JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 19, Number 4, Winter 2007

AN INVERSION FORMULA IN ENERGY DEPENDENT SCATTERING

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ABSTRACT. The paper discusses the inverse scattering problem to recover the potentials of an energy dependent Schrödinger equation from the scattering data. A new inversion formula is developed, by which the potentials are recovered directly through the solution of a Marchenko equation. It requires no differentiability assumptions on potentials.

1. Introduction. This paper is concerned with inverse scattering problems for the energy dependent Schrödinger equation

$$f'' + [k^2 - (U(x) + 2kQ(x))]f = 0, \quad x \in I, \quad '' = \frac{d^2}{dx^2}, \tag{1.1}$$

where $I = [0, \infty)$ or $I = \mathbf{R}$ and $U(x), Q(x)$ are real-valued functions defined on I. As a special case where $U(x) \equiv -Q(x)^2$, equation (1.1) contains the time-independent Klein-Gordon equation for a particle of zero mass and of energy k subject to a static potential $Q(x)$. In the case of $Q = 0$, equation (1.1) is the radial (or one-dimensional) time-independent Schrödinger equation, where k^2 denotes the energy and $U(x)$ is the interaction potential. Besides the quantum scattering theory, equation (1.1) appears in a class of inverse spectral problems; recently the author has applied an inverse scattering theory for (1.1) with $U = 0$ to the reconstruction of an oceanic flow from the data of an observable property in the ocean. That will be left to a later paper [**10**].

The validity of the Marchenko method (see Marchenko [**13**] and Chadan and Sabatier [**3**]) in the inverse scattering theory for relativistic scattering problems was first suggested in Cornille [**4**] and Weiss and Scharf [**19**]. More systematically, Jaulent and Jean [**8, 9**] and Jaulent [**5, 6**] treated the inverse scattering problem for equation (1.1) and

Partly supported by Grant-in-Aid for Scientific Research No. 15540201, Japan Society for Promotion of Science.
Received by the editors on July 10, 2006.

have established a procedure through which the potentials U and Q are recovered from the scattering data, under the assumption that U and Q are real-valued, differentiable functions belonging to spaces of integrable functions together with the derivatives, when there are no bound states. They derived a nonlinear differential equation for $\int_x^\infty Q(\eta)\,d\eta$ from the Marchenko equation and, based upon it, proved that the potentials are uniquely determined from the scattering data. Sattinger and Szmigielski [**15**] simplified the Jaulent-Jean procedure to show the existence of isospectral flows in a nonlinear evolution equation. Though we confine ourselves to the case $U(x), Q(x)$ real-valued, the Marchenko method has been applied also to the case where $U(x)$ is real and $Q(x)$ is purely imaginary, see Jaulent [**7**], Aktosun, Klaus, and van der Mee [**1, 2**], and furthermore to the energy dependent Schrödinger equation with a positive mass parameter, see Kaup [**11**], Tsutsumi [**17**], Sattinger and Szmigielski [**16**], van der Mee and Pivovarchik [**18**].

The purpose of the present paper is to develop a new recovery formula of potentials from the scattering data. The formula has two advantages. In the first place it requires no auxiliary differential equations for the unknown potentials; we can recover the potentials directly from the solution of a Marchenko equation. Secondly it requires no differentiability assumptions on potentials; we just assume that

$$(1 + |x|)U(x),\ Q(x) \in L^1(I), \tag{1.2}$$

$$Q(x) \in BC(I), \tag{1.3}$$

where $L^1(I)$ denotes the space of integrable functions on I, while $BC(I)$ denotes that of bounded, continuous functions there.

Let us describe the recovery formula in the case of $I = [0, \infty)$ without embarking on the details, which we shall defer to later sections. First we assume (1.2). Then, for each k in $\operatorname{Im} k \geq 0$, equation (1.1) admits a unique solution $f(x, k)$ with the asymptotic behavior

$$f(x, k) = e^{ikx}[1 + o(1)], \quad x \to \infty.$$

The solution, which is referred to as the Jost solution, can be expressed as

$$f(x, k) = f(x, 0)e^{ikx} - ik \int_x^\infty K(x, t)e^{ikt}\,dt,\ \ \operatorname{Im} k \geq 0, \tag{1.4}$$

in terms of a continuous, bounded function $K(x,t)$ defined on $0 \le x \le t < \infty$ with the derivative $K_t(x,t)$ that belongs to $L^1(x,\infty)$ as a function of t. The function $K(x,t)$, which is referred to as the transformation kernel, is connected with the potential Q through the relation

$$f(x,0) + K(x,x) = e^{i\int_x^\infty Q(\eta)\,d\eta}, \quad 0 \le x < \infty. \tag{1.5}$$

The inverse problem we discuss here is: to recover $U(x)$ and $Q(x)$ in (1.1) from the scattering data

$$S(k) := \frac{\overline{f(0,k)}}{f(0,k)}$$

on the real axis, where $\overline{f(0,k)}$ denotes the complex conjugate of $f(0,k)$. Let us assume that $f(0,0) \neq 0$. Then, the data $S(k)$ can be expressed as

$$S(k) = C + \int_{-\infty}^{\infty} F(t)e^{-ikt}\,dt, \quad k \in \mathbf{R} \tag{1.6}$$

in terms of a complex constant C with absolute value 1 and a function $F(t) \in L^1(\mathbf{R})$. The pair $(C, F(t))$ is uniquely determined from $S(k)$.

We deal with the inverse problem in the absence of bound states. In other words, we suppose that $f(0,k)$ has no zeros in the upper half-plane $\operatorname{Im} k > 0$. Then the transformation kernel $K(x,t)$ and $F(t)$ are connected by a Marchenko equation of the following form:

$$\overline{K(x,t)} + \int_x^\infty K(x,r)F(r+t)\,dr + f(x,0)\int_x^\infty F(r+t)\,dr = 0, \quad x \le t. \tag{1.7}$$

This integral equation admits a unique solution $K(x,\cdot)$ in the space $BC[x,\infty)$ for each $x \ge 0$. The solvability holds even if the third term in (1.7) is replaced by a bounded continuous function. In particular, the integral equation

$$\overline{\Delta(x,t)} + \int_x^\infty \Delta(x,r)F(r+t)\,dr + \int_x^\infty F(r+t)\,dr = 0, \quad x \le t, \tag{1.8}$$

has a unique solution $\Delta(x,t)$ in the space. Since $f(x,0)$ is real-valued and the solution of (1.7) is unique, we obtain

$$K(x,t) = f(x,0)\Delta(x,t). \tag{1.9}$$

We now employ the assumption (1.3). Then the derivative $K_t(x,t)$ is connected with the potentials U, Q through the formula

$$2K_t(x,x)e^{-i\int_x^\infty Q(\eta)\,d\eta} = \int_x^\infty [U(r) + Q(r)^2]\,dr - iQ(x), \quad 0 \le x < \infty. \tag{1.10}$$

This equality is a key to our approach; by a computation with (1.5), (1.9) and (1.10) we deduce the formula

$$\frac{2\Delta_t(x,x)}{1+\Delta(x,x)} = \int_x^\infty [U(r) + Q(r)^2]\,dr - iQ(x), \quad 0 \le x < \infty. \tag{1.11}$$

This formula recovers the potential U, Q in (1.1) with $I = [0,\infty)$ from the solution $\Delta(x,t)$ of Marchenko's equation of the form (1.8). Actually, $Q(x)$ is determined from the imaginary part of it:

$$Q(x) = -2\ \mathrm{Im}\,\frac{\Delta_t(x,x)}{1+\Delta(x,x)},$$

and, in turn, $U(x)$ is determined by taking the derivative of the real part of it:

$$U(x) = -2\frac{d}{dx}\ \mathrm{Re}\left(\frac{\Delta_t(x,x)}{1+\Delta(x,x)}\right) - Q(x)^2.$$

In this way, for the Schrödinger equation (1.1) in the case $I = [0,\infty)$, we arrive at the following result:

Theorem 1.1. *If, for given data $S(k)$ on $\mathbf{R}$, there exists a pair (U,Q) of real-valued functions $U(x)$, $Q(x)$ with* (1.2), (1.3) *for which $f(0,k)$ satisfies*

$$f(0,k) \neq 0, \quad \mathrm{Im}\,k > 0, \quad k = 0; \qquad \frac{\overline{f(0,k)}}{f(0,k)} = S(k), \quad k \in \mathbf{R},$$

then (U,Q) *is recovered from* $S(k)$ *by* (1.11)*, where* $\Delta(x,t)$ *is the solution of integral equation* (1.8) *with* F *defined by* (1.6) *from* $S(k)$*.*

There are three particular cases where: (I) $Q(x)\equiv 0$. In this case, (1.5) combined with (1.9) yields $f(x,0)(1+\Delta(x,x))=1$. Therefore, (1.11) is rewritten as

$$U(x) = -2\frac{d}{dx}K_t(x,x).$$

This is a well-known formula (see [**13**, page 224]; note that $K_t(x,t)$ in our terminology is no other than $K(x,t)$ there) in the original Marchenko theory concerning the (nonrelativistic) inverse scattering problem. When $Q(x)\equiv 0$, the transformation kernel $K(x,t)$ and the function $F(t)$ in (1.6) are real-valued because the scattering data $S(k)$ has the symmetric relation: $S(-k)=\overline{S(k)}$. Hence, the original Marchenko equation is deduced from (1.7) by differentiating it and performing an integration by parts. In this way we can reproduce the inverse scattering theory to restore the interaction potential $U(x)$ in classical quantum mechanics when there are no bound states.

(II) $U(x)\equiv -Q(x)^2$. In this case, formula (1.11) yields

$$Q(x) = 2i\frac{\Delta_t(x,x)}{1+\Delta(x,x)}.$$

This gives a recovery formula of the potential in the Klein-Gordon equation from the solution $\Delta(x,t)$ of (1.8), originally from the scattering data $S(k)$, which in $k>0$ and $k<0$ describes the scattering of the particle and the anti-particle, respectively. With regard to (1.11), the Klein-Gordon equation is a peculiar one in the sense that the lefthand side of it becomes purely imaginary.

(III) $U(x)\equiv 0$. In this case, $Q(x)$ is determined directly from (1.5) as

$$Q(x) = i\frac{d}{dx}\log(1+K(x,x)).$$

Unlike in the general case, where $U\neq 0$, neither the assumption (1.3) nor the condition $f(0,k)\neq 0$, $\operatorname{Im} k>0$, $k=0$, need be imposed on

$Q(x)$ because the former is necessary only for (1.10) and the latter is automatically fulfilled in this case, as will be shown in Appendix C, where we shall study conditions on the potentials for the assumption of the absence of bound states to be guaranteed, and prove, by a homotopy trick, that if there are no bound states for a pair $(U, 0)$ then the situation is the same for any pair (U, Q) with arbitrary $Q \in L^1(0, \infty)$.

When $I = \mathbf{R}$ we can follow the same steps as for the case of $I = [0, \infty)$ and obtain an inversion formula (formulated as Theorem 5.5) by which the potentials $U(x)$ and $Q(x)$ on $\mathbf{R}$ can be recovered from a reflection coefficient in the S-matrix. It is also derived with no differentiability assumptions on the potentials.

2. Transformation kernel. In this section we shall establish the representation (1.4) of the Jost solution to (1.1) and relation (1.5) for the kernel $K(x, t)$, under the assumption (1.2). If $(1 + |x|)U(x)$, $Q(x) \in L^1(0, 1)$, then, for each k in $\operatorname{Im} k \geq 0$, the equation

$$f'' + [k^2 - (U(x) + 2kQ(x))]f = 0, \quad 0 < x < \infty, \tag{2.1}$$

admits a unique solution $f(x, k)$ with asymptotic behavior

$$f(x, k) = e^{ikx}[1 + o(1)], \quad x \to \infty. \tag{2.2}$$

The solution $f(x, k)$ satisfies

$$f'(x, k) = ike^{ikx}[1 + o(1)], \quad x \to \infty, \tag{2.3}$$

uniformly in $\operatorname{Im} k \geq 0$. Moreover, for each $x \geq 0$, it is holomorphic with respect to k in the upper half plane $\operatorname{Im} k > 0$ and is continuous in its closure $\operatorname{Im} k \geq 0$. The solution $f(x, k)$ is referred to as the Jost solution of (1.1).

The following lemma establishes the representation (1.4) of the Jost solution and yields relation (1.5) of the transformation kernel $K(x, t)$. Let us use the notation

$$\sigma(x) := \int_x^\infty [(1 + |\eta|)|U(\eta)| + 2|Q(\eta)|]\, d\eta. \tag{2.4}$$

Lemma 2.1. *Under the assumption* $(1+x)U(x), Q(x) \in L^1(0,\infty)$, *the Jost solution* $f(x,k)$ *can be expressed as* (1.4) *in terms of a continuous, bounded function* $K(x,t)$ *defined on* $0 \le x \le t < \infty$. *The kernel* $K(x,t)$ *is uniquely determined from* $U(x), Q(x)$ *and*

(1) $K(x,t)$ *tends to* 0 *as* $t \to \infty$.

(2) *For every* $x \in [0,\infty)$, *the transformation kernel* $K(x,t)$ *is differentiable with respect to* t *almost everywhere, and the derivative* $K_t(x,t)$ *satisfies the inequality*

$$\int_x^\infty |K_t(x,t)|\,dt \le M\sigma(x) \tag{2.5}$$

with some constant M.

(3) $K(x,t)$ *satisfies relation* (1.5).

(4) *The following identity holds*:

$$f(x,k) = e^{i\int_x^\infty Q(\eta)\,d\eta} e^{ikx} + \int_x^\infty K_t(x,t)e^{ikt}\,dt, \quad \operatorname{Im} k \ge 0. \tag{2.6}$$

In particular,

$$f(0,k) = e^{i\int_0^\infty Q(\eta)\,d\eta} + \int_0^\infty K_t(0,t)e^{ikt}\,dt, \quad \operatorname{Im} k \ge 0. \tag{2.7}$$

Proof. As was shown in Jaulent and Jean [**8**, Lemma 4.1], if $A(x,t)$ satisfies the integral equation

$$\begin{aligned} A(x,t) = {}& \frac{1}{2}\int_{(x+t)/2}^\infty U(s)e^{i\int_s^\infty Q(\eta)\,d\eta}\,ds \\ & - \frac{i}{2}Q\left(\frac{x+t}{2}\right)e^{i\int_{(x+t)/2}^\infty Q(\eta)\,d\eta} \\ & + \frac{1}{2}\int_{(x+t)/2}^\infty U(s)\,ds\int_s^{t+s-x} A(s,u)\,du \\ & + \frac{1}{2}\int_x^{(x+t)/2} U(s)\,ds\int_{t+x-s}^{t+s-x} A(s,u)\,du \\ & + i\int_x^\infty Q(s)A(s,t+s-x)\,ds \\ & - i\int_x^{(x+t)/2} Q(s)A(s,t+x-s)\,ds, \end{aligned} \tag{2.8}$$

then the Jost solution $f(x,k)$ is represented as

$$f(x,k)=e^{i\int_x^\infty Q(\eta)\,d\eta}e^{ikx}+\int_x^\infty A(x,t)e^{ikt}\,dt,\quad \operatorname{Im}k\ge 0. \tag{2.9}$$

We solve equation (2.8) in the space $L^1(x,\infty)$ for each $x\ge 0$. For the purpose we set

$$\begin{aligned}A_0(x,t)=\frac{1}{2}\int_{(x+t)/2}^\infty U(s)e^{i\int_s^\infty Q(\eta)\,d\eta}\,ds\\ -\frac{i}{2}Q\left(\frac{x+t}{2}\right)e^{i\int_{(x+t)/2}^\infty Q(\eta)\,d\eta},\end{aligned}$$

$$\begin{aligned}A_n(x,t)=\frac{1}{2}\int_{(x+t)/2}^\infty &U(s)\,ds\int_s^{t+s-x}A_{n-1}(s,u)\,du\\ +\frac{1}{2}\int_x^{(x+t)/2}&U(s)\,ds\int_{t+x-s}^{t+s-x}A_{n-1}(s,u)\,du\\ +i\int_x^\infty &Q(s)A_{n-1}(s,t+s-x)\,ds\\ -i\int_x^{(x+t)/2}&Q(s)A_{n-1}(s,t+x-s)\,ds.\end{aligned} \tag{2.10}$$

By the assumption $(1+|x|)U(x), Q(x)\in L^1(0,\infty)$, we have $\int_x^\infty |A_0(x,t)|\,dt \le M_0\sigma(x)$ with some constant M_0. Since

$$\begin{aligned}\int_x^\infty dt\left\{\int_{(x+t)/2}^\infty ds\int_s^{t+s-x}du+\int_x^{(x+t)/2}ds\int_{t+x-s}^{t+s-x}du\right\}\\ =\int_x^\infty ds\int_s^\infty du\int_{u-s+x}^{u-x+s}dt,\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{2}\int_x^\infty dt\bigg\{\int_{(x+t)/2}^\infty |U(s)|\,ds\int_s^{t+s-x}|A_{n-1}(s,u)|\,du\\ +\int_x^{(x+t)/2}|U(s)|\,ds\int_{t+x-s}^{t+s-x}|A_{n-1}(s,u)|\,du\bigg\}\\ =\int_x^\infty (s-x)|U(s)|\,ds\int_s^\infty |A_{n-1}(s,u)|\,du.\end{aligned}$$

Moreover, we get

$$\int_x^\infty dt\biggl\{\int_x^\infty |Q(s)||A_{n-1}(s,t+s-x)|\,ds \\ + \int_x^{(x+t)/2} |Q(s)||A_{n-1}(s,t+x-s)|\,ds\biggr\} \\ = 2\int_x^\infty |Q(s)|\,ds \int_s^\infty |A_{n-1}(s,u)|\,du.$$

Hence, if

$$\int_x^\infty |A_{n-1}(x,t)|\,dt \le \frac{M_0}{(n-1)!}\,\sigma(x)^n, \tag{2.11}$$

then

$$\begin{aligned}\int_x^\infty |A_n(x,t)|\,dt &\le \int_x^\infty [(s-x)|U(s)| + 2|Q(s)|]\,ds \int_s^\infty |A_{n-1}(s,u)|\,du \\ &\le \frac{M_0}{(n-1)!}\int_x^\infty [(1+s)|U(s)| + 2|Q(s)|]\sigma(s)^n\,ds \\ &= \frac{M_0}{n!}\sigma(x)^{n+1}.\end{aligned}$$

Therefore, $A_n(x,t)$ belongs to $L^1(x,\infty)$ as a function of t for each $x \ge 0$ and satisfies (2.11) for $n = 1, 2, \dots$. By (2.11), the series $\sum_{n=0}^\infty A_n(x,t)$ converges in $L^1(x,\infty)$ and, moreover, $A(x,t)$ defined by this series satisfies (2.8) and

$$\int_x^\infty |A(x,t)|\,dt \le M_0\sigma(x)e^{\sigma(x)}. \tag{2.12}$$

We now define a function $K(x,t)$ by

$$K(x,t) = -\int_t^\infty A(x,\eta)\,d\eta. \tag{2.13}$$

Assertion (1) is direct from this definition. Moreover, we have $K_t(x,t) = A(x,t)$ for almost every t and, by noting (2.12) and setting $M = M_0e^{\sigma(0)}$, we obtain (2.5).

Since $K(x,t) \to 0$ as $t \to \infty$, by performing an integration by parts in (2.9), it follows that, for $\operatorname{Im} k \geq 0$,

$$\begin{aligned} f(x,k) &= e^{i\int_x^\infty Q(\eta)\,d\eta} e^{ikx} + \int_x^\infty K_t(x,t) e^{ikt}\,dt \\ &= \left(e^{i\int_x^\infty Q(\eta)\,d\eta} - K(x,x)\right) e^{ikx} \\ &\quad - ik \int_x^\infty K(x,t) e^{ikt}\,dt. \end{aligned}$$

Therefore, upon setting $k = 0$, we obtain (1.5) and (1.4).

The uniqueness of $K(x,t)$ follows from the uniqueness theorem for the Fourier transform. Actually, if $\int_x^\infty K_1(x,t)e^{ikt}\,dt = \int_x^\infty K_2(x,t)e^{ikt}\,dt$, then, for each $\eta > 0$,

$$\int_x^\infty (K_1(x,t) - K_2(x,t)) e^{-\eta t} e^{i\xi t}\,dt = 0, \quad -\infty < \xi < \infty,$$

where $(K_1(x,t)-K_2(x,t))e^{-\eta t} \in L^1(x,\infty)$. Hence, $K_1(x,t) = K_2(x,t)$. $\square$

3. Separation formula. In this section we shall derive formula (1.10) which separates Q from (U,Q) in the imaginary part.

Lemma 3.1. *We assume, in addition to* $(1+|x|)U(x), Q(x) \in L^1(0,\infty)$, *that* $Q(x) \in BC[0,\infty)$, *i.e., that* $Q(x)$ *is a bounded, continuous function on* $[0,\infty)$. *Then*:

(1) $K_t(x,t)$ *is a continuous, bounded function on* $0 \leq x \leq t < 1$ *as well as* $K_t(x,\cdot) \in L^1(x,\infty)$.

(2) $K_t(x,t)$ *satisfies relation* (1.10).

Proof. (1) The function $K_t(x,t) = A(x,t)$ was obtained by $A(x,t) = \sum_{n=0}^\infty A_n(x,t)$ as a limit in $L^1(x,\infty)$. Because of (2.8) and the assumption that $Q(x)$ is continuous on $[0,\infty)$, it is shown by induction that $A_n(x,t)$ is a continuous function on $0 \leq x \leq t < \infty$.

We use notation (2.4) and take M_1 so large that

$$\frac{1}{2}\int_0^\infty |U(r)|\,dr + \frac{1}{2}\sup_{0\leq x<\infty} |Q(x)| \leq M_1, \quad \sigma(0) + 1 \leq M_1$$

and that (2.11) holds with M_1 instead of M_0. Then we obtain by induction the estimate

$$(3.1)\quad |A_n(x,t)| \le \frac{M_1^{n+1}}{n!}\,\sigma(x)^n, \quad 0 \le x \le t < \infty; \quad n = 0,1,2,\dots .$$

In fact, clearly $|A_0(x,t)| \le M_1$, and moreover, assuming that (3.1) holds for $n-1$, we have, for $0 \le x \le t < \infty$,

$$\begin{aligned}
|A_n(x,t)| &\le \int_x^\infty |U(s)|\,ds \int_s^\infty |A_{n-1}(s,u)|\,du \\
&\quad + 2\int_x^\infty |Q(s)|\frac{M_1^n}{(n-1)!}\sigma(s)^{n-1}\,ds \\
&\le \int_x^\infty |U(s)|\frac{M_1}{(n-1)!}\sigma(s)^n \\
&\quad + 2\int_x^\infty |Q(s)|\frac{M_1^n}{(n-1)!}\sigma(s)^{n-1}\,ds \\
&\le \{\sigma(0)M_1 + M_1^n\}\frac{1}{(n-1)!}\int_x^\infty \{|U(s)| + 2|Q(s)|\}\sigma(s)^{n-1}\,ds \\
&\le \{\sigma(0)M_1 + M_1^n\}\frac{1}{n!}\sigma(x)^n \\
&\le \{\sigma(0)+1\}M_1^n\frac{1}{n!}\sigma(x)^n \\
&\le \frac{M_1^{n+1}}{n!}\sigma(x)^n.
\end{aligned}$$

Estimate (3.1) shows that $\sum_{n=0}^\infty A_n(x,t)$ converges uniformly in $0 \le x \le t < \infty$. Hence, $A(x,t)$ is continuous in $0 \le x \le t < \infty$ and is majorized by $M_1 e^{M_1\sigma(0)}$ there.

(2) From (2.8) and assertion (1) of this lemma, we have

$$\begin{aligned}
(3.2)\quad K_t(x,x) - i\int_x^\infty Q(s)K_t(s,s)\,ds &= \frac{1}{2}\int_x^\infty U(s)e^{i\int_s^\infty Q(\eta)\,d\eta}\,ds \\
&\quad - \frac{i}{2}Q(x)e^{i\int_x^\infty Q(\eta)\,d\eta}.
\end{aligned}$$

It is easy to see that if $h(x) \in BC[0,\infty)$, then the integral equation

$$\phi(x) - i\int_x^\infty Q(s)\phi(s)\,ds = h(x), \quad 0 \le x < \infty$$

admits a unique solution in the space $BC[0,\infty)$, and the solution is given by

$$\phi(x) = h(x) + i\int_x^\infty Q(r)e^{-i\int_r^\infty Q(\eta)\,d\eta}h(r)\,dr\ e^{i\int_x^\infty Q(\eta)\,d\eta}.$$

Hence equation (3.2) is solved as

$$\begin{aligned} K_t(x,x) = {} & \frac{1}{2}\int_x^\infty U(s)e^{i\int_s^\infty Q(\eta)\,d\eta}\,ds - \frac{i}{2}Q(x)e^{i\int_x^\infty Q(\eta)\,d\eta} \\ & + \frac{i}{2}\int_x^\infty Q(r)e^{-i\int_r^\infty Q(\eta)\,d\eta}\,dr \\ & \times \int_r^\infty U(s)e^{i\int_s^\infty Q(\eta)\,d\eta}\,ds\ e^{i\int_x^\infty Q(\eta)\,d\eta} \\ & + \frac{1}{2}\int_x^\infty Q(r)^2\,dr\ e^{i\int_x^\infty Q(\eta)\,d\eta}. \end{aligned}$$

But, by integrating by parts, the second term of the righthand side can be rewritten as

$$-\frac{1}{2}\int_x^\infty U(s)e^{i\int_s^\infty Q(\eta)\,d\eta}\,ds + \frac{1}{2}\int_x^\infty U(r)\,dre^{i\int_x^\infty}Q(\eta)\,d\eta.$$

Therefore, we obtain

$$K_t(x,x) = \frac{1}{2}\bigg\{-iQ(x) + \int_x^\infty U(r)\,dr + \int_x^\infty Q(r)^2\,dr\bigg\}e^{i\int_x^\infty Q(\eta)\,d\eta}.$$

This yields (1.10). $\square$

4. Inversion formula: on the half-line. In this section we shall establish inversion formula (1.11) for equation (1.1) in the case where $I = [0,\infty)$. For each $k \in \mathbf{R}$, the complex conjugate function $\overline{f(x,k)}$ satisfies (2.1) for each $k \in \mathbf{R}$ as well as $f(x,k)$, because U and Q are real-valued. With the aid of asymptotic behaviors (2.2) and (2.3), their Wronskian is computed as

$$W[f(x,k),\overline{f(x,k)}] = -2ik, \quad k \in \mathbf{R}. \tag{4.1}$$

Here the Wronskian is defined as $W[f, g] := fg' - f'g$. Identity (4.1) leads to

$$f(0, k) \neq 0, \quad k \in \mathbf{R} \setminus \{0\}. \tag{4.2}$$

We assume that $f(0, 0) \neq 0$ and define the scattering data $S(k)$ by

$$S(k) := \frac{\overline{f(0, k)}}{f(0, k)}, \quad k \in \mathbf{R}. \tag{4.3}$$

Then, by (2.7), the Wiener-Lévy theorem (see, e.g., Paley and Wiener [**14**, page 63]) and the convolution theorem, it follows that $S(k)$ is represented as

$$S(k) = e^{-2i \int_0^\infty Q(\eta)\, d\eta} + \int_{-\infty}^{\infty} F(t) e^{-ikt}\, dt, \quad k \in \mathbf{R} \tag{4.4}$$

in terms of a function $F(t)$ in $L^1(\mathbf{R})$. Thus, we obtain expression (1.6). In view of the Riemann-Lebesgue lemma, the constant C is determined from $S(k)$ as the limit of $S(k)$ as $|k| \to \infty$. Hence, by means of the uniqueness theorem for the Fourier transform, $F(t)$ is uniquely determined from $S(k)$.

We now derive a Marchenko equation (1.7). It is an integral form of the Marchenko equation, see equation (4.10), that was derived by Jaulent and Jean [**8**] under a differentiability assumption on Q.

Lemma 4.1. *Let* $(1 + |x|)U(x), Q(x) \in L^1(0, \infty)$*, and assume that*

$$f(0, 0) \neq 0; \qquad f(0, k) \neq 0, \quad \operatorname{Im} k > 0. \tag{4.5}$$

Then the transformation kernel $K(x, t)$ *and a function* $F(t) \in L^1(\mathbf{R})$ *defined by* (4.4) *satisfy integral equation* (1.7)*, i.e.,*

$$\overline{K(x, t)} + \int_x^\infty K(x, r) F(r + t)\, dr + f(x, 0) \int_x^\infty F(r + t)\, dr = 0$$
$$x \leq t.$$

Proof. Let $\varphi(x, k)$ be a solution of (2.1) satisfying the initial condition

$$\varphi(0, k) = 0, \qquad \varphi'(0, k) = 1. \tag{4.6}$$

The function $\varphi(x,k)$ is represented as

$$2ik\varphi(x,k) = e^{-i\int_0^x Q(\eta)\,d\eta}e^{ikx} - e^{i\int_0^x Q(\eta)\,d\eta}e^{-ikx} - \int_{-x}^{x} H(x,t)e^{ikt}\,dt, \tag{4.7}$$

in terms of a function $H(x,t)$ belonging to $L^1(-x,x)$ as a function of t for each $x \geq 0$. We leave the proof of this fact to Appendix A.

Since, in view of (4.1), $f(x,k)$ and $\overline{f(x,k)}$ form a fundamental system of solutions to equation (2.1) for any $k \in \mathbf{R}\setminus\{0\}$, we obtain

$$\varphi(x,k) = -\frac{1}{2ik}\{f(0,k)\overline{f(x,k)} - \overline{f(0,k)}f(x,k)\}, \quad k \in \mathbf{R}\setminus\{0\},$$

and hence, we arrive at

$$-2ik\frac{\varphi(x,k)}{f(0,k)} = \overline{f(x,k)} - S(k)f(x,k), \quad k \in \mathbf{R}. \tag{4.8}$$

This remains valid for $k=0$ because of $S(0)=1$ and $f(x,0)=\overline{f(x,0)}$.

Insertion of (2.6) and (4.4) in (4.8) shows that

$$\begin{aligned}
&-2ik\frac{\varphi(x,k)}{f(0,k)} + e^{-i\int_0^\infty Q(\eta)\,d\eta}\{e^{-i\int_0^x Q(\eta)\,d\eta}e^{ikx} - e^{i\int_0^x Q(\eta)\,d\eta}e^{-ikx}\}\\
&= -e^{i\int_x^\infty Q(\eta)\,d\eta}e^{ikx}\int_{-\infty}^{\infty} F(t)e^{-ikt}\,dt - e^{-2i\int_0^\infty Q(\eta)\,d\eta}\int_x^\infty K_t(x,t)e^{ikt}\,dt\\
&\quad + \int_x^\infty \overline{K_t(x,t)}e^{-ikt}\,dt - \int_{-\infty}^{\infty} F(t)e^{-ikt}\,dt\int_x^\infty K_t(x,t)e^{ikt}\,dt.
\end{aligned}$$

Let us denote the lefthand side by $l(x,k,t)$:

$$\begin{aligned}
l(x,t,k) := &-2ik\frac{\varphi(x,k)}{f(0,k)}\\
&+ e^{-i\int_0^\infty Q(\eta)\,d\eta}\{e^{-i\int_0^x Q(\eta)\,d\eta}e^{ikx} - e^{i\int_0^x Q(\eta)\,d\eta}e^{-ikx}\},
\end{aligned}$$

and rewrite the righthand side. Then we obtain

$$\begin{aligned}
l(x,t,k) = &-e^{i\int_x^\infty Q(\eta)\,d\eta}\int_{-\infty}^{\infty} F(x+t)e^{-ikt}\,dt \\
&- e^{-2i\int_0^\infty Q(\eta)\,d\eta}\int_{-\infty}^{-x} K_t(x,-t)e^{-ikt}\,dt \\
&+ \int_x^\infty \overline{K_t(x,t)}e^{-ikt}\,dt \\
&- \int_{-\infty}^{\infty}\bigg(\int_x^\infty K_t(x,r)F(r+t)\,dr\bigg)e^{-ikt}\,dt.
\end{aligned}$$

On the other hand, by (4.7) and (2.7), $l(x,t,k)$ is rewritten as

$$\begin{aligned}
l(x,t,k) &= \frac{1}{f(0,k)}\bigg(\int_{-x}^{x} H(x,t)e^{ikt}\,dt + \int_0^\infty K_t(0,t)e^{ikt}\,dt \\
&\qquad \times e^{-i\int_0^\infty Q(\eta)\,d\eta}\{e^{-i\int_0^x Q(\eta)\,d\eta}e^{ikx} - e^{i\int_0^x Q(\eta)\,d\eta}e^{-ikx}\}\bigg) \\
&= \frac{\int_{-x}^\infty H_1(x,t)e^{ikt}\,dt}{e^i\int_0^\infty Q(\eta)\,d\eta + \int_0^\infty K_t(0,t)e^{ikt}\,dt},
\end{aligned}$$

where $H_1(x,t)$ is a function of t in $L^1(-x,\infty)$. By (4.2) and the assumption (4.5), it follows that $f(0,k)\neq 0$ for $\operatorname{Im} k \geq 0$. Therefore, by the Paley-Wiener theorem, see Paley and Wiener [**14**, Theorem 18], there exists a function $H_2(t)\in L^1(0,\infty)$ such that

$$\begin{aligned}
\frac{1}{f(0,k)} &= \frac{1}{e^i\int_0^\infty Q(\eta)\,d\eta} + \int_0^\infty K_t(0,t)e^{ikt}\,dt \\
&= e^{-i\int_0^\infty Q(\eta)\,d\eta} + \int_0^\infty H_2(t)e^{ikt}\,dt,
\end{aligned}$$

and hence, by the convolution theorem, the function $l(x,t,k)$ is rewritten as

$$\begin{aligned}
l(x,t,k) &= \int_{-x}^\infty H_1(x,t)e^{ikt}\,dt\bigg(e^{-i\int_0^\infty Q(\eta)\,d\eta} + \int_0^\infty H_2(t)e^{ikt}\,dt\bigg) \\
&= \int_{-x}^\infty H_3(x,t)e^{ikt}\,dt = \int_{-\infty}^{x} H_3(x,-t)e^{-ikt}\,dt.
\end{aligned}$$

Comparing this with (4.9) leads to

$$\overline{K_t(x,t)} - \int_x^\infty K_t(x,r)F(r+t)\,dr - e^{i\int_x^\infty Q(\eta)\,d\eta} F(x+t) = 0, x < t. \tag{4.10}$$

By integrating both sides of this equation with respect to t from t to ∞ and noting the assertion (1) in Lemma 2.1, we get

$$\overline{K(x,t)} + \int_x^\infty K_t(x,r)\,dr \int_{r+t}^\infty F(\eta)\,d\eta + e^{i\int_x^\infty Q(\eta)\,d\eta} \int_{x+t}^\infty F(\eta)\,d\eta = 0.$$

Hence, integrating by parts and using (1.5), we obtain (1.7). □

We now consider an integral equation

$$\overline{K(x,t)} + \int_x^\infty K(x,r)F(r+t)\,dr = J(x,t), \quad x \le t < \infty, \tag{4.11}$$

with a function $J(x,\cdot)$ in the space $BC[x,\infty)$. As is easily seen, a solution $K(x,\cdot)$ in $L^\infty(x,\infty)$ belongs also to $BC[0,\infty)$. Hence, by the Riesz-Schauder theory, for the solvability of (4.11), it suffices to show that the homogeneous adjoint equation

$$\overline{L(x,t)} + \int_x^\infty L(x,r)F(r+t)\,dr = 0, \quad x < t < \infty \tag{4.12}$$

has no nonzero solutions in the space $L^1(x,\infty)$, where we view this space as a real linear space. Provided that $F(t) \in L^1(\mathbf{R})$ is a function for which $S(k)$ in (1.6) with some constant C satisfies $|S(k)| = 1$ and $\operatorname{ind} S(k) = 0$ (assumption (4.5) leads to this circumstance), there are no nontrivial solutions to (4.12) in $L^1(x,\infty)$. We shall include a proof of this uniqueness in Appendix B for the convenience of the reader, though it is essentially due to Jaulent [**5**, page 370].

The discussion above is summarized as:

Lemma 4.2. *Under the same assumptions as in Lemma* 4.1*, given a function* $J(x,\cdot) \in BC[x,\infty)$*, equation* (4.11) *admits a unique solution* $K(x,\cdot)$ *in the space* $BC[x,\infty)$ *for each* $x \ge 0$.

This lemma assures that there exists a unique solution $\Delta(x,\cdot)$ of equation (1.8) in the space $BC[x,\infty)$. Since $f(x,0)$ is real-valued, which follows from the assumption that $U(x)$ is real-valued and the uniqueness of the Jost solutions, the function $K(x,t)=f(x,0)\Delta(x,t)$ satisfies (1.7). This, combined with the uniqueness of solutions to (4.11), shows that the solution of equation (1.7) is given by (1.9). In other words, the transformation kernel $K(x,t)$ of (2.1) is written as (1.9) in terms of the solution $\Delta(x,t)$ of a Marchenko equation (1.8). This observation, together with (1.5), yields

$$f(x,0)(1+\Delta(x,x))=e^{i\int_x^\infty Q(\eta)\,d\eta}. \tag{4.13}$$

Notice that, as a consequence of this relation, we get

$$f(x,0)\neq 0,\quad 1+\Delta(x,x)\neq 0,\quad 0\le x<\infty,$$

provided that $(1+|x|)U(x),Q(x)\in L^1(0,\infty)$ and (4.5).

Use of Lemma 3.1 now shows that the derivative $K_t(x,t)$ satisfies (1.10) provided that $Q(x)\in BC[0,\infty)$. Insertion of (1.9) in (1.10) leads to

$$2f(x,0)\Delta_t(x,x)e^{-i\int_x^\infty Q(\eta)\,d\eta}=\int_x^\infty [U(r)+Q(r)^2]\,dr-iQ(x),\qquad 0\le x<\infty.$$

Hence, by (4.13), we finally obtain (1.11) and Theorem 1.1.

We conclude this section with the following observation, which makes it clear how the original inverse scattering theory (where $Q(x)\equiv 0$) is included in that with energy dependent potentials.

Theorem 4.3. *Under the assumptions and notations as in Theorem* 1.1*, the following are equivalent*:

(i) $Q(x)\equiv 0$;

(ii) $S(-k)=\overline{S(k)}$, $k\in\mathbf{R}$;

(iii) $F(t)$ *is real-valued*;

(iv) $\Delta(x,t)$ *is real-valued.*

Proof. (i) $\Rightarrow$ (ii). Assumption (i) leads to the relation $f(x,k) = \overline{f(x,-k)}$ of the Jost function. From this relation and definition (4.3), we easily deduce (ii).

(ii) $\Rightarrow$ (iii). This follows from (1.6), the Riemann-Lebesgue lemma, and the uniqueness theorem for the Fourier transform.

(iii) $\Rightarrow$ (iv). Taking the complex conjugate of (1.8) shows that $\overline{\Delta(x,t)}$ satisfies (1.8). Because of the uniqueness of solutions to it, we get $\overline{\Delta(x,t)} = \Delta(x,t)$.

(iv) $\Rightarrow$ (i). This is immediate from formula (1.11). $\square$

5. Inversion formula: on the full line. In this section we shall establish inversion formula (1.11) for equation (1.1) in the case where $I = \mathbf{R}$, i.e.,

$$f'' + [k^2 - (U(x) + 2kQ(x))]f = 0, \quad -\infty < x < \infty. \tag{5.1}$$

Here $U(x), Q(x)$ are real-valued functions defined on $\mathbf{R}$. Let $(1 + |x|)U(x)$, $Q(x) \in L^1(\mathbf{R})$, and let $f^{\pm}(x,k)$ be solutions of (5.1) for each k in $\operatorname{Im} k \geq 0$ with the asymptotics

$$\begin{aligned} f^{\pm}(x,k) &= e^{\pm ikx}[1 + o(1)], \\ f^{\pm\prime}(x,k) &= \pm ike^{\pm ikx}[1 + o(1)], \quad x \to \pm\infty. \end{aligned} \tag{5.2}$$

These solutions are obtained by applying the method of successive approximations to the following integral equations:

$$f^{\pm}(x,k) = e^{\pm ikx} - \int_x^{\pm\infty} \frac{\sin k(x-t)}{k}[U(t) + 2kQ(t)]f^{\pm}(t,k)\,dt. \tag{5.3}$$

The solutions $f^{\pm}(x,k)$ are holomorphic in the upper half-plane $\operatorname{Im} k > 0$ and continuous in its closure $\operatorname{Im} k \geq 0$. In view of Lemma 2.1 there exist continuous, bounded functions $K^{\pm}(x,t)$ by which $f^{\pm}(x,k)$ are represented as

$$f^{\pm}(x,k) = e^{\pm i\int_x^{+\infty} Q(\eta)\,d\eta} e^{\pm ikx} + \int_x^{\pm\infty} K_t^{\pm}(x,t)e^{\pm ikt}\,dt \tag{5.4}$$

$$\operatorname{Im} k \geq 0,$$

where the almost everywhere existing derivatives $K_t^{\pm}(x,\cdot)$ belong to $L^1(x,\pm\infty)$.

The Jost solutions of (5.1) analytic in the lower half-plane can be defined as

$$g^{\pm}(x,k) = \overline{f^{\pm}(x,\bar{k})}. \tag{5.5}$$

These solutions have the asymptotics

$$\begin{aligned} g^{\pm}(x,k) &= e^{\mp ikx}[1+o(1)], \\ g^{\pm\prime}(x,k) &= \mp ike^{\mp ikx}[1+o(1)], \quad x \to \pm\infty. \end{aligned} \tag{5.6}$$

By (5.2) and (5.6), the Wronskian of each pair of the solutions $f^{\pm}(x,k)$, $g^{\pm}(x,k)$ is computed as

$$W[f^{\pm}(x,k), g^{\pm}(x,k)] = \mp 2ik, \quad k \in \mathbf{R}.$$

Hence, for real $k \neq 0$,

$$f^{-}(x,k) = a(k)g^{+}(x,k) + b(k)f^{+}(x,k), \tag{5.7}$$

where

$$\begin{aligned} a(k) &:= \frac{W[f^{-}(x,k), f^{+}(x,k)]}{2ik}, \\ b(k) &:= \frac{W[f^{-}(x,k), g^{+}(x,k)]}{-2ik}. \end{aligned} \tag{5.8}$$

It should be noted that $a(k)$ is a boundary value on the real line of a holomorphic function in the upper half-plane.

By (5.7) and (5.5), we have, for real $k \neq 0$,

$$g^{-}(x,k) = \overline{a(k)}f^{+}(x,k) + \overline{b(k)}g^{+}(x,k). \tag{5.9}$$

Insertion of this equality and (5.7) in $W[f^{-}(x,k), g^{-}(x,k)] = 2ik$ shows that

$$|a(k)|^2 = 1 + |b(k)|^2. \tag{5.10}$$

In a similar fashion for (5.7) one can show that

$$f^{+}(x,k) = c(k)f^{-}(x,k) + d(k)g^{-}(x,k),$$

where

$$c(k) = \frac{W[f^{+}(x,k), g^{-}(x,k)]}{2ik}, \qquad d(k) = \frac{W[f^{-}(x,k), f^{+}(x,k)]}{2ik}. \tag{5.11}$$

This, together with (5.8) and (5.5), shows that

$$d(k) = a(k), \quad c(k) = -\overline{b(k)}. \tag{5.12}$$

Following the usual manner (see Marchenko [**13**, Section 3.5], Chadan and Sabatier [**3**, Chapter 17]) we now introduce the functions:

$$s_{11}(k) = \frac{1}{a(k)}, \qquad s_{12}(k) = \frac{b(k)}{a(k)}, \qquad s_{21}(k) = \frac{c(k)}{d(k)}, \qquad s_{22}(k) = \frac{1}{d(k)}. \tag{5.13}$$

Then, by (5.7), (5.6) and (5.2), we get

$$\psi^{\leftarrow}(x,k) := \frac{f^{-}(x,k)}{a(k)} \sim \begin{cases} e^{-ikx} + s_{12}(k)e^{ikx} & x \to \infty, \\ s_{11}(k)e^{-ikx} & x \to -\infty. \end{cases}$$

The function $\psi^{\leftarrow}(x,k)$ is referred to as the scattering solution to the left; $s_{12}(k)e^{ikx}$ represents the reflected wave of the probability amplitude $s_{12}(k)$ of the unit wave e^{-ikx} incoming from the right. Similarly, by (4.11), (4.7) and (4.3), we get the scattering solution to the right:

$$\psi^{\rightarrow}(x,k) := \frac{g^{+}(x,k)}{d(k)} \sim \begin{cases} e^{ikx} + s_{21}(k)e^{-ikx} & x \to -\infty, \\ s_{22}(k)e^{ikx} & x \to \infty. \end{cases}$$

The coefficients $s_{12}(k)$ and $s_{21}(k)$ are called the reflection coefficients; while $s_{11}(k)$ $(= s_{22}(k))$ is called the transmission coefficient. Relations (5.10) and (5.12) are written as

$$\begin{gathered} s_{11}(k) = s_{22}(k), \\ |s_{11}(k)|^2 + |s_{12}(k)|^2 = 1, \\ s_{11}(k)\overline{s_{21}(k)} + s_{12}(k)\overline{s_{22}(k)} = 0 \end{gathered} \tag{5.14}$$

in terms of the reflection coefficients and the transmission coefficient. This shows that the S-matrix

$$\begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}$$

is unitary.

The inverse problem we shall discuss in this section is to recover $U(x)$ and $Q(x)$ in (5.1) from the reflection coefficient $s_{12}(k)$. The first step in our approach to this problem is to establish integral expressions of the functions $a(k), b(k)$ defined in (5.8):

Lemma 5.1. *Let* $(1+|x|)U(x), Q(x) \in L^1(\mathbf{R})$. *Then*:

(1) $a(k)$ *is expressed as*

$$a(k) = \frac{1-ik}{2ik}\left(-2e^{i\int_{-\infty}^{\infty} Q(\eta)\,d\eta} + \int_{-\infty}^{0} G(t)e^{-ikt}\,dt \right) \tag{5.15}$$

in terms of a function $G(t) \in L^1(-\infty, 0)$.

(2) $b(k)$ *is expressed as*

$$b(k) = \frac{1-ik}{2ik}\int_{-\infty}^{\infty} H(t)e^{-ikt}\,dt \tag{5.16}$$

in terms of a function $H(t) \in L^1(\mathbf{R})$.

Proof. By equation (5.3) it follows that, for real $k \neq 0$,

$$\begin{aligned} f^-(x,k) = e^{-ikx} &- \frac{e^{-ikx}}{2ik}\int_{-\infty}^{\infty} [U(t) + 2kQ(t)]f^-(t,k)e^{ikt}\,dt \\ &+ \frac{e^{ikx}}{2ik}\int_{-\infty}^{\infty} [U(t)+2kQ(t)]f^-(t,k)e^{-ikt}\,dt + o(1), \\ &x \to \infty. \end{aligned}$$

On the other hand, by (5.9) and (5.5), we have

$$f^-(x,k) = a(k)e^{-ikx} + b(k)e^{ikx} + o(1), \quad x \to \infty.$$

A comparison of corresponding terms shows that

$$a(k)=1-\frac{1}{2ik}\int_{-\infty}^{\infty}[U(x)+2kQ(x)]f^{-}(x,k)e^{ikx}dx, \tag{5.17}$$

$$b(k)=\frac{1}{2ik}\int_{-\infty}^{\infty}[U(x)+2kQ(x)]f^{-}(x,k)e^{-ikx}dx, \tag{5.18}$$

for real $k \neq 0$.

(1) Insertion in (5.17) of the expression

$$f^{-}(x,k)e^{ikx} = e^{i\int_{-\infty}^{x} Q(\eta)\,d\eta} - \int_{-\infty}^{0} K_t^{-}(x,t+x)e^{-ikt}\,dt,$$

see (5.4), leads to

$$\begin{aligned} a(k) = 1 + i\int_{-\infty}^{\infty} Q(x)e^{i\int_{-\infty}^{x} Q(\eta)\,d\eta}\,dx \\ - \frac{1}{2ik}\int_{-\infty}^{\infty} U(x)e^{i\int_{-\infty}^{x} Q(\eta)\,d\eta}\,dx \\ + \frac{1}{2ik}\int_{-\infty}^{0} e^{-ikt}\,dt \times \int_{-\infty}^{\infty} U(x)K_t^{-}(x,t+x)\,dx \\ + \frac{1}{i}\int_{-\infty}^{0} e^{-ikt}\,dt \times \int_{-\infty}^{\infty} Q(x)K_t^{-}(x,t+x)\,dx \end{aligned}$$

Hence, observing that

$$1 + i\int_{-\infty}^{\infty} Q(x)e^{i\int_{-\infty}^{x} Q(\eta)\,d\eta}\,dx = e^{i\int_{-\infty}^{\infty} Q(\eta)\,d\eta},$$

and setting

$$\begin{aligned} \alpha &:= -2e^{i\int_{-\infty}^{\infty} Q(\eta)\,d\eta}, \\ \beta &:= -\int_{-\infty}^{\infty} U(x)e^{i\int_{-\infty}^{x} Q(\eta)\,d\eta}\,dx, \\ G_V(t) &:= \int_{-\infty}^{\infty} V(x)K_t^{-}(x,t+x)\,dx, \\ V &= U, Q, \end{aligned}$$

we obtain

$$a(k) = -\frac{\alpha}{2} + \frac{\beta}{2ik} + \frac{1}{2ik}\int_{-\infty}^{0} G_U(t)e^{-ikt}\,dt + \frac{1}{i}\int_{-\infty}^{0} G_Q(t)e^{-ikt}\,dt. \tag{5.19}$$

The function G_U belongs to $L^1(-\infty,0)$, because

$$\int_{-\infty}^{0} dt \int_{-\infty}^{\infty} |U(x)K_t^-(x,t+x)|\,dx = \int_{-\infty}^{\infty} |U(x)|\,dx \int_{-\infty}^{x} |K_t^-(x,t)|\,dt < \infty,$$

by estimate (2.5) for $K_t^-(x,t)$. Similarly, $G_Q \in L^1(0,\infty)$.

By (5.19) we have

$$\begin{aligned} \frac{2ik}{1-ik}a(k) &= \alpha + \frac{\beta-\alpha}{1-ik}\int_{-\infty}^{0} G_U(t)e^{-ikt}\,dt \\ &\quad + \frac{2k}{1-ik}\int_{-\infty}^{0} G_Q(t)e^{-ikt}\,dt \\ &= \alpha + \int_{-\infty}^{0}\{(\beta-\alpha)e^t + e^t * G_U \\ &\quad + 2iG_Q - 2ie^t * G_Q\}e^{-ikt}\,dt. \end{aligned}$$

Setting

$$G(t) = (\beta-\alpha)e^t + e^t * G_U + 2iG_Q - 2ie^t * G_Q$$

completes the proof of (1).

(2) Insertion of (5.4) in (5.18) leads to

$$\begin{aligned} 2ikb(k) &= \int_{-\infty}^{\infty} U(x)\Big(e^{i\int_{-\infty}^{x} Q(\eta)\,d\eta}e^{-ikx} \\ &\quad - \int_{-\infty}^{x} K_t^-(x,t)e^{-ikt}\,dt\Big)e^{-ikx}\,dx \\ &\quad + 2k\int_{-\infty}^{\infty} Q(x)\Big(e^{i\int_{-\infty}^{x} Q(\eta)\,d\eta}e^{-ikx} \\ &\quad - \int_{-\infty}^{x} K_t^-(x,t)e^{-ikt}\,dt\Big)e^{-ikx}\,dx. \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} U(x)e^{-ikx}\,dx \int_{-\infty}^{x} K_t^-(x,t)e^{-ikt}\,dt = \int_{-\infty}^{\infty} e^{-ikt}\,dt \int_{t/2}^{\infty} U(x)K_t^-(x,t-x)\,dx,$$

we get

$$2ikb(k) = \int_{-\infty}^{\infty} H_U(t)e^{-ikt}\,dt + 2k\int_{-\infty}^{\infty} H_Q(t)e^{-ikt}\,dt, \tag{5.20}$$

where

$$H_V(t) := \frac{1}{2}V\left(\frac{t}{2}\right)e^{i\int_{-\infty}^{t/2} Q(\eta)\,d\eta} - \int_{t/2}^{\infty} V(x)K_t^-(x,t-x)\,dx, \quad V = U, Q.$$

The function H_U belongs to $L^1(\mathbf{R})$ because

$$\int_{-\infty}^{\infty} dt \int_{t/2}^{\infty} |U(x)K_t^-(x,t-x)|\,dx = \int_{-\infty}^{\infty} |U(x)|\,dx \int_{-\infty}^{x} |K_t^-(x,t)|\,dt < \infty.$$

Similarly, $H_Q \in L^1(\mathbf{R})$.

Equation (5.20) yields

$$\begin{aligned}\frac{2ik}{1-ik}b(k) &= \frac{1}{1-ik}\int_{-\infty}^{\infty} H_U(t)e^{-ikt}\,dt \\ &\quad + \frac{2k}{1-ik}\int_{-\infty}^{\infty} H_Q(t)e^{-ikt}\,dt \\ &= \int_{-\infty}^{0} e^t e^{-ikt}\,dt \int_{-\infty}^{\infty} H_U(t)e^{-ikt}\,dt \\ &\quad + 2i\left(1 - \int_{-\infty}^{0} e^t e^{-ikt}\,dt\right)\int_{-\infty}^{\infty} H_Q(t)e^{-ikt}\,dt \\ &= \int_{-\infty}^{\infty} \{e^t * H_U + 2iH_Q - 2ie^t * H_Q\}e^{-ikt}\,dt,\end{aligned}$$

where $e^t * H_U$ denotes the convolution $\int_{-\infty}^0 H_U(t-s)e^s\,ds$. Setting

$$H(t) = e^t * H_U + 2iH_Q - 2ie^t * H_Q$$

leads us to (2). $\square$

By (5.8) and (5.10), the Wronskian $W[f^-(x,k), f^+(x,k)]$ does not vanish at real $k \neq 0$. We assume that it does not vanish either at $k = 0$. Under this assumption, we have the following result, which asserts that the reflection coefficient $s_{12}(k)$ is the Fourier image of a function in $L^1(\mathbf{R})$.

Lemma 5.2. *Let $(1+|x|)U(x), Q(x) \in L^1(\mathbf{R})$, and assume that $W[f^-(x,k), f^+(x,k)]$ does not vanish at $k = 0$. Then the reflection coefficient $s_{12}(k)$ is expressed as*

$$s_{12}(k) = -\int_{-\infty}^{\infty} R(t)e^{-ikt}\,dt \tag{5.21}$$

in terms of a function $R \in L^1(\mathbf{R})$.

Proof. By definition (5.13) and Lemma 5.1, we have

$$s_{12}(k) = \frac{b(k)}{a(k)} = \frac{-\int_{-\infty}^{\infty} H(t)e^{-ikt}\,dt}{-2e^{i\int_{-\infty}^{\infty} Q(\eta)\,d\eta} + \int_{-\infty}^0 G(t)e^{-ikt}\,dt}.$$

By assumption, the denominator on the righthand side does not vanish on $\mathbf{R}$. Therefore, by the Wiener-Lévy theorem, there exists a function $G_1 \in L^1(\mathbf{R})$ such that

$$\begin{aligned}\frac{1}{-2e^{i\int_{-\infty}^{\infty} Q(\eta)\,d\eta} + \int_{-\infty}^0 G(t)e^{-ikt}\,dt}\\ = -\frac{1}{2}e^{-i\int_{-\infty}^{\infty} Q(\eta)\,d\eta} + \int_{-\infty}^{\infty} G_1(t)e^{-ikt}\,dt.\end{aligned} \tag{5.22}$$

This, with the aid of the convolution theorem, yields (5.21). $\square$

We now assume that

$$W[f^-(x,k), f^+(x,k)] \neq 0, \quad \operatorname{Im} k > 0, \ k = 0. \tag{5.23}$$

This means that there are no bound states in equation (5.1). A Marchenko equation of the same form as (1.7) can be derived under this assumption.

Lemma 5.3. *Let* $(1+|x|)U(x)$, $Q(x) \in L^1(\mathbf{R})$, *and assume* (5.23). *Then* $K^+(x,t)$ *and the function* $R(t) \in L^1(\mathbf{R})$ *defined by* (5.21) *satisfy the integral equation*

$$\begin{aligned}\overline{K^+(x,t)} + \int_x^\infty K^+(x,r)R(r+t)\,dr \\ + f^+(x,0)\int_x^\infty R(r+t)\,dr = 0, \quad x \le t.\end{aligned} \tag{5.24}$$

Proof. Relation (5.7) leads to

$$\begin{aligned}\frac{f^-(x,k)}{a(k)} - e^{-i\int_x^\infty Q(\eta)\,d\eta} e^{-ikx} \\ = g^+(x,k) - e^{-i\int_x^\infty Q(\eta)\,d\eta} e^{-ikx} + s_{12}(k) f^+(x,k)\end{aligned} \tag{5.25}$$

for $k \in \mathbf{R}$. It follows from (5.4) and (5.5) that

$$g^+(x,k) - e^{-i\int_x^\infty Q(\eta)\,d\eta} e^{-ikx} = \int_x^\infty \overline{K_t^+(x,t)} e^{-ikt}\,dt.$$

Hence, by use of (5.4) and (5.21), the righthand side of (5.25) is computed as

$$\begin{aligned} g^+(x,k) &- e^{-i\int_x^\infty Q(\eta)\,d\eta} e^{-ikx} + s_{12}(k) f^+(x,k) \\ &= \int_x^\infty \overline{K_t^+(x,t)} e^{-ikt}\,dt \\ &\quad - \int_{-\infty}^\infty R(t) e^{-ikt}\,dt \left(e^{i\int_x^\infty Q(\eta)\,d\eta} e^{ikx} + \int_x^\infty K_t^+(x,t) e^{ikt}\,dt \right) \\ &= \int_x^\infty \overline{K_t^+(x,t)} e^{-ikt}\,dt - e^{i\int_{-\infty}^\infty Q(\eta)\,d\eta} \int_{-\infty}^\infty R(x+t) e^{-ikt}\,dt \\ &\quad - \int_{-\infty}^\infty \left(\int_x^\infty K_t^+(x,r) R(t+r)\,dr \right) e^{-ikt}\,dt. \end{aligned}$$

This shows that, for each $x \in \mathbf{R}$, the right side of (5.25) is the Fourier transform of the function

$$\overline{K_t^+(x,t)} - \int_x^\infty K_t^+(x,r)R(r+t)\,dr - e^{i\int_x^\infty Q(\eta)\,d\eta}R(x+t), \tag{5.26}$$

which belongs to $L^1(\mathbf{R})$ as a function of t.

We next compute the lefthand side of (5.25). Assumption (5.23) enables us to apply the Paley-Wiener theorem to (5.22) and to find that the function $G_1(t)$ in (5.22) must vanish for $t > 0$. In addition, by (5.4), we have

$$f^-(x,k) = \left(e^{i\int_{-\infty}^x Q(\eta)\,d\eta} - \int_{-\infty}^0 K_t^-(x,x+t)e^{-ikt}\,dt\right)e^{-ikx}.$$

Hence, with the aid of (5.15), the first term of (5.25) is computed as

$$\begin{aligned}\frac{f^-(x,k)}{a(k)} &= \frac{2ik}{1-ik}\left(\alpha^{-1} + \int_{-\infty}^0 G_1(t)e^{-ikt}\,dt\right)\\ &\quad\times\left(e^{i\int_{-\infty}^x Q(\eta)\,d\eta} - \int_{-\infty}^0 K_t^-(x,x+t)e^{-ikt}\,dt\right)e^{-ikx}\\ &= \left(-2 + 2\int_{-\infty}^0 e^t e^{-ikt}\,dt\right)\left(\alpha^{-1} + \int_{-\infty}^0 G_1(t)e^{-ikt}\,dt\right)\\ &\quad\times\left(e^{i\int_{-\infty}^x Q(\eta)\,d\eta} - \int_{-\infty}^0 K_t^-(x,x+t)e^{-ikt}\,dt\right)e^{-ikx}\\ &= \left(e^{-i\int_x^\infty Q(\eta)\,d\eta} + \int_{-\infty}^0 \Omega(x,t)e^{-ikt}\,dt\right)e^{-ikx},\end{aligned}$$

where $\Omega(x,\cdot)$ is a function in $L^1(-\infty,0)$ for each $x \in \mathbf{R}$. Thus we have

$$\frac{f^-(x,k)}{a(k)} - e^{-i\int_x^\infty Q(\eta)\,d\eta}e^{-ikx} = \int_{-\infty}^x \Omega(x,t-x)e^{-ikt}\,dt,$$

where $\Omega(x,\cdot - x) \in L^1(-\infty,x)$. This, combined with the fact that the righthand side of (5.25) is the Fourier transform of (5.26), shows that

$$\begin{aligned}\overline{K_t^+(x,t)} - \int_x^\infty K_t^+(x,r)R(r+t)\,dr&\\ - e^{i\int_x^\infty Q(\eta)\,d\eta}R(x+t) = 0,&\quad x < t.\end{aligned} \tag{5.27}$$

Integrating both sides of this equation, we get

$$\overline{K^+(x,t)} + \int_x^\infty K_t^+(x,r)\,dr \int_{r+t}^\infty R(s)\,ds + e^{i\int_x^\infty Q(\eta)\,d\eta} \int_{x+t}^\infty R(s)\,ds = 0, \quad x \le t.$$

Performing an integration by parts and observing that

$$-K^+(x,x) + e^{i\int_x^\infty Q(\eta)\,d\eta} = f^+(x,0),$$

we obtain (5.24). □

Our next task is to prove the solvability of a Marchenko equation (5.24) in the space $BC[x,\infty)$. This is reduced to showing that the corresponding homogeneous equation

$$\overline{L(x,t)} + \int_x^\infty L(x,r)R(r+t)\,dr = 0, \quad x < t < \infty, \tag{5.28}$$

has only the trivial solution in the space $L^1(x,\infty)$, viewed as a real linear space. As is verified in an analogous manner to that in Marchenko [**13**, page 221], a solution $L(x,\cdot)$ of (5.28) in $L^1(x,\infty)$ belongs also to $L^2(x,\infty)$. Therefore, the absence of nontrivial solutions to (5.28) follows from a lemma in Jaulent and Jean [**9**, page 124]. We have shown:

Lemma 5.4. *Under the same assumption as in Lemma* 5.3*, equation* (5.24) *has a unique solution* $K^+(x,\cdot)$ *in the space* $BC[x,\infty)$*. The solution is given by*

$$K^+(x,t) = f^+(x,0)\Delta^+(x,t) \tag{5.29}$$

in terms of the solution $\Delta^+(x,t)$ *of*

$$\overline{\Delta^+(x,t)} + \int_x^\infty \Delta^+(x,r)R(r+t)\,dr + \int_x^\infty R(r+t)\,dr = 0, \quad x \le t. \tag{5.30}$$

If we suppose that $Q(x) \in BC(\mathbf{R})$, in addition to $(1+|x|)U(x), Q(x) \in L^1(\mathbf{R})$, then, by Lemma 3.1, we get

$$2K_t^+(x,x)e^{-i\int_x^\infty Q(\eta)\,d\eta} = \int_x^\infty [U(r)+Q(r)^2]\,dr - iQ(x),\ x \in \mathbf{R}.$$

Since the discussion in Section 4 for (1.11) now applies verbatim, we can derive the formula

$$\frac{2\Delta_t^+(x,x)}{1+\Delta^+(x,x)} = \int_x^\infty [U(r)+Q(r)^2]\,dr - iQ(x),\ x \in \mathbf{R}, \tag{5.31}$$

and draw the following conclusion.

Theorem 5.5. *If there exists a pair $(U(x), Q(x))$ of real-valued functions with* (1.2), (1.3) (*where* $I = \mathbf{R}$) *and* (5.23) *which has a given function $s_{12}(k)$ on $\mathbf{R}$ as its reflection coefficient, then (U, Q) is recovered from $s_{12}(k)$ by* (5.31), *where $\Delta^+(x,t)$ is the solution of integral equation* (5.30) *with R defined by* (5.21) *from $s_{12}(k)$.*

APPENDIX

A. We shall establish the representation (4.7). Let $\varphi(x,k)$ be a solution of (2.1) satisfying condition (4.6).

Lemma A.1. *Suppose that $U(x), Q(x)$ are locally integrable functions in the interval $[0,\infty)$. Then, for any $k \in \mathbf{C}$, the solution $\varphi(x,k)$ can be represented as*

$$\varphi(x,k) = \psi_0(x)\frac{\sin kx}{k} + \frac{1}{2}\int_{-x}^{x} \Phi(x,t)e^{ikt}\,dt, \tag{A.1}$$

where $\psi_0(x)$ is a solution of $\psi_0''(x) = U(x)\psi_0(x)$ satisfying $\psi_0(0) = 1$, $\psi_0'(0) = 0$ and the integral kernel $\Phi(x,t)$ is a continuous function defined on $-x \le t \le x$. The kernel possesses the following properties:

(1) $\Phi(x,t)$ *has the following boundary values*:

$$\begin{aligned} \Phi(x,x) &= -\psi_0(x) + e^{-i\int_0^x Q(\eta)\,d\eta}, \\ \Phi(x,-x) &= -\psi_0(x) + e^{i\int_0^x Q(\eta)\,d\eta}. \end{aligned} \tag{A.2}$$

(2) $\Phi(x,t)$ *has the partial derivative* $\Phi_t(x,t)$, *which belongs to* $L^1(-x,x)$ *as a function of* t *for each* $x \geq 0$.

Proof. A function $\varphi(x,k)$ is the solution of (2.1) satisfying (4.6) if and only if $\varphi(x,k)$ satisfies the integral equation

$$\varphi(x,k) = \frac{\sin kx}{k} + \int_0^x \frac{\sin k(x-s)}{k}[U(s)+2kQ(s)]\varphi(s,k)\,ds. \tag{A.3}$$

Substituting in (A.3) the righthand member of (A.1) instead of $\varphi(x,k)$, we get

$$\begin{aligned}
\psi_0(x)\frac{\sin kx}{k} &+ \frac{1}{2}\int_{-x}^{x} \Phi(x,t)e^{ikx}\,dt \\
&= \frac{\sin kx}{k} + \frac{1}{k}\int_0^x \frac{\sin k(x-s)\sin ks}{k}[U(s)+2kQ(s)]\psi_0(s)\,ds \\
&\quad + \frac{1}{2k}\int_0^x \sin k(x-s)U(s)\,ds\int_{-s}^{s}\Phi(s,t)e^{ikt}\,dt \\
&\quad + \int_0^x \sin k(x-s)Q(s)\,ds\int_{-s}^{s}\Phi(s,t)e^{ikt}\,dt.
\end{aligned}$$

Let J_2, J_3, J_4 denote the second, third and fourth terms on the right-hand side of (A.4). Observing that

$$\frac{\sin k(x-s)\sin ks}{k} = -\frac{i}{4}\left\{\int_{2s-x}^{x} e^{ikt}\,dt - \int_{-x}^{x-2s} e^{ikt}\,dt\right\},$$

and noticing the formula for reversing the order of integration

$$\int_0^x ds\int_{2s-x}^{x} dt - \int_0^x ds\int_{-x}^{x-2s} dt = \int_{-x}^{x} dt\int_{(x-t)/2}^{(x+t)/2} ds,$$

we obtain

$$\begin{aligned}
J_2 &= -\frac{i}{4k}\int_{-x}^{x} e^{ikt}\,dt\int_{(x-t)/2}^{(x+t)/2}[U(s)+2kQ(s)]\psi_0(s)\,ds \\
&= -\frac{i}{4k}\int_{-x}^{x} e^{ikt}\,dt\int_{(x-t)/2}^{(x+t)/2} U(s)\psi_0(s)\,ds \\
&\quad - \frac{i}{2}\int_{-x}^{x} e^{ikt}\,dt\int_{(x-t)/2}^{(x+t)/2} Q(s)\psi_0(s)\,ds.
\end{aligned}$$

But, by an integration by parts, it follows that

$$
\begin{aligned}
&\frac{i}{4k}\int_{-x}^{x} e^{ikt}\,dt\int_{0}^{(x-t)/2} U(s)\psi_0(s)\,ds\\
&\qquad= \frac{i}{4k}e^{-ikx}\int_{-x}^{x} d\eta\int_{0}^{(x-\eta)/2} U(s)\psi_0(s)\,ds\\
&\qquad\quad -\frac{1}{4}\int_{-x}^{x} e^{ikt}\,dt\int_{t}^{x} d\eta\int_{0}^{(x-\eta)/2} U(s)\psi_0(s)\,ds\\
&\qquad= \frac{i}{2k}e^{-ikx}\int_{0}^{x}(x-s)U(s)\psi_0(s)\,ds\\
&\qquad\quad -\frac{1}{2}\int_{-x}^{x} e^{ikt}\,dt\int_{0}^{(x-t)/2}\left(\frac{x-t}{2}-s\right)U(s)\psi_0(s)\,ds,
\end{aligned}
$$

and therefore that

$$
\begin{aligned}
&-\frac{i}{4k}\int_{-x}^{x} e^{ikt}\,dt\int_{(x-t)/2}^{(x+t)/2} U(s)\psi_0(s)\,ds\\
&\qquad= \frac{i}{4k}\int_{-x}^{x} e^{ikt}\,dt\int_{0}^{(x-t)/2} U(s)\psi_0(s)\,ds\\
&\qquad\quad -\frac{i}{4k}\int_{-x}^{x} e^{-ikt}\,dt\int_{0}^{(x-t)/2} U(s)\psi_0(s)\,ds\\
&\qquad= \frac{i}{2k}e^{-ikx}\int_{0}^{x}(x-s)U(s)\psi_0(s)\,ds\\
&\qquad\quad -\frac{i}{2k}e^{ikx}\int_{0}^{x}(x-s)U(s)\psi_0(s)\,ds\\
&\qquad\quad -\frac{1}{2}\int_{-x}^{x} e^{ikt}\,dt\int_{0}^{(x-t)/2}\left(\frac{x-t}{2}-s\right)U(s)\psi_0(s)\,ds\\
&\qquad\quad -\frac{1}{2}\int_{-x}^{x} e^{-ikt}\,dt\int_{0}^{(x-t)/2}\left(\frac{x-t}{2}-s\right)U(s)\psi_0(s)\,ds\\
&\qquad= \frac{\sin kx}{k}\int_{0}^{x}(x-s)U(s)\psi_0(s)\,ds\\
&\qquad\quad -\frac{1}{2}\int_{-x}^{x} e^{ikt}\,dt\int_{0}^{(x-t)/2}\left(\frac{x-t}{2}-s\right)U(s)\psi_0(s)\,ds\\
&\qquad\quad -\frac{1}{2}\int_{-x}^{x} e^{ikt}\,dt\int_{0}^{(x+t)/2}\left(\frac{x+t}{2}-s\right)U(s)\psi_0(s)\,ds.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
J_2 = {} & \frac{\sin kx}{k} \int_0^x (x-s) U(s) \psi_0(s)\, ds \\
& - \frac{1}{2} \int_{-x}^{x} e^{ikt}\, dt \int_0^{(x-t)/2} \left(\frac{x-t}{2} - s \right) U(s) \psi_0(s)\, ds \\
& - \frac{1}{2} \int_{-x}^{x} e^{ikt}\, dt \int_0^{(x+t)/2} \left(\frac{x+t}{2} - s \right) U(s) \psi_0(s)\, ds \\
& - \frac{i}{2} \int_{-x}^{x} e^{ikt}\, dt \int_{(x-t)/2}^{(x+t)/2} Q(s) \psi_0(s)\, ds.
\end{aligned} \tag{A.5}
$$

On the other hand,

$$
J_3 = \frac{1}{2k} \int_0^x \sin k(x-s) U(s)\, ds \int_{-s}^{s} \Phi(s,\tau) e^{ik\tau}\, d\tau
$$

is rewritten as

$$
\begin{aligned}
J_3 &= \frac{1}{4} \int_0^x U(s)\, ds \int_{-s}^{s} \Phi(s,\tau)\, d\tau \int_{\tau+s-x}^{\tau+x-s} e^{ikt}\, dt \\
&= \frac{1}{4} \int_0^x U(s)\, ds \int_{-x}^{x} e^{ikt}\, dt \int_{\max(-s,t+s-x)}^{\min(s,t+x-s)} \Phi(s,\tau)\, d\tau \\
&= \frac{1}{4} \int_{-x}^{x} e^{ikt}\, dt \int_0^x U(s)\, dsZ \int_{\max(-s,t+s-x)}^{\min(s,t+x-s)} \Phi(s,\tau)\, d\tau.
\end{aligned}
$$

This leads to

$$
J_3 = \frac{1}{8} \int_{-x}^{x} e^{ikt}\, dt \int_0^{x+t} d\xi \int_0^{x-t} d\eta\, U\!\left(\frac{\xi+\eta}{2} \right) \Phi\!\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \right). \tag{A.6}
$$

Moreover,

$$
\begin{aligned}
J_4 &= -\frac{i}{2} \int_0^x Q(s)\, ds \int_{-s}^{s} \Phi(s,t) e^{ik(t+x-s)}\, dt \\
&\quad + \frac{i}{2} \int_0^x Q(s)\, ds \int_{-s}^{s} \Phi(s,t) e^{ik(t+s-x)}\, dt \\
&= -\frac{i}{2} \int_0^x Q(s)\, ds \int_{x-2s}^{x} \Phi(s, t+s-x) e^{ikt}\, dt \\
&\quad + \frac{i}{2} \int_0^x Q(s)\, ds \int_{-x}^{2s-x} \Phi(s, t+x-s) e^{ikt}\, dt.
\end{aligned}
$$

Hence, by changing the order of integration, we have

$$\text{(A.7)}\quad J_4=\frac{1}{2}\int_{-x}^{x}e^{ikt}\,dt\Big\{-i\int_{(x-t)/2}^{x}Q(s)\Phi(s,t+s-x)\,ds \\ +i\int_{(x+t)/2}^{x}Q(s)\Phi(s,t+x-s)\,ds\Big\}.$$

By (A.5), (A.6) and (A.7), equation (A.4) can be rewritten as

$$\begin{aligned}&\psi_0(x)\frac{\sin kx}{k}+\frac{1}{2}\int_{-x}^{x}\Phi(x,t)e^{ikt}\,dt\\ &\qquad=\left(1+\int_0^x(x-s)U(s)\psi_0(s)\,ds\right)\frac{\sin kx}{k}+\frac{1}{2}\int_{-x}^{x}e^{ikt}\,dt\\ &\times\Big\{-\int_0^{(x-t)/2}\left(\frac{x-t}{2}-s\right)U(s)\psi_0(s)\,ds\\ &-\int_0^{(x+t)/2}\left(\frac{x+t}{2}-s\right)U(s)\psi_0(s)\,ds-i\int_{(x-t)/2}^{(x+t)/2}Q(s)\psi_0(s)\,ds\\ &+\frac{1}{4}\int_0^{x+t}d\xi\int_0^{x-t}d\eta\;U\left(\frac{\xi+\eta}{2}\right)\Phi\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)\\ &-i\int_{(x-t)/2}^{x}Q(s)\Phi(s,t+s-x)\,ds+i\int_{(x+t)/2}^{x}Q(s)\Phi(s,t+x-s)\,ds\Big\}.\end{aligned}$$

Since $\psi_0(x)$ satisfies

$$\text{(A.8)}\qquad \psi_0(x)=1+\int_0^x(x-s)U(s)\psi_0(s)\,ds,$$

if a function $\Phi(x,t)$ satisfies the equation

$$\begin{aligned}\Phi(x,t) = &-\int_0^{(x-t)/2}\left(\frac{x-t}{2}-s\right)U(s)\psi_0(s)\,ds\\ &-\int_0^{(x+t)/2}\left(\frac{x+t}{2}-s\right)U(s)\psi_0(s)\,ds\\ &-i\int_{(x-t)/2}^{(x+t)/2}Q(s)\psi_0(s)\,ds\\ &+\frac{1}{4}\int_0^{x+t}d\xi\int_0^{x-t}d\eta\;U\left(\frac{\xi+\eta}{2}\right)\Phi\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)\\ &-i\int_{(x-t)/2}^{x}Q(s)\Phi(s,t+s-x)\,ds\\ &+i\int_{(x+t)/2}^{x}Q(s)\Phi(s,t+x-s)\,ds,\end{aligned}\tag{A.9}$$

then $\varphi(x,k)$ defined by (A.1) satisfies equation (A.3) and so is a solution of (2.1) satisfying (4.6).

To solve (A.9) by the method of successive approximation, we put

$$\begin{aligned}\Phi_0(x,t) = &-\int_0^{(x-t)/2}\left(\frac{x-t}{2}-s\right)U(s)\psi_0(s)\,ds\\ &-\int_0^{(x+t)/2}\left(\frac{x+t}{2}-s\right)U(s)\psi_0(s)\,ds\\ &-i\int_{(x-t)/2}^{(x+t)/2}Q(s)\psi_0(s)\,ds,\end{aligned}$$

$$\begin{aligned}\Phi_n(x,t) = &\frac{1}{4}\int_0^{x+t}d\xi\int_0^{x-t}d\eta\;U\left(\frac{\xi+\eta}{2}\right)\Phi_{n-1}\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)\\ &-i\int_{(x-t)/2}^{x}Q(s)\Phi_{n-1}(s,t+s-x)\,ds\\ &+i\int_{(x+t)/2}^{x}Q(s)\Phi_{n-1}(s,t+x-s)\,ds.\end{aligned}$$

It is easy to show that, under the assumption that $U(x), Q(x)$ are locally integrable functions, the series $\sum_{n=0}^{\infty}\Phi_n(x,t)$ converges uniformly in

$-x \le t \le x$, and its sum $\Phi(x,t)$ is a continuous, bounded solution of (A.9). By (A.8) and (A.9), we obtain

$$\frac{d}{dx}(\psi_0(x) + \Phi(x,x)) = -iQ(x)(\psi_0(x) + \Phi(x,x)),$$
$$\frac{d}{dx}(\psi_0(x) + \Phi(x,-x)) = iQ(x)(\psi_0(x) + \Phi(x,-x)).$$

On solving these equations under the condition $\Phi(0,0) = 0$, we get (A.2). By the same manner as in the proof of Lemma 2.1, one can prove assertion (2). The proof is complete. $\square$

Corollary A.2. *Under the same assumption and the same notation as in Lemma* A.1*, the following holds*:

$$2ik\varphi(x,k) = e^{-i\int_0^x Q(\eta)\,d\eta} e^{ikx} - e^{i\int_0^x Q(\eta)\,d\eta} e^{-ikx} - \int_{-x}^{x} \Phi_t(x,t) e^{ikt}\,dt.$$

Proof. By (A.1), after performing an integration by parts, we obtain

$$2ik\varphi(x,k) = 2i\psi_0(x)\sin kx - [\Phi(x,t)e^{ikt}]_{-x}^{x} - \int_{-x}^{x} \Phi_t(x,t)e^{ikt}\,dt.$$

This, together with (A.2), completes the proof. $\square$

B. We shall present the proof of the following:

Lemma B.1. *Let $F \in L^1(\mathbf{R})$, and assume that the function $S(k)$ defined by* (1.6) *with a complex constant C of absolute value* 1 *satisfies $|S(k)| = 1$, $k \in \mathbf{R}$, and*

$$\text{(B.1)} \qquad \operatorname{ind} S(k) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d[\arg S(k)] = \frac{1}{2\pi}[\arg S(k)]_{-\infty}^{\infty} = 0.$$

Then equation (4.12) *has no nontrivial solutions in $L^1(x,\infty)$.*

Proof. Let x be a fixed positive number, and let $L(x,t)$ be a solution of (4.12) in $L^1(x,\infty)$. In analogy to Marchenko [**13**, page 221], one can show that $L(x,\cdot)$ belongs also to $L^2(x,\infty)$. We take the Fourier transform of (4.12) and set

$$\hat{L}(k) = \int_x^\infty L(x,t)e^{ikt}\,dt.$$

Then, by (1.6), we have

$$\overline{\hat{L}(k)} + \hat{L}(k)(S(k) - C) = \int_{-\infty}^x M(t)e^{-ikt}\,dt,$$

where $M(t)$ is a function on $(-\infty, x)$ defined by

$$M(t) = \int_x^\infty L(x,r)F(r+t)\,dr, \quad t \le x.$$

Taking the inner product of this identity and $\overline{\hat{L}(k)}$, we have

$$(\overline{\hat{L}(k)}, \overline{\hat{L}(k)}) + (S(k)\hat{L}(k), \overline{\hat{L}(k)}) = 0, \tag{B.2}$$

since, by the Parseval equality,

$$(\hat{L}(k), \overline{\hat{L}(k)}) = \left(\int_{-\infty}^x M(t)e^{-ikt}\,dt, \overline{\hat{L}(k)} \right) = 0.$$

Here $(f(k), g(k))$ denotes the standard inner product in the space $L^2(\mathbf{R})$. Because of $|S(k)| = 1$, equality (B.2) yields

$$(\overline{\hat{L}(k)} + S(k)\hat{L}(k), \overline{\hat{L}(k)} + S(k)\hat{L}(k)) = 0.$$

Hence $\overline{\hat{L}(k)} + S(k)\hat{L}(k) = 0$.

By the assumption (B.1) and the Wiener-Hopf factorization, see, e.g., Kreĭn [**12**, Theorem 2.1], the function $S(k)$ is expressed as

$$S(k) = C\frac{\exp(\int_{-\infty}^0 F_-(t)e^{-ikt}\,dt)}{\exp(\int_0^\infty F_+(t)e^{-ikt}\,dt)},$$

in terms of functions $F_\pm \in L^1(0,\pm\infty)$. This, combined with $\overline{\hat{L}(k)} + S(k)\hat{L}(k) = 0$, leads to

$$\overline{\hat{L}(k)} \exp\left(\int_0^\infty F_+(t)e^{-ikt}\,dt\right) = -C\hat{L}(k)\exp\left(\int_{-\infty}^0 F_-(t)e^{-ikt}\,dt\right).$$

The function on the lefthand (righthand) side of this identity is holomorphic and bounded in the lower, respectively the upper, half plane and is extended continuously up to the real axis. Hence, by Morera's theorem, it has a continuation to the whole complex plane as an entire, bounded function. In view of Liouville's theorem, this implies that the function must be a constant, which is easily seen to be 0 by the Riemann-Lebesgue lemma. Thus, we have $\hat{L}(k) = 0$ and so $L(x,\cdot) = 0$ in $L^1(x,\infty)$. $\square$

C. Throughout the paper we assume that there are no bound states. We here ask under what conditions on the potentials such an assumption is valid and the absence of bound states is guaranteed. As a general answer we present the following:

Proposition C.1. *Let* $(1+x)U(x) \in L^1(0,\infty)$, *and assume that the Jost solution* $f_0(x,k)$ *of*

$$f'' + [k^2 - U(x)]f = 0, \quad 0 < x < 1, \tag{C.1}$$

satisfies $f_0(0,0) \neq 0$. *Moreover, let* $f(x,k)$ *be the Jost solution of* (1.1) *with* $I = [0,\infty)$ *where* $Q(x) \in L^1(0,\infty)$. *Then the number of zeros, counted with multiplicities, of* $f(0,k)$ *in the upper-half plane* $\operatorname{Im} k > 0$ *equals that of* $f_0(0,k)$ *there. In particular, if* $f_0(0,k) \neq 0$ $(\operatorname{Im} k > 0$, $k = 0)$ *then* $f(0,k) \neq 0$ $(\operatorname{Im} k > 0$, $k = 0)$.

Proof. Noting that $f(x,0) = f_0(x,0)$, we have $f(0,0) \neq 0$ by the assumption $f_0(0,0) \neq 0$. This, together with (4.2), means that $f(0,k)$ has no zeros for real k. Moreover, in view of (2.7) and the Riemann-Lebesgue lemma, $f(0,k)$ tends to a nonzero number (of absolute value 1) as $|k| \to \infty$, and so, a curve γ defined by

$$z = f(0,k), \quad -\infty \le k \le \infty,$$

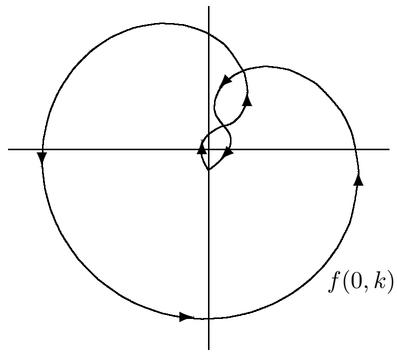

FIGURE 1. Winding number.

is a continuous, oriented, closed curve in the complex plane from a point on the unit circle to itself, not passing through the origin $z=0$. Hence we can define the index

$$\text{(C.2)} \qquad \operatorname{ind} f(0,k) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d[\arg f(0,k)] = \frac{1}{2\pi}[\arg f(0,k)]_{-\infty}^{\infty}.$$

This number indicates how many times the curve γ winds around the origin in the counterclockwise direction, see Figure 1.

On the other hand, by the argument principle, the index $\operatorname{ind} f(0,k)$ gives the number of zeros, counted with multiplicities, of $f(0,k)$ in the region $\operatorname{Im} k > 0$, since $f(0,k)$ is holomorphic in the region and is continuous in its closure. Therefore, to prove the proposition, it suffices to show that

$$\text{(C.3)} \qquad \operatorname{ind} f(0,k) = \operatorname{ind} f_0(0,k).$$

To show (C.3), let $0 \le \tau \le 1$, and let $f = f(x,k;\tau)$ be the Jost solution of

$$f'' + [k^2 - (U(x) + 2k\tau Q(x))]f = 0, \quad 0 \le x < \infty,$$

for each $\tau \in [0,1]$. Since $f(x,0;\tau) = f_0(x,0)$ and the discussion above for $f(0,k)$ applies verbatim, the winding number

$$\operatorname{ind} f(0,k;\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\,[\arg(f(0,k;\tau)]$$

can be defined for each $\tau \in [0,1]$. We shall show (C.3) by contradiction. If $\operatorname{ind} f(0,k;1)$ ($= \operatorname{ind} f(0,k)$) were not equal to $\operatorname{ind} f(0,k;0)$, then we can define a number τ_0 by

$$\tau_0 := \inf\{\tau \in [0,1] \mid \operatorname{ind} f(0,k;\tau) \neq \operatorname{ind} f_0(0,k)\}. \tag{C.4}$$

Since $f(0,0;\tau_0) = f_0(0,0) \neq 0$, the curve $\gamma(\tau_0)$ defined by

$$z = f(0,k;\tau_0), \quad -\infty \leq k \leq \infty,$$

does not pass through the origin $z = 0$. But $f(0,k;\tau)$ is continuous in τ, and therefore, the winding numbers $\operatorname{ind} f(0,k;\tau)$ are invariant under small perturbations of τ, in other words, $\operatorname{ind} f(0,k;\tau) = \operatorname{ind} f(0,k;\tau_0)$ for τ sufficiently near τ_0. This is incompatible with definition (C.4). Thus, we obtain (C.3). The proof is complete. $\square$

By Proposition C.1, exploring the absence of bound states for (1.1) can be reduced to the question whether there are bound states for a corresponding equation with $Q(x) \equiv 0$. In particular, in the case of $U(x) \equiv 0$ in (C.1), it is clear from $f_0(x,k) = e^{ikx}$ that $\operatorname{ind} f_0(0,k) = 0$. Hence, we can draw the following conclusion from Proposition C.1.

Proposition C.2. *Let $f(x,k)$ be the Jost solution of*

$$f'' + [k^2 - 2kQ(x)]f = 0, \quad 0 < x < \infty,$$

where $Q(x) \in L^1(0,\infty)$. Then $f(0,k) \neq 0$ for any k in $\operatorname{Im} k \geq 0$.

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