estimates now imply (b), and this completes the proof. □

Lemma 5.6. *With η^n as above, a constant C exists independent of k and may depend on T such that*

$$\|\eta^n\|^2 + k\sum_{j=1}^{n}\|\eta^j\|_1^2 \leq C(T)k\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2.$$

Proof. Taking $\chi = \eta^n$ in (5.1), use of identity $(\bar{\partial}_t\eta^n,\eta^n)_h = (1/2)\bar{\partial}_t[\|\eta^n\|_h^2] + (k/2)\|\bar{\partial}_t\eta^n\|_h^2$ leads to

$$\frac{1}{2}\,\bar{\partial}_t[\|\eta^n\|_h^2] + A_h(t_n;\eta^n,\eta^n) + \frac{k}{2}\,\|\bar{\partial}_t\eta^n\|_h^2$$
$$\leq \|\tau^n\|_{-1,h}\|\eta^n\|_1 + Ck\sum_{j=0}^{n-1}\|\eta^j\|_1\|\eta^n\|_1 + \|Q_B^n(u_h)\|_{-1,h}\|\eta^n\|_1.$$

Sum n from 1 to m and apply Young's inequality to obtain

$$\|\eta^m\|^2 + k\sum_{n=1}^{m}\|\eta^n\|_1^2 \leq C\Bigg[k\sum_{n=1}^{m}\|\tau^n\|_{-1,h}^2$$
$$+ k^2\sum_{n=1}^{m-1}\sum_{j=1}^{n}\|\eta^j\|_1^2 + k\sum_{n=1}^{m}\|Q_B^n(u_h)\|_{-1,h}^2\Bigg].$$

Use Lemmas 5.2, 5.5 and then apply the discrete Gronwall's lemma to complete the rest of the proof. □

Below we shall obtain an estimate for η^n in $L^\infty(L^2)$.

Lemma 5.7. *With η^n as above, we have for η^n, $n \geq 1$,*

$$t_n^2\|\eta^n\|^2 + k\sum_{j=1}^{n} t_j^2\|\eta^j\|_1^2 \leq C(T)t_nk^2\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2.$$

Proof. Setting $\chi = t_n^2\eta^n$ in (5.1) and using identity

$$t_n^2(\bar{\partial}_t\eta^n, \eta^n)_h = \frac{1}{2}\,\bar{\partial}_t[t_n^2\|\eta^n\|_h^2] + \frac{kt_n^2}{2}\,\|\bar{\partial}_t\eta^n\|_h^2 - t_{n-1}\|\eta^{n-1}\|_h^2 - \frac{k}{2}\,\|\eta^{n-1}\|_h^2,$$

we obtain

$$\frac{1}{2}\,\bar{\partial}_t[t_n^2\|\eta^n\|_h^2] + t_n^2A_h(t_n;\eta^n,\eta^n) + \frac{kt_n^2}{2}\,\|\bar{\partial}_t\eta^n\|_h^2 = t_n^2(\tau^n,\eta^n)_h + k\sum_{j=0}^{n-1} t_n^2B_h(t_n,t_j;\eta^j,\eta^n) + Q_B^n(u_h)(t_n^2\eta^n) + t_{n-1}\|\eta^{n-1}\|_h^2 + \frac{k}{2}\|\eta^{n-1}\|_h^2.$$

For $n = 1$, we have at once

$$t_1^2\|\eta^1\|^2 + kt_1^2\|\eta^1\|_1^2 \leq Ck[t_1^2\|\tau^1\|_{-1,h}^2 + \|Q_B^1(u_h)\|_{-1,h}^2]. \tag{5.9}$$

For $n \geq 2$, we note that

$$k\sum_{j=0}^{n-1} t_n^2B_h(t_n,t_j;\eta^j,\eta^n) = k\sum_{j=1}^{n-1} t_n^2B_h(t_n,t_j;\bar{\partial}_t\hat{\eta}^j,\eta^n) = t_n^2B_h(t_n,t_{n-1};\hat{\eta}^{n-1},\eta^n) - k\sum_{j=1}^{n-1} t_n^2(\bar{\partial}_2B_h)(t_n,t_j;\hat{\eta}^{j-1},\eta^n),$$

and hence, summing for n from 2 to m we have

$$\begin{aligned} t_m^2\|\eta^m\|^2+&(2c-\varepsilon)k\sum_{n=2}^{m} t_n^2\|\eta^n\|_1^2 \\ &\leq t_1^2\|\eta^1\|^2 + C(\varepsilon)k\sum_{n=2}^{m}[t_n^2\|\tau^n\|_{-1,h}^2 + \|\hat{\eta}^{n-1}\|_1^2] \\ &\quad + C(\varepsilon)k\sum_{n=2}^{m}\left[k\sum_{j=1}^{n-1}\|\hat{\eta}^{j-1}\|_1^2 + \|Q_B^n(u_h)\|_{-1,h}^2\right] \\ &\quad + Ck\sum_{n=2}^{m}[t_{n-1}\|\eta^{n-1}\|^2 + k^2\|\eta^{n-1}\|^2]. \end{aligned}$$

Choose ε so that $(2c-\varepsilon) > 0$. Now add $kt_1^2\|\eta^1\|_1^2$ to both sides of the above inequality and use (5.9). Finally, an application of Lemmas 5.2–5.6 completes the rest of the proof. □

Remark 5.1. In fact we now have from Lemmas 5.4 and 5.7

$$k\sum_{j=1}^{n} t_j\|\eta^j\|^2 + t_n\|\hat{\eta}^n\|_1^2 \leq C(T)t_nk^2\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2. \tag{5.10}$$

In order to prove the $L^\infty(H^1)$ estimate of η^n, the following lemmas will prove convenient.

Lemma 5.8. *A constant C exists independent of k such that the following estimate*

$$k\sum_{j=1}^{n}\|\bar{\partial}_t\eta^j\|^2 + \|\eta^n\|_1^2 \leq C(T)\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2$$

holds.

Proof. Take $\chi = \bar\partial_t \eta^n$ in (5.1) and obtain

$$\begin{aligned}\|\bar\partial_t\eta^n\|_h^2 + \frac{1}{2}\,\bar\partial_t[A_h(t_n;\eta^n,\eta^n)] + \frac{k}{2}\,A_h(t_n;\bar\partial_t\eta^n,\bar\partial_t\eta^n)\\ = (\tau^n,\bar\partial_t\eta^n)_h + k\sum_{j=0}^{n-1} B_h(t_n,t_j;\eta^j,\bar\partial_t\eta^n)\\ + \bar\partial_t[Q_B^n(u_h)(\eta^n)] - (\bar\partial_t Q_B^n(u_h))(\eta^{n-1})\\ + \frac{1}{2}\,(\bar\partial A_h)(t_n;\eta^{n-1},\eta^{n-1}).\end{aligned} \tag{5.11}$$

For $n = 1$, we find that

$$k\|\bar\partial_t\eta^1\|^2 + \|\eta^1\|_1^2 \le C[k\|\tau^1\|^2 + \|Q_B^1(u_h)\|_{-1,h}^2]. \tag{5.12}$$

For $n \ge 2$, we note that

$$\begin{aligned}k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1} B_h(t_n,t_j;\eta^j,\bar\partial_t\eta^n) &= k\sum_{n=2}^{m} B_h(t_n,t_{n-1};\hat\eta^{n-1},\bar\partial_t\eta^n)\\ &\quad - k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}(\bar\partial_2 B_h)(t_n,t_j;\hat\eta^{j-1},\bar\partial_t\eta^n)\\ &= I_1 + I_2.\end{aligned}$$

The terms I_1 and I_2 can be rewritten as

$$\begin{aligned}I_1 = B_h(t_m,t_{m-1};\hat\eta^{m-1},\eta^m) - k\sum_{n=2}^{m} B_h(t_n,t_{n-1};\eta^{n-1},\eta^{n-1})\\ - k\sum_{n=2}^{m}[(\bar\partial_1 B_h)(t_n,t_{n-1};\hat\eta^{n-2},\eta^{n-1})\\ + (\bar\partial_2 B_h)(t_{n-1},t_{n-1};\hat\eta^{n-2},\eta^{n-1})],\end{aligned}$$

and

$$\begin{aligned}I_2 &= -k^2\sum_{j=1}^{m-1}\sum_{n=j+1}^{m}(\bar\partial_2 B_h)(t_n,t_j;\hat\eta^{j-1},\bar\partial_t\eta^n)\\ &= -k\sum_{j=1}^{m-1}(\bar\partial_2 B_h)(t_m,t_j;\hat\eta^{j-1},\eta^m) + k\sum_{j=1}^{m-1}(\bar\partial_2 B_h)(t_j,t_j;\hat\eta^{j-1},\eta^j)\\ &\quad + k^2\sum_{n=2}^{m-1}\sum_{j=1}^{n-1}(\bar\partial_{21} B_h)(t_n,t_j;\hat\eta^{j-1},\eta^{n-1}),\end{aligned}$$

where $(\bar{\partial}_{21}B_h)(t_n,t_j;\cdot,\cdot) = k^{-1}[(\bar{\partial}_2 B_h)(t_n,t_j;\cdot,\cdot) - (\bar{\partial}_2 B_h)(t_{n-1},t_j;\cdot,\cdot)]$ is the backward difference quotient of $(\bar{\partial}_2 B)$ with respect to the first variable. Hence, summing (5.11) with respect to n from 2 to m and using (5.12), we obtain

(5.13)
$$\begin{aligned} k\sum_{n=1}^{m}\|\bar{\partial}_t\eta^n\|^2 + \|\eta^m\|_1^2 & \\ \leq C\bigg[k\sum_{n=1}^{m}\|\tau^n\|^2 &+ k\sum_{n=1}^{m-1}\|\hat{\eta}^n\|_1^2 + \|Q_B^1(u_h)\|_{-1,h}^2 + \|Q_B^m(u_h)\|_{-1,h}^2 \\ &+ k\sum_{n=2}^{m}\|\bar{\partial}_t(Q_B^n(u_h))\|_{-1,h}^2 + k\sum_{n=1}^{m-1}\|\eta^n\|_1^2 + \|\hat{\eta}^{m-1}\|_1^2\bigg]. \end{aligned}$$

As in Pani et al. [**10**, p. 1100], rewrite

$$\bar{\partial}_t(Q_B^n(u_h))(\chi) = k^{-1}Q_B^{n-1,n}(u_h)(\chi) + Q_{\bar{\partial}_1 B}^{n-1}(u_h)(\chi), \tag{5.14}$$

where $(\bar{\partial}_1, B_h)$ is the backward difference quotient of B_h with respect to the first variable. Apply Lemma 5.2 (replacing B_h by $\bar{\partial}_1 B_h$ for the second term) to obtain

$$\begin{aligned} \|\bar{\partial}_t(Q_B^n(u_h))\|_{-1,h} &\leq k^{-1}\|Q_B^{n-1,n}(u_h)\|_{-1,h} + \|Q_{\bar{\partial}_1 B}^{n-1}(u_h)\|_{-1,h} \\ &\leq C\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)\|u_0\|_1. \end{aligned}$$

Note that $\|\tau^n\|^2 \leq (1/k)\int_{t_{n-1}}^{t_n} s^2\|u_{hss}(s)\|^2\,ds$, and hence use Theorem 2.2 to find that

$$k\sum_{n=1}^{m}\|\tau^n\|^2 \leq \int_0^{t_m} s^2\|u_{hss}(s)\|^2\,ds \leq C\|u_0\|_1^2.$$

Using Lemma 5.2, for the third and fourth terms on the righthand side of (5.13), Lemma 5.1 and (5.10), now leads to

$$k\sum_{n=1}^{m}\|\bar{\partial}_t\eta^n\|^2 + \|\eta^m\|_1^2 \leq C\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2 + k\sum_{n=1}^{m-1}\|\eta^n\|_1^2.$$

Finally, apply the discrete Gronwall's lemma to complete the rest of the proof. □

Lemma 5.9. *With η^n as above, there is a constant C such that*
$$k\sum_{j=1}^{n} t_j\|\bar{\partial}_t\eta^j\|^2 + t_n\|\eta^n\|_1^2 \le Ck\biggl(1+\Bigl(\log\frac{1}{k}\Bigr)^{1/2}\biggr)^2\|u_0\|_1^2$$
holds.

Proof. The proof will proceed as in Lemma 5.8 taking $\chi = t_n\bar{\partial}_t\eta^n$ in (5.1). For the sake of clarity, we shall present a short proof of this lemma. In view of Lemma 5.8, it is enough to consider the first and fifth terms on the right of (5.13) as these terms lead to a loss of accuracy in k. The first term in the present case is of the form
$$k\sum_{n=1}^{m} t_n\|\tau^n\|^2 \le C\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}(s-t_{n-1})^2 t_n\|u_{hss}(s)\|^2\,ds.$$
Since $(s-t_{n-1})t_n \le sk$ for $s\in[t_{n-1},t_n]$, we have
$$k\sum_{n=1}^{m} t_n\|\tau^n\|^2 \le Ck\int_0^{t_m} s^2\|u_{hss}(s)\|^2\,ds \le Ck\|u_0\|_1^2.$$
Similarly, the fifth term in the present case is of the form $k\sum_{n=2}^{m} t_{n-1}\|\bar{\partial}_t(Q_B^n(u_h))\|_{-1,h}^2$. Note that
$$\begin{aligned} t_{n-1}^{1/2}|Q_B^{n-1,n}(u_h)(\chi)| &= \Bigl|\int_{t_{n-1}}^{t_n}(s-t_n)t_{n-1}^{1/2}\frac{\partial}{\partial s}[B_h(t_n,s;u_h(s),\chi)]\,ds\Bigr| \\ &\le Ck\int_{t_{n-1}}^{t_n} s^{1/2}(\|u_h(s)\|_1+\|u_{hs}(s)\|_1)\,ds\|\chi\|_1. \end{aligned}$$
Then, use of Theorem 2.2 yields
$$\begin{aligned} k^{-1}\sum_{n=2}^{m} t_{n-1}\|Q_B^{n-1,n}(u_h)\|_{-1,h}^2 & \\ &\le Ck\sum_{n=2}^{m}\int_{t_{n-1}}^{t_n} s(\|u_h(s)\|_1^2+\|u_{hs}(s)\|_1^2)\,ds \\ &\le Ck\int_0^{t_m} s(\|u_h(s)\|_1^2+\|u_{hs}(s)\|_1^2)\,ds \\ &\le Ck\|u_0\|_1^2. \end{aligned}$$

Applying Lemma 5.2, replacing B_h by $\bar{\partial}_1 B_h$, we obtain

$$t_{n-1}\|Q^{n-1}_{\bar{\partial}_1 B}(u_h)\|_{-1,h} \le Ck\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)\|u_0\|_1.$$

Hence,

$$\begin{aligned} k\sum_{n=2}^{m} t_{n-1}\|\bar{\partial}_t(Q^n_B(u_h))\|^2_{-1,h} \\ &\le Ck\|u_0\|_1^2 + Ck^2\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2 \\ &\le Ck\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2, \end{aligned}$$

and this completes the proof. □

The following lemma yields an estimate for $\|\eta^n\|_1$.

Lemma 5.10. *The following estimate holds for η^n, $n \ge 1$,*

$$k\sum_{j=1}^{n} t_j^3\|\bar{\partial}_t\eta^j\|^2 + t_n^3\|\eta^n\|_1^2 \le Ct_nk^2\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)^2\|u_0\|_1^2.$$

Proof. The proof is very similar to that of Lemma 5.9. Setting $\chi = t_n^3\bar{\partial}_t\eta^n$ in (5.1) and using identities

$$\begin{aligned} 2t_n^3A_h(t_n;\eta^n,\bar{\partial}_t\eta^n) &= \bar{\partial}_t[t_n^3A_h(t_n;\eta^n,\eta^n)] + t_n^3kA_h(t_n;\bar{\partial}_t\eta^n,\bar{\partial}_t\eta^n) \\ &\quad - t_{n-1}^3(\bar{\partial}A_h)(t_n;\eta^{n-1},\eta^{n-1}) \\ &\quad - (3t_{n-1}^2 + 3kt_{n-1} + k^2)A_h(t_n;\eta^{n-1},\eta^{n-1}), \end{aligned}$$

and

$$\begin{aligned} t_n^3Q^n_B(u_h)(\bar{\partial}_t\eta^n) &= \bar{\partial}_t[t_n^3Q^n_B(u_h)(\eta^n)] - t_{n-1}^3(\bar{\partial}_tQ^n_B(u_h))(\eta^{n-1}) \\ &\quad - (3t_{n-1}^2 + 3kt_{n-1} + k^2)Q^n_B(u_h)(\eta^{n-1}), \end{aligned}$$

we proceed as in Lemma 5.9 and apply Lemmas 5.1–5.9 to obtain the desired estimates, and this completes the proof. □

We are now in a position to prove the main result of this section.

Theorem 5.1. *Let u be the exact solution of* (1.1) *and U^n the backward Euler approximation defined by* (1.8). *Then, for $n \geq 1$, we have*

(a) $\|U^n - u(t_n)\| \leq C(T)t_n^{-1/2}(h^2 + k(1+\log(1/k))^{1/2}))\|u_0\|_1$,

(b) $\|U^n - u(t_n)\|_1 \leq C(T)t_n^{-1}(h + k(1+(\log(1/k))^{1/2}))\|u_0\|_1$.

Further, for any $\varepsilon > 0$, there is a constant C_ε such that

(c)

$$\|U^n - u(t_n)\|_\infty \leq C_\varepsilon h^{2-\varepsilon} t_n^{-1}\|u_0\|_1 + C|\log h|^{1/2} k t_n^{-1}(1+(\log(1/k))^{1/2})\|u_0\|_1.$$

Proof. We write $U^n - u(t_n)$ as $U^n - u(t_n) = \eta^n + e(t_n)$. The estimates (a) and (b) follow from Theorem 4.1 and Lemmas 5.7 and 5.10. For the estimation of (c), we have from (2.5) and Lemma 5.10

$$\|\eta^n\|_\infty \leq C|\log h|^{1/2}\|\eta^n\|_1 \leq C|\log h|^{1/2} k t_n^{-1}\left(1+\left(\log\frac{1}{k}\right)^{1/2}\right)\|u_0\|_1,$$

and this in combination with Theorem 4.2 completes the proof. □

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REFERENCES

1. J.R. Cannon and Y. Lin, *Nonclassical H^1 projections and Galerkin methods for nonlinear parabolic integro-differential equations*, Calcolo **25** (1988), 187–201.

2. ———, *A priori L^2 error estimates for finite element methods for nonlinear diffusion equations with memory*, SIAM J. Numer. Anal. **27** (1990), 595–607.

3. C.M. Chen and V. Thomée, *The lumped mass finite element method for a parabolic problem*, J. Austral. Math. Soc. Ser. B **26** (1985), 329–354.

4. S.H. Chou and Q. Li, *The effect of numerical integration in finite element methods for nonlinear parabolic equations*, Numer. Meth. PDE **6** (1990), 263–274.

5. M. Crouzeix and V. Thomée, *On the discretization in time of semilinear parabolic equations with nonsmooth initial data*, Math. Comp. **49** (1987), 35–93.

6. M.N. Le Roux and V. Thomée, *Numerical solution of semilinear integro-differential equations of parabolic type with nonsmooth data*, SIAM J. Numer. Anal. **26** (1989), 1291–1309.

7. Y. Lin, V. Thomée and L.B. Wahlbin, *Ritz-Volterra projections to finite element spaces and applications to integro-differential and related equations*, SIAM J. Numer. Anal. **28** (1991), 1047–1070.

8. Y. Lin and T. Zhang, *The stability of Ritz-Volterra projection and error estimates for finite element methods for a class of integro-differential equations of parabolic type*, Appl. Math. **36** (1991), Springer, New York, 123–133.

9. Y.Y. Nie and V. Thomée, *A lumped mass finite-element method with quadrature for a nonlinear parabolic problem*, IMA J. Numer. Anal. **5** (1985), 371–396.

10. A.K. Pani and T.E. Peterson, *Finite element method with numerical quadrature for parabolic integro-differential equation*, SIAM J. Numer. Anal. **33** (1996), 1084–1105.

11. A.K. Pani and R.K. Sinha, *Error estimates for semidiscrete Galerkin approximation to a time dependent parabolic integro-differential equation with nonsmooth data*, Research Report MRR 10-95, Dept. of Math., IIT, Bombay (India).

12. A.K. Pani, V. Thomée and L.B. Wahlbin, *Numerical methods for hyperbolic and parabolic integro-differential equations*, J. Integral Equations Appl. **4** (1992), 533–584.

13. P.A. Raviart, *The use of numerical integration in finite element methods for solving parabolic equations*, in *Topics in numerical analysis* (J.J.H. Miller, ed.), Academic Press, New York, 1973.

14. M. Schechter, *On L^p estimates and regularity*, I, Amer. J. Math. **85** (1963), 1–13.

15. L.R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), 483–493.

16. V. Thomée, J.C. Chao Xu and N.Y. Zhang, *Superconvergence of the gradient in piecewise linear finite-element approximation to a parabolic problem*, SIAM J. Numer. Anal. **26** (1989), 553–573.

17. V. Thomée and N.Y. Zhang, *Error estimates for semidiscrete finite element methods for parabolic integro-differential equations*, Math. Comp. **53** (1989), 121–139.

18. N.Y. Zhang, *On fully discrete Galerkin approximations for partial integro-differential equations of parabolic type*, Math. Comp. **60** (1993), 133–166.

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