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THE EXTRAPOLATION METHOD FOR TWO-DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS BASED ON THE ASYMPTOTIC EXPANSION OF ITERATED GALERKIN SOLUTIONS

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ABSTRACT. In this paper we study the numerical solution of two-dimensional Volterra integral equations by Galerkin and the iterated Galerkin method. Asymptotic error expansion of the iterated Galerkin solution is obtained. We show that when piecewise polynomials of $\pi_{p-1,q-1}$ are used, the iterated Galerkin solution admits an error expansion in powers of the stepsizes h and k, beginning with terms in h^{2p} and k^{2q}. Thus, Richardson's extrapolation can be performed based on this error expansion, and this will increase the accuracy of the numerical solution greatly. The theoretical results are confirmed by some numerical experiments.

1. Introduction. In this paper we are concerned with the Galerkin method and the iterated Galerkin method for the two-dimensional Volterra integral equation of the second kind

$$u(x,y) = g(x,y)+\int_0^x\int_0^y K(x,y,t,s)u(t,s)\,dt\,ds, \quad (x,y)\in D, \tag{1.1}$$

where $g(x,y)$, $K(x,y,t,s)$ are given continuous functions defined, respectively, on $D=[0,X]\times[0,Y]$ and $E=\{(x,y,t,s): 0\le t\le x\le X,\ 0\le s\le y\le Y\}$. It follows from the classical theory of Volterra (see, for example, [**2**], [**3**]) that (1.1) possesses a unique solution $u^*(x,y)\in C(D)$. Especially when g and K are r times continuously differentiable on D and E, respectively, then u^* is r times continuously differentiable on D.

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Let

$$(Ku)(x,y) = \int_0^x \int_0^y K(x,y,t,s)u(t,s)\,dt\,ds.$$

Then equation (1.1) becomes

$$u = Ku + g. \tag{1.2}$$

The study of superconvergence properties of numerical solutions for Volterra integral equations and methods for accelerating the convergence orders has received considerable attention since the early 1980's (see [**1**], [**2**], [**4**]–[**7**], [**11**], [**12**] and the references therein). In a recent paper [**5**], Brunner, Lin and Zhang studied the Richardson extrapolation method and two defect correction schemes by an interpolation post-processing technique for the numerical solution of one-dimensional linear Volterra integral equations by iterated finite element methods. Brunner and Kauthen [**3**] introduced collocation and iterated collocation methods for two-dimensional linear Volterra integral equations. They gave an analysis of global and local convergence properties of the collocation methods and the iterated collocation methods, and derived results on attainable orders of global convergence and local superconvergence. In [**7**], [**6**], asymptotic error expansions of iterated collocation solutions for two-dimensional linear and nonlinear Volterra integral equations were obtained, respectively. In this paper we discuss the Galerkin method and the iterated Galerkin method for equation (1.1). The asymptotic error expansion for the iterated Galerkin solution of (1.1) is obtained. We show that when piecewise polynomials of $\pi_{p-1,q-1}$ are used, the iterated Galerkin solution admits an error expansion in powers of the stepsizes h and k, beginning with terms in h^{2p} and k^{2q}. Thus, Richardson's extrapolation can be performed on the numerical solution, and this will increase the accuracy of the numerical solution greatly. The theoretical results are confirmed by some numerical experiments.

2. The asymptotic error expansion. Let $\Delta_M^{(1)}$ and $\Delta_N^{(2)}$ denote, respectively, equidistant partitions of $[0,X]$ and $[0,Y]$

$$\Delta_M^{(1)} : 0 = x_0 < x_1 < \cdots < x_M = X,$$

and

$$\Delta_N^{(2)} : 0 = y_0 < y_1 < \cdots < y_N = Y,$$

$h = (x_{i+1} - x_i) = X/M$, $k = (y_{j+1} - y_j) = Y/N$. $h = ck$, c is a constant. These partitions define a grid for D.

$$\Delta_{M,N} = \Delta_M^{(1)} \times \Delta_N^{(2)} = \{(x_m, y_n) : 0 \le m \le M,\ 0 \le n \le N\}.$$

Set

$$\begin{aligned} I_0^{(1)} &= [x_0, x_1], \quad I_m^{(1)} = (x_m, x_{m+1}], \quad m = 1, 2, \dots, M-1, \\ I_0^{(2)} &= [y_0, y_1], \quad I_n^{(2)} = (y_n, y_{n+1}], \quad n = 1, 2, \dots, N-1, \end{aligned}$$

and

$$I_{m,n} = I_m^{(1)} \times I_n^{(2)}, \quad m = 0, 1, \dots, M-1, \quad n = 0, 1, \dots, N-1.$$

We denote the finite element space by

$$\begin{aligned} S_{p-1,q-1}^{(-1)}(\Delta_{M,N}) = \{u : u|_{I_{m,n}} = u_{m,n} \in \pi_{p-1,q-1}, \\ 0 \le m \le M-1,\ 0 \le n \le N-1\}. \end{aligned}$$

Here $\pi_{p-1,q-1}$ denotes the space of real polynomials of degree $p-1$ in x and degree $q-1$ in y.

We use the superscript (-1) in the notation for the above finite element space to emphasize that it is not a subspace of $C(D)$.

The Galerkin method for solving (1.2) is defined as follows. Find $u^{hk} \in S_{p-1,q-1}^{(-1)}(\Delta_{M,N})$ such that

$$(u^{hk}, v) = (g, v) + (Ku^{hk}, v), \quad \forall\, v \in S_{p-1,q-1}^{(-1)}(\Delta_{M,N}), \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the usual inner product in $L_2(D)$.

Let P_{hk} denote the orthogonal projection of $L_2(D)$ onto $S_{p-1,q-1}^{(-1)}(\Delta_{M,N})$. Then problem (2.1) can be equivalently rewritten: Find $u^{hk} \in S_{p-1,q-1}^{(-1)}(\Delta_{M,N})$ such that

$$u^{hk} = P_{hk} g + P_{hk} K u^{hk}. \tag{2.2}$$

Therefore, the iterated Galerkin solution, $\tilde{u}^{hk}$, corresponding to the above Galerkin solution u^{hk}, is given by

$$\tilde{u}^{hk}(x, y) = g(x, y) + (Ku^{hk})(x, y), \quad (x, y) \in D. \tag{2.3}$$

For the iterated Galerkin solution $\tilde{u}^{hk}$, it is straightforward to show that

$$(I - KP_{hk})\tilde{u}^{hk} = g, \tag{2.4}$$

and that

$$P_{hk}\tilde{u}^{hk} = u^{hk}. \tag{2.5}$$

We first give an explicit formula for $P_{hk}u$. Denote the inner product in the real Hilbert space $L_2[0,1]$ by

$$(u \mid v) = \int_0^1 u(t)v(t)\,dt. \tag{2.6}$$

Let $\varphi_0, \varphi_1, \ldots$ be a sequence of orthogonal polynomials associated with the inner product (2.6), i.e., $\varphi_i(t)$ is a polynomial of degree of i, and

$$(\varphi_i \mid \varphi_j) = \delta_{ij}, \quad i, j \geq 0.$$

In fact, let $L_0(t) = 1$,

$$L_i(t) = \frac{1}{2^i i!}\left[\frac{d^i}{dt^i}(t^2-1)^i\right], \quad i \geq 1$$

be the Legendre polynomial of degree i. Then the polynomial φ_i is related to the Legendre polynomial L_i by

$$\varphi_i(t) = \sqrt{2i+1}\,L_i(2t-1).$$

Now set

$$\psi_j(s) = \sqrt{2j+1}\,L_j(2s-1).$$

Define

$$\varphi_{im}(x) = \begin{cases} h^{-1/2}\varphi_i((x-x_m)/h) & x \in [x_m, x_{m+1}] \\ 0 & x \in [0,X] \setminus [x_m, x_{m+1}] \end{cases}$$

and

$$\psi_{jn}(y) = \begin{cases} k^{-1/2}\psi_j((y-y_n)/k) & y \in [y_n, y_{n+1}] \\ 0 & y \in [0,Y] \setminus [y_n, y_{n+1}]. \end{cases}$$

Then the functions $\{\varphi_{im}(x)\psi_{jn}(y)\}$, $0 \le i \le p-1$, $0 \le m \le M-1$, $0 \le j \le q-1$ and $0 \le n \le N-1$, form an orthogonal basis for $S^{(-1)}_{p-1,q-1}(\Delta_{M,N})$, therefore

$$(2.7)\qquad (P_{hk}u)(x,y) \;=\; \sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(\varphi_{im}\psi_{jn},u)\varphi_{im}(x)\psi_{jn}(y).$$

Lemma 1. *Let $u(x,y) \in C^{r+1}(D)$, $r \ge \max(p,q)$ be an integer. Then, for any $(x,y) \in (x_m,x_{m+1}) \times (y_n,y_{n+1})$, $m = 0,1,\dots,M-1$, $n = 0,1,\dots,N-1$, we have*

(2.8)

$$P_{hk}u(x,y) \;=\; \sum_{l=0}^{r}\sum_{\mu=0}^{l} h^{\mu}k^{l-\mu}u^{(\mu,l-\mu)}(x,y)\Phi_{\mu}\left(\frac{x-x_m}{h}\right)\Psi_{l-\mu}\left(\frac{y-y_n}{k}\right) + O(h^{r+1}+k^{r+1}),$$

where

$$\Phi_{\mu}(\tau) \;=\; \int_0^1 \sum_{i=0}^{p-1}\varphi_i(\xi)\varphi_i(\tau)\,\frac{(\xi-\tau)^{\mu}}{\mu!}\,d\xi$$

and

$$\Psi_{\nu}(\theta) \;=\; \int_0^1 \sum_{j=0}^{q-1}\psi_j(\eta)\psi_j(\theta)\,\frac{(\eta-\theta)^{\nu}}{\nu!}\,d\eta.$$

Proof. Let $(x,y) \in (x_m,x_{m+1}) \times (y_n,y_{n+1})$. From (2.7) we have

$$P_{hk}u(x,y)-u(x,y) = \sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\varphi_{im}(x)\psi_{jn}(y) \cdot \int_{x_m}^{x_{m+1}}\int_{y_n}^{y_{n+1}}\varphi_{im}(t)\psi_{jn}(s)[u(t,s)-u(x,y)]\,dt\,ds.$$

Let $t = x_m + \xi h$, $s = y_n + \eta k$, $x = x_m + \tau h$ and $y = y_n + \theta k$, then, expanding $u(x_m+\xi h, y_n+\eta k)$ in Taylor series at $\xi = \tau$, $\eta = \theta$ and writing it as polynomials in h and k we obtain

$$\begin{aligned}
&P_{hk}u(x,y) - u(x,y)\\
&\quad = \int_0^1 \int_0^1 \sum_{i=0}^{p-1} \varphi_i(\xi)\varphi_i(\tau) \sum_{j=0}^{q-1} \psi_j(\eta)\psi_j(\theta)\\
&\qquad\qquad \cdot [u(x_m + \xi h, y_n + \eta k) - u(x_m + \tau h, y_n + \theta k)]\, d\xi\, d\eta\\
&\quad = \sum_{l=1}^{r} \sum_{\mu=0}^{l} h^\mu k^{l-\mu} u^{(\mu,l-\mu)}(x,y) \int_0^1 \sum_{i=0}^{p-1} \varphi_i(\xi)\varphi_i(\tau)\, \frac{(\xi-\tau)^\mu}{\mu!}\, d\xi\\
&\qquad\qquad \cdot \int_0^1 \sum_{j=0}^{q-1} \psi_j(\eta)\psi_j(\theta)\, \frac{(\eta-\theta)^{l-\mu}}{(l-\mu)!}\, d\eta + O(h^{r+1} + k^{r+1})\\
&\quad = \sum_{l=1}^{r} \sum_{\mu=0}^{l} h^\mu k^{l-\mu} u^{(\mu,l-\mu)}(x,y) \Phi_\mu\left(\frac{x-x_m}{h}\right) \Psi_{l-\mu}\left(\frac{y-y_n}{k}\right)\\
&\qquad + O(h^{r+1} + k^{r+1}).
\end{aligned}$$

The lemma is proved.

Using the Christoffel-Darboux identity [**8**, p. 342],

$$\sum_{i=0}^{p-1} \varphi_i(\xi)\varphi_i(\tau) \;=\; \frac{a_{p-1}}{a_p} \cdot \frac{\varphi_p(\xi)\varphi_{p-1}(\tau) - \varphi_{p-1}(\xi)\varphi_p(\tau)}{\xi - \tau}, \tag{2.9}$$

where a_p denotes the leading coefficient of the polynomial $\varphi_p(x)$. Note that $\varphi_0, \varphi_1, \dots$ is a sequence of orthogonal polynomials, and it is easy to show that

$$\Phi_\mu(\tau) \;=\; 0, \quad 1 \le \mu \le p-1,$$

and

$$\Psi_\nu(\tau) \;=\; 0, \quad 1 \le \nu \le q-1.$$

From Lemma 1 we can obtain the following corollary.

Corollary 1. *Let* $u(x,y) \in C^{r+1}(D)$, $r \ge \max(p,q)$ *be an integer. Then, for any* $(x,y) \in (x_m, x_{m+1}) \times (y_n, y_{n+1})$, $m = 0, 1, \dots, M-1$,

$n = 0, 1, \dots, N-1$, we have

$$
\begin{aligned}
(P_{hk} - I)u(x,y) \; &= \sum_{\mu=p}^{r} h^{\mu} u^{(\mu,0)}(x,y)\Phi_{\mu}\left(\frac{x - x_m}{h}\right) \\
&+ \sum_{\nu=q}^{r} k^{\nu} u^{(0,\nu)}(x,y)\Psi_{\nu}\left(\frac{y - y_n}{k}\right) \\
&+ \sum_{\mu=p}^{r-q} \sum_{\nu=q}^{r-\mu} h^{\mu} k^{\nu} u^{(\mu,\nu)}(x,y)\Phi_{\mu}\left(\frac{x-x_m}{h}\right)\Psi_{\nu}\left(\frac{y-y_n}{k}\right) \\
&+ O(h^{r+1} + k^{r+1}),
\end{aligned}
$$

where $\Phi_{\mu}(\tau)$, $\Psi_{\nu}(\theta)$ are defined in Lemma 1.

Lemma 2. *Let $V(x,y) = \sum_{i=0}^{r} \sum_{j=0}^{r-i} h^i k^j V_{i,j}(x,y)$, $V_{i,j}(x,y) \in C^{r+1-i-j}(D)$, $i = 0, 1, \dots, r$, $j = 0, 1, \dots, r-i$. Then, for any $(x,y) \in (x_m, x_{m+1}) \times (y_n, y_{n+1})$, $m = 0, 1, \dots, M-1$, $n = 0, 1, \dots, N-1$, we have*

$$
\begin{aligned}
P_{hk}V(x,y) \; &= \; V_{0,0}(x,y) + \sum_{i=0}^{r} \sum_{j=0}^{r-i} h^i k^j \tilde{V}_{i,j}\left(x, y, \frac{x - x_m}{h}, \frac{y - y_n}{k}\right) \\
&+ O(h^{r+1} + k^{r+1}),
\end{aligned}
$$

where $\tilde{V}_{0,0}(x,y,t,s) = 0$, for $i \neq 0$ or $j \neq 0$, $\tilde{V}_{i,j}(x,y,t,s) = \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} V_{i-\mu,j-\nu}^{(\mu,\nu)}(x,y)\Phi_{\mu}(t)\Psi_{\nu}(s)$.

Proof. For any $(x,y) \in (x_m, x_{m+1}) \times (y_n, y_{n+1})$, from Lemma 1 we have

$$
\begin{aligned}
P_{hk}V(x,y) \; &= \; \sum_{i=0}^{r} \sum_{j=0}^{r-i} h^i k^j P_{hk} V_{i,j}(x,y) \\
&= \sum_{i=0}^{r} \sum_{j=0}^{r-i} h^i k^j \sum_{\mu=0}^{r-i-j} \sum_{\nu=0}^{r-i-j-\mu} h^{\mu} k^{\nu} V_{i,j}^{(\mu,\nu)}(x,y) \\
&\cdot \Phi_{\mu}\left(\frac{x - x_m}{h}\right)\Psi_{\nu}\left(\frac{y - y_n}{k}\right) + O(h^{r+1} + k^{r+1}).
\end{aligned}
\tag{2.10}
$$

Write (2.10) as polynomials in h and k:

$$P_{hk}V(x,y) = \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j \sum_{\mu=0}^{i}\sum_{\nu=0}^{j} V_{i-\mu,j-\nu}^{(\mu,\nu)}(x,y) \cdot \Phi_\mu\left(\frac{x-x_m}{h}\right)\Psi_\nu\left(\frac{y-y_n}{k}\right) + O(h^{r+1}+k^{r+1}).$$

Now let $\tilde{V}_{0,0}(x,y,t,s)=0$ and, for $i\neq 0$ or $j \neq 0$,

$$\tilde{V}_{i,j}(x,y,t,s) = \sum_{\mu=0}^{i}\sum_{\nu=0}^{j} V_{i-\mu,j-\nu}^{(\mu,\nu)}(x,y)\Phi_\mu(t)\Psi_\nu(s)$$

we can obtain Lemma 2.

Lemma 3 (Euler-MacLaurin summation formula). *Let* $f(x,y)\in C^{r+1}(D)$, $0\le\tau\le 1$, $0\le\theta\le 1$. *Then*

$$hk\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} f(x_\mu+\tau h, y_\nu+\theta k) = \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j \,\frac{B_i(\tau)}{i!}\frac{B_j(\theta)}{j!}\,[f^{(i-1,j-1)}(x,y)]_{x=0,y=0}^{x_m\ \ y_n} + O(h^{r+1}+k^{r+1})$$

where $B_j(t)$ *are Bernoulli polynomials,* $[f^{(-1,-1)}(x,y)]_{x=0,y=0}^{x_m\ \ y_n} = \int_0^{x_m}\int_0^{y_n} f(x,y)\,dx\,dy$, $[f(x,y)]_{x=0,y=0}^{x_m\ \ y_n} = f(x_m,y_n)-f(x_m,0)-f(0,y_n)+f(0,0)$.

Proof. To establish the above expression, we use the general one-dimensional Euler-MacLaurin summation formula

$$(*)\qquad h\sum_{\mu=0}^{m-1} g(x_\mu+\tau h) = \sum_{i=0}^{r} h^i\,\frac{B_i(\tau)}{i!}\,[g^{(i-1)}(x,y)]_{x=0}^{x_m} + O(h^{r+1})$$

valid for $0\le\tau\le 1$, see [**10**, p. 377]. Here B_i is the Bernoulli polynomial of degree i, $[g^{(-1)}(x,y)]_{x=0}^{x_m} = \int_0^{x_m} g(x)\,dx$, for $i\ge 0$, $[g^{(i)}(x)]_{x=0}^{x_m} = g^{(i)}(x_m)-g^{(i)}(0)$.

Using the summation formula $(*)$, we find that

$$
\begin{aligned}
hk \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} f(x_\mu + \tau h, y_\nu + \theta k)& \\
&= k \sum_{\nu=0}^{n-1} \sum_{i=0}^{r} h^i \, \frac{B_i(\tau)}{i!} \, [f^{(i-1,0)}(x,y)]_{x=0}^{x_m} + O(h^{r+1}) \\
&= \sum_{i=0}^{r} \sum_{j=0}^{r-i} h^i k^j \, \frac{B_i(\tau)}{i!} \, \frac{B_j(\theta)}{j!} \, [f^{(i-1,j-1)}(x,y)]_{x=0,y=0}^{x_m \quad y_n} \\
&\quad + O(h^{r+1} + k^{r+1}),
\end{aligned}
$$

which gives the result.

Next we turn to discussing the main theorem in this section, the asymptotic expansion theorem.

Theorem 1. *Suppose that the hypotheses of Lemma* 2 *are satisfied and* $K(x,y,t,s) \in C^{r+1}(E)$. *Then, for any* $(x,y) \in \Delta_{M,N}$, *we have*

(2.11)

$$
\begin{aligned}
&(KP_{hk}V)(x,y) \\
&\quad = \int_0^x \int_0^y K(x,y,t,s) V_{0,0}(t,s) \, dt \, ds \\
&\qquad + \sum_{i=0}^{r} \sum_{j=0}^{r-i} h^i k^j \sum_{\alpha=0}^{i} \sum_{\beta=0}^{j} \int_0^1 \int_0^1 \frac{B_\alpha(\xi)}{\alpha!} \, \frac{B_\beta(\eta)}{\beta!} \\
&\qquad \cdot \left[\frac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1} \partial s^{\beta-1}} \left(K(x,y,t,s) \tilde{V}_{i-\alpha,j-\beta}(t,s,\xi,\eta) \right) \right]_{t=0,s=0}^{x_m \quad y_n} d\xi \, d\eta \\
&\qquad + O(h^{r+1} + k^{r+1}).
\end{aligned}
$$

Proof. Let $(x,y) = (x_m, y_n)$ be a point on the grid $\Delta_{M,N}$. According to Lemma 2, we have

$$
\begin{aligned}
(KP_{hk}V)(x,y) &= \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}\int_{x_\mu}^{x_{\mu+1}}\int_{y_\nu}^{y_{\nu+1}} K(x,y,t,s)P_{hk}V(t,s)\,dt\,ds \\
&= \int_0^x\int_0^y K(x,y,t,s)V_{0,0}(t,s)\,dt\,ds \\
&\quad + \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}\bigg\{\int_{x_\mu}^{x_{\mu+1}}\int_{y_\nu}^{y_{\nu+1}} \\
&\qquad\qquad \cdot\left[K(x,y,t,s)\tilde{V}_{i,j}\left(t,s,\frac{t-x_\mu}{h},\frac{s-y_\nu}{k}\right)\right]dt\,ds\bigg\} \\
&\quad + O(h^{r+1}+k^{r+1}).
\end{aligned}
$$

Let $t = x_\mu + \xi h$, $s = y_\nu + \eta k$, then

(2.12)

$$
\begin{aligned}
&(KP_{hk}V)(x,y) \\
&\quad = (KV_{0,0})(x,y) \\
&\qquad + \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j \int_0^1\int_0^1 hk \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} \\
&\qquad\qquad \cdot [K(x,y,x_\mu+\xi h, y_\nu+\eta k)\tilde{V}_{i,j}(x_\mu+\xi h, y_\nu+\eta k,\xi,\eta)]\,d\xi\,d\eta \\
&\qquad + O(h^{r+1}+k^{r+1}).
\end{aligned}
$$

Using Lemma 3 we find

$$
\begin{aligned}
&hk\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} K(x,y,x_\mu+\xi h,y_\nu+\eta k)\tilde{V}_{i,j}(x_\mu+\xi h,y_\nu+\eta k,\xi,\eta) \\
&\quad = \sum_{\alpha=0}^{r-i-j}\sum_{\beta=0}^{r-i-j-\alpha} h^\alpha k^\beta \frac{B_\alpha(\xi)}{\alpha!}\,\frac{B_\beta(\eta)}{\beta!} \\
&\qquad\qquad \cdot\left[\frac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1}\partial s^{\beta-1}}\big(K(x,y,t,s)\tilde{V}_{i,j}(t,s,\xi,\eta)\big)\right]_{t=0,s=0}^{x_m\quad y_n} \\
&\qquad + O(h^{r+1-i-j}+k^{r+1-i-j}).
\end{aligned}
$$

Substituting this expression into (2.12) and writing it as polynomials in h and k, we can obtain Theorem 1.

From Lemma 2 we know

$$V(x,y) = \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j V_{i,j}(x,y).$$

Subtracting this expression from (2.11), we obtain

(2.13′)

$$\begin{aligned}
&V(x,y) - (KP_{hk}V)(x,y)\\
&\quad = V_{0,0}(x,y) - \int_0^x\int_0^y K(x,y,t,s)V_{0,0}(t,s)\,dt\,ds\\
&\qquad + \sum_{p=1}^{r}\sum_{i+j=p} h^i k^j \biggl(V_{i,j}(x,y) - \sum_{\alpha=0}^{i}\sum_{\beta=0}^{j}\int_0^1\int_0^1 \frac{B_\alpha(\xi)}{\alpha!}\,\frac{B_\beta(\eta)}{\beta!}\\
&\qquad \cdot \left[\frac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1}\partial s^{\beta-1}}\bigl(K(x,y,t,s)(\tilde V_{i-\alpha,j-\beta}(t,s,\xi,\eta))\bigr)\right]_{t=0,s=0}^{x\quad y} d\xi\,d\eta\biggr)\\
&\qquad + O(h^{r+1}+k^{r+1}).
\end{aligned}$$

Note that

$$\begin{aligned}
&V_{i,j}(x,y) - \sum_{\alpha=0}^{i}\sum_{\beta=0}^{j}\int_0^1\int_0^1 \frac{B_\alpha(\xi)}{\alpha!}\,\frac{B_\beta(\eta)}{\beta!}\\
&\quad \cdot \left[\frac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1}\partial s^{\beta-1}}\bigl(K(x,y,t,s)(\tilde V_{i-\alpha,j-\beta}(t,s,\xi,\eta))\bigr)\right]_{t=0,s=0}^{x\quad y} d\xi\,d\eta\\
&= V_{i,j}(x,y) - \int_0^x\int_0^y K(x,y,t,s)V_{i,j}(t,s)\,dt\,ds\\
&\quad - \sum_{\alpha=0}^{i}\sum_{\beta=0}^{j}\int_0^1\int_0^1 \frac{B_\alpha(\xi)}{\alpha!}\,\frac{B_\beta(\eta)}{\beta!}\\
&\qquad\qquad \cdot \biggl[\frac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1}\partial s^{\beta-1}}\bigl(K(x,y,t,s)(\tilde V_{i-\alpha,j-\beta}(t,s,\xi,\eta)\\
&\qquad\qquad\qquad - (1-\operatorname{sgn}(\alpha+\beta))V_{i,j}(t,s))\bigr)\biggr]_{t=0,s=0}^{x\quad y} d\xi\,d\eta.
\end{aligned}$$

Now we choose $V_{0,0}(x,y) = u^*(x,y)$ and $V_{i,j}(x,y)$, $i \neq 0$ or $j \neq 0$, to satisfy the following linear Volterra integral equations

(2.13)

$$
\begin{aligned}
V_{i,j}(x,y) &- \int_0^x \int_0^y K(x,y,t,s) V_{i,j}(t,s)\,dt\,ds \\
&= \sum_{\alpha=0}^{i} \sum_{\beta=0}^{j} \int_0^1 \int_0^1 \frac{B_\alpha(\xi)}{\alpha!}\,\frac{B_\beta(\eta)}{\beta!} \\
&\qquad \cdot \Bigg[\frac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1} \partial s^{\beta-1}} \big(K(x,y,t,s)(\tilde{V}_{i-\alpha,j-\beta}(t,s,\xi,\eta) \\
&\qquad\qquad - (1-\operatorname{sgn}(\alpha+\beta)) V_{i,j}(t,s)) \big) \Bigg]_{t=0,\,s=0}^{x\quad y} d\xi\,d\eta
\end{aligned}
$$

where

$$
\operatorname{sgn}(x) \;=\; \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0. \end{cases}
$$

From $(2.13')$ we have

$$
\begin{aligned}
V(x,y) - (KP_{hk}V)(x,y) \;&=\; g(x,y) + O(h^{r+1}+k^{r+1}), \\
(x,y) \;&\in\; \Delta_{M,N}.
\end{aligned}
\tag{2.14}
$$

In the following, we will prove that the righthand side of equation (2.13) does not contain the unknown function $V_{i,j}(t,s)$ and its derivatives.

For any $0 \le i \le r$, $0 \le j \le r-i$, let:

(i) $\alpha \ne 0$ or $\beta \ne 0$.

In this case, $1-\operatorname{sgn}(\alpha+\beta)=0$. From the definition of $\tilde{V}_{i-\alpha,j-\beta}(t,s,\xi,\eta)$, we know that $\tilde{V}_{i-\alpha,j-\beta}(t,s,\xi,\eta)$ does not contain the function $V_{i,j}(t,s)$ and its derivatives.

(ii) $\alpha = 0$ and $\beta = 0$.

$$\begin{aligned}\tilde{V}_{i,j}(t,s,\xi,\eta)-V_{i,j}(t,s) &= \sum_{\mu=0}^{i}\sum_{\nu=0}^{j} V_{i-\mu,j-\nu}^{(\mu,\nu)}(t,s)\Phi_\mu(\xi)\Psi_\nu(\eta)-V_{i,j}(t,s)\\ &= \sum_{\nu=1}^{j} V_{i,j-\nu}^{(0,\nu)}(t,s)\Psi_\nu(\eta)\\ &\quad+\sum_{\mu=1}^{i}\sum_{\nu=0}^{j} V_{i-\mu,j-\nu}^{(\mu,\nu)}(t,s)\Phi_\mu(\xi)\Psi_\nu(\eta)\end{aligned}$$

$\tilde{V}_{i,j}(t,s,\xi,\eta) - V_{i,j}(t,s)$ does not contain $V_{i,j}(t,s)$ and its derivatives.

Combining (i) and (ii), it follows that the righthand side of equation (2.13) does not contain the function $V_{i,j}(t,s)$ and its derivatives.

Remark 1. For any $0 \le i \le r$, $0 \le j \le r-i$, it is easily seen that the righthand side of equation (2.13) is $r+1-i-j$ times continuously differentiable when $g(x,y) \in C^{r+1}(D)$ and $K(x,y,t,s) \in C^{r+1}(E)$. It now follows from the classical theory of Volterra (see, e.g., [**2**]) that (2.13) possesses a unique solution $V_{i,j}(x,y) \in C^{r+1-i-j}(D)$.

Theorem 2. *Let $g(x,y) \in C^{r+1}(D)$, $K(x,y,t,s) \in C^{r+1}(E)$ and $u^*(x,y)$ be a solution of* (1.1). *Then, for sufficiently large M and N, the iterated Galerkin solution in mesh points $\tilde{u}^{hk}(x,y)$, $(x,y) \in \Delta_{M,N}$, can be expanded as*

(2.15)
$$\begin{aligned}\tilde{u}^{hk}(x,y) \;=\; u^*(x,y) &+ \sum_{i=p}^{[r/2]} h^{2i}V_{2i,0}(x,y) + \sum_{j=q}^{[r/2]} k^{2j}V_{0,2j}(x,y)\\ &+\sum_{i=p}^{[r/2]-q}\;\sum_{j=q}^{[r/2]-i} h^{2i}k^{2j}V_{2i,2j}(x,y) + O(h^{r+1}+k^{r+1}),\\ &\qquad (x,y)\in\Delta_{M,N}\end{aligned}$$

where the $V_{i,j}(x,y)$, $i \ne 0$ or $j \ne 0$, satisfy the equations of (2.13).

Proof. For any mesh point $(x,y) = (x_m, y_n) \in \Delta_{M,N}$, let $\eta(x,y) =$

$V(x,y) - \tilde{u}^{hk}(x,y)$. Subtracting (2.4) from (2.14), we obtain

$$(I - KP_{hk})\,\eta\,(x,y) \;=\; O(h^{r+1} + k^{r+1}), \quad (x,y) \in \Delta_{M,N}. \tag{2.16}$$

The operator series KP_{hk} converges uniformly to K as $h \to 0^+$ and $k \to 0^+$. Note that $(I-K)^{-1}$ exists and is uniformly bounded. It follows that $(I - KP_{hk})^{-1}$ exists and is uniformly bounded for all sufficiently small values h and k. So we have, for any mesh point,

$$\tilde{u}^{hk}(x,y) \;=\; \sum_{i=0}^{r}\sum_{j=0}^{r-i} h^i k^j V_{i,j}(x,y) + O(h^{r+1}+k^{r+1}), \qquad (x,y) \;\in\; \Delta_{M,N}.$$

Thus, to complete the proof, we need only verify that $V_{i,j}(x,y) = 0$ if i is odd or $i \le 2p-1$, j is odd or $j \le 2q-1$.

Let $R_{p-1}(\tau,\xi) \;=\; \sum_{m=0}^{p-1} \varphi_m(\tau)\varphi_m(\xi)$. It is easily seen that the righthand side of equation (2.13) is related to the sum of products of

$$\int_0^1\int_0^1 B_\alpha(\xi) R_{p-1}(\tau,\xi)(\tau-\xi)^\mu \, d\tau\, d\xi$$
$$0 \le \alpha \le i, \quad 0 \le \mu \le i - \alpha,$$
$$\int_0^1\int_0^1 B_\beta(\eta) R_{q-1}(\theta,\eta)(\theta-\eta)^\nu \, d\theta\, d\eta,$$
$$\theta \le \beta \le j, \quad 0 \le \nu \le j - \beta,$$

and the derivatives of $V_{i-\alpha-\mu,j-\beta-\nu}(x,y)$.

Note that

$$L_m(-\tau) \;=\; (-1)^m L_m(\tau),$$
$$\varphi_m(\tau) \;=\; \sqrt{2m+1}\, L_m(2\tau-1),$$
$$B_\alpha(\xi) \;=\; (-1)^\alpha B_\alpha(1-\xi),$$

it follows that

$$\varphi_m(1-\tau) \;=\; (-1)^m \varphi_m(\tau),$$
$$R_{p-1}(1-\tau, 1-\xi) \;=\; R_{p-1}(\tau,\xi).$$

Hence, if $\tau' = 1-\tau$ and $\xi' = 1-\xi$, then

$$B_\alpha(\xi) R_{p-1}(\tau,\xi)(\tau-\xi)^\mu \;=\; (-1)^{\alpha+\mu} B_\alpha(\xi') R_{p-1}(\tau',\xi')(\tau'-\xi')^\mu,$$

which leads to the formula

$$A_{\alpha\mu} = \int_0^1 \int_0^1 B_\alpha(\xi) R_{p-1}(\tau,\xi)(\tau-\xi)^\mu \, d\tau \, d\xi = (-1)^{\alpha+\mu} A_{\alpha\mu}.$$

Now suppose that i is odd. If $\alpha+\mu$ is odd, then $A_{\alpha\mu} = 0$. If $\alpha+\mu$ is even, then $i-\alpha-\mu < i$ is odd. By using the induction method we know that $V_{i-\alpha-\mu,j-\beta-\nu}(x,y) = 0$. Therefore, we know that the righthand term of equation (2.13) is equal to zero whenever i is odd. So $V_{i,j}(x,y) = 0$ whenever i is odd.

To prove $V_{i,j}(x,y) = 0$ for $i \le 2p-1$, use the Christoffel-Darboux identity

$$R_{p-1}(\tau,\xi) = \frac{a_{p-1}}{a_p} \cdot \frac{\varphi_p(\xi)\varphi_{p-1}(\tau) - \varphi_{p-1}(\xi)\varphi_p(\tau)}{\xi-\tau},$$

and note that $\varphi_0(\tau), \varphi_1(\tau), \ldots$ is a sequence of orthogonal polynomials. Thus we can show that

$$A_{\alpha\mu} = \int_0^1 \int_0^1 B_\alpha(\xi) R_{p-1}(\tau,\xi)(\tau-\xi)^\mu \, d\tau \, d\xi = 0, \\ 0 \le \alpha \le i, \quad 0 \le \mu \le i-\alpha,$$

hence $V_{i,j}(x,y) = 0$ whenever $i \le 2p-1$.

Similarly we have $V_{i,j}(x,y) = 0$ if j is odd or $j \le 2q-1$. The theorem is thus proved.

For the two-dimensional nonlinear Volterra integral equation of the second kind,

(2.17)
$$u(x,y) = g(x,y) + \int_0^x \int_0^y K(x,y,t,s,u(t,s)) \, dt \, ds, \quad (x,y) \in D$$

where $g(x,y)$, $K(x,y,t,s,u)$ are given continuous functions defined, respectively, on $D = [0,X] \times [0,Y]$ and $E = \{(x,y,t,s,u) : 0 \le t \le x \le X, 0 \le s \le y \le Y, -\infty < u < +\infty\}$, with $K(x,y,t,s,u)$ nonlinear in u. We can obtain a similar asymptotic error expansion of the iterated Galerkin solution for equation (2.17).

We choose $W_{0,0}(x,y) = u^*(x,y)$ and $W_{i,j}(x,y)$, $i \neq 0$ or $j \neq 0$, to satisfy the following linear Volterra integral equations

(2.18)
$$\begin{aligned}
&W_{i,j}(x,y) - \int_0^x \int_0^y K_u(x,y,t,s,u^*(t,s))W_{i,j}(t,s)\,dt\,ds \\
&= \sum_{\alpha=0}^{i} \sum_{\beta=0}^{j} \int_0^1 \int_0^1 \frac{B_\alpha(\xi)}{\alpha!} \frac{B_\beta(\eta)}{\beta!} \\
&\qquad \cdot \bigg[\frac{\partial^{\alpha+\beta-2}}{\partial t^{\alpha-1} \partial s^{\beta-1}} \big(K_u(x,y,t,s,u^*(t,s))(\tilde{W}_{i-\alpha,j-\beta}(t,s,\xi,\eta) \\
&\quad - (1-\operatorname{sgn}(\alpha+\beta))W_{i,j}(t,s)) + f_{i-\alpha,j-\beta}(x,y,t,s,\xi,\eta) \big) \bigg]_{t=0,s=0}^{x \quad y} d\xi\, d\eta.
\end{aligned}$$

where $\tilde{W}_{0,0}(x,y,t,s) = 0$ for $i \neq 0$ or $j \neq 0$, $\tilde{W}_{i,j}(x,y,t,s) = \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} W_{i-\mu,j-\nu}^{(\mu,\nu)}(x,y)\Phi_\mu(t)\Psi_\nu(s)$.

$$\begin{aligned}
f_{i,j}(x,y,t,s,\xi,\eta) = &\sum_{p=2}^{i+j} \frac{1}{p!} \left(\frac{\partial}{\partial u} \right)^p K(x,y,t,s,u^*(t,s)) \\
&\cdot \bigg(\sum_{\alpha_1+\cdots+\alpha_p=i} \sum_{\beta_1+\cdots+\beta_p=j} \prod_{n=1}^{p} \tilde{W}_{\alpha_n,\beta_n}(t,s,\xi,\eta) \bigg).
\end{aligned}$$

In analogy to Theorem 2, we obtain the following results.

Theorem 3. *Let $g(x,y) \in C^{r+1}(D)$, $K(x,y,t,s,u) \in C^{r+1}(E)$ and $u^*(x,y)$ be a solution of (2.17). Then, for sufficiently large M and N, the iterated Galerkin solution at the mesh points, $\tilde{u}^{hk}(x,y)$, $(x,y) \in \Delta_{M,N}$, can be expanded as*

$$\begin{aligned}
\tilde{u}^{hk}(x,y) \;=\; &u^*(x,y) + \sum_{i=p}^{[r/2]} h^{2i} W_{2i,0}(x,y) + \sum_{j=q}^{[r/2]} k^{2j} W_{0,2j}(x,y) \\
&+ \sum_{i=p}^{[r/2]-q} \sum_{j=q}^{[r/2]-i} h^{2i} k^{2j} W_{2i,2j}(x,y) \qquad (2.19) \\
&+ O(h^{r+1} + k^{r+1}), \quad (x,y) \in \Delta_{M,N}
\end{aligned}$$

where the $W_{i,j}(x,y)$, $i \neq 0$ *or* $j \neq 0$, *satisfy the equations of* (2.18).

Remark 2. Note that the asymptotic error expansion (2.19) (or (2.15)) holds only at the points of the mesh $\Delta_{M,N}$. The Richardson extrapolation method can be used only at the same points. But by virtue of (3.2), similar to [**5**], we can also construct the global extrapolation approximation of order 4 or higher order by an interpolation post-processing method [**9**].

Remark 3. Based on asymptotic expansion, similarly to [**5**], an iterative correction method for the interpolation post-processing for the iterated Galerkin solution $\tilde{u}^{hk}(x,y)$ of equation (1.1) (or (2.17)), corresponding to the piecewise constant finite element solution can also be given. The iterative technique is of high precision in that the $(n-1)$-fold application of the iterative correction method will lead to a global convergence rate of $O(h^{2n}+k^{2n})$.

Remark 4. In [**6**], asymptotic error expansion of the iterated collocation solution at mesh points for two-dimensional nonlinear Volterra integral equations was obtained. We showed that when piecewise polynomials of $\pi_{p-1,q-1}$ are used, the collocation points are the Gauss points, the iterated collocation solution admits an error expansion in even powers of the stepsizes h and k, beginning with terms in h^{2p} and k^{2q}. In this paper we prove that the iterated Galerkin solution at the mesh points for equation (1.1) (or (2.17)) can also be expanded in even powers of the stepsizes h and k. These theoretical results are confirmed by some numerical experiments in Section 3.

3. Numerical illustration.

Example 3.1 (Brunner [**3**]). Consider the linear Volterra integral equation

$$u(x,y) = g(x,y) + \int_0^x \int_0^y k(x,y,t,s)u(t,s)\,dt\,ds, \tag{3.1}$$

$$(x,y) \in [0,1]\times[0,1]$$

where $k(x,y,t,s) = \exp(-(x-t))\cos(y-s)$, $g(x,y) = \exp(-x)(\cos(y)-$

$x(\sin(y) + y\cos(y))/2)$. Its exact solution is $u^*(x,y) = \exp(-x)\cos(y)$.

Example 3.2. Consider the nonlinear Volterra integral equation

$$\begin{aligned} u(x,y) \;&=\; g(x,y) + \int_0^x \int_0^y (x+y-t-s)u^2(t,s)\,dt\,ds, \\ &(x,y) \;\in\; [0,1]\times[0,1] \end{aligned} \tag{3.2}$$

where
$$g(x,y) \;=\; x + y - xy(x^3 + 4x^2y + 4xy^2 + y^3)/12.$$
Its exact solution is $u^*(x,y) = x + y$.

The solutions of (3.1) and (3.2) will be approximated by the iterated Galerkin method in the space $S^{(-1)}_{p-1,q-1}(\Delta_{M,N})$, with $p = q = 1$. This space is the piecewise constant finite element space. We choose uniform partitions with $M = N$, $h = k = 1/N$, $N = 2, 4, 8, 16, 32, 64$. The maximum absolute errors of Example 3.1 and Example 3.2 are given in Table 1 and Table 2, respectively.

Denoting by $\tilde{u}^{hk}(x,y)$ the iterated Galerkin approximation with respect to this partition, using Theorem 2 (or Theorem 3) for $m = 1, 2, \ldots$, we derive the Richardson extrapolation formulas

$$\tilde{u}^{hk}_m(x,y) \;=\; \frac{4^m \tilde{u}^{h/2,k/2}_{m-1}(x,y) - \tilde{u}^{hk}_{m-1}(x,y)}{4^m - 1}, \quad (x,y) \in \Delta_{M,N}, \tag{3.3}$$

where $\tilde{u}^{hk}_0(x,y) = \tilde{u}^{hk}(x,y)$.

These Richardson extrapolations yield a series of new approximations which generate approximations of higher and higher order. In fact, from the asymptotic error expansion, it is easily seen that the function $\tilde{u}^{hk}_m$ approximates u^* with accuracy of order $O(h^{2+2m} + k^{2+2m})$.

The following tables exhibit a summary of the predicted convergence orders.

For ease of notation, we define in the following tables: $E^{(m)}_N = \max\{|u^*(x,y) - \tilde{u}^{hk}_m(x,y)| : (x,y) \in \Delta_{M,N}\}$ and $\alpha^{(i)} = \log_2(E^{(i)}_N / E^{(i)}_{2N})$ has been used as an estimate of the convergence order.

TABLE 1. Maximum absolute errors of Example 3.1.

N	$E_N^{(0)}$	$\alpha^{(0)}$	$E_N^{(1)}$	$\alpha^{(1)}$	$E_N^{(2)}$	$\alpha^{(2)}$
2	3.1407E-3	2.0027	4.8694E-5	3.9727	1.0005E-7	5.9750
4	7.8370E-4	2.0050	3.1016E-6	4.0094	1.5906E-9	5.9857
8	1.9525E-4	1.9967	1.9259E-7	4.0020	2.5100E-11	5.9946
16	4.8925E-5	2.0001	1.2020E-8	4.0011	3.9366E-13	
32	1.2231E-5	2.0002	7.5073E-10			
64	3.0573E-6					

TABLE 2. Maximum absolute errors of Example 3.2.

N	$E_N^{(0)}$	$\alpha^{(0)}$	$E_N^{(1)}$	$\alpha^{(1)}$	$E_N^{(2)}$	$\alpha^{(2)}$
2	7.8266E-2	2.1342	2.3170E-3	4.0993	1.0277E-5	6.2009
4	1.7829E-2	2.0332	1.3518E-4	4.0225	1.3970E-7	6.0478
8	4.3558E-3	2.0083	8.3177E-6	4.0055	2.1116E-9	6.0137
16	1.0827E-3	2.0021	5.1788E-7	4.0013	3.2682E-11	
32	2.7029E-4	2.0005	3.2338E-8			
64	6.7548E-5					

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