JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 19, Number 3, Fall 2007

AN INTEGRAL EQUATION FOR MAXWELL'S EQUATIONS IN A LAYERED MEDIUM WITH AN APPLICATION TO THE FACTORIZATION METHOD

ANDREAS KIRSCH

ABSTRACT. In the first part of this paper we study the direct scattering problem for time harmonic electromagnetic fields in a layered medium where arbitrary incident fields are scattered by a medium described by a space dependent permittivity and conductivity. We derive an integral equation and prove a theorem of Riesz-Fredholm type. In the second part we investigate the Factorization Method for the corresponding inverse problem with magnetic dipoles as incident fields. This is the problem to recover the support of the contrast from field measurements.

1. Introduction. This paper studies the direct and the inverse scattering problem for electromagnetic time harmonic fields where the scattering medium is described by a space-dependent permittivity and conductivity imbedded in a two-layered space. Following the standard notations we combine these quantities by introducing a space-dependent and complex valued relative permittivity ε_r. This scattering problem is motivated by a project[1] supported by the German Federal Ministry of Education and Research. Here, the two layers correspond to air (upper layer) and soil (lower layer), and the mine is modeled by a different permittivity and/or conductivity.

For a mathematical treatment of the direct problem we first refer to the monographs of Monk [**20**] and Nédélec [**21**] who proposed variational formulations with nonlocal boundary conditions on an artificial boundary based on the representation formula of Stratton-Chu type, cf. [**6**], or the exterior "capacity operator," respectively. We refer also to [**7**] and [**16**] for earlier results of this kind. Although these approaches are presented for perfect conductors it is quite obvious how to carry them over to the case of an inhomogeneous medium which we characterize by the change of the electric permittivity $\varepsilon\varepsilon_r$ with respect to the

Received by the editors on March 24, 2006, and in revised form on November 2, 2006.

background medium of permittivity ε. The dimensionless factor ε_r is complex valued (in order to include conducting media) and depends on $x \in \mathbf{R}^3$. For smooth relative permittivities ε_r and homogeneous background media in [**6**] an equivalent integral equation is derived from the Stratton-Chu formula to which the Riesz-Fredholm theory is applied. In contrast to this approach we derive a quite simple integral equation for the magnetic field rather than for the electric field. With this approach we are able to allow quite general electric permittivities while the magnetic permeability is assumed to be that of the background medium.

In the second part of this paper we study the inverse problem to recover the support of the contrast $\varepsilon_r - 1$ from the magnetic field data corresponding to a large number of magnetic dipoles as incident fields. Here, ε_r denotes the relative electric permittivity The Factorization Method belongs to the class of probe or sampling methods, see [**23**] for a recent survey article. In our opinion, the Factorization Method is a particularly elegant and satisfying approach from both the mathematical and numerical point of view.

The idea is to consider the measured "response matrix" as the kernel of an integral operator $\mathcal{M}$ and solve, for every point z in space (or in a sufficiently large region enclosing the unknown medium), an integral equation of the first kind. The righthand side is just the field of a magnetic dipole at z for the (known) background medium. In this form the method is known as the Linear Sampling method, proposed first in [**5**] for scalar problems and later applied to several electromagnetic inverse scattering problems, see [**2–4, 8–10, 17**].

The difficulties in the rigorous mathematical justification of the Linear Sampling Method led to the development of the Factorization Method which replaces, roughly speaking, the integral operator with its square root. The number of inverse problems for which the Factorization method is applicable is limited since a factorization of $\mathcal{M}$ in the form $\mathcal{M} = \mathcal{H}^*\mathcal{T}\mathcal{H}$–which is the essential tool of the Factorization Method, see, e.g., [**12, 13, 15**]–does not hold in many important cases. In this paper we consider one of these cases where the incident fields are given by magnetic dipoles located on a surface in the upper layer of a two-layered medium. The incident fields are scattered by an inhomogeneity given by a space-dependent relative permittivity ε_r which can be complex valued to allow for absorption. The inverse problem is to recover the shape Ω of the support of ε_r from measurements of the scattered fields on some surface in the upper layer.

The paper is organized as follows. After a short physical derivation of the scattering problem with magnetic dipoles in Section 2 we study in Section 3 the mathematical questions concerning the direct scattering problem for nonsmooth data. For L^2-source functions and L^∞-contrasts ε_r we derive a new integral equation for the unknown magnetic field and prove equivalence with a variational formulation and a result of Riesz-Fredholm type. Section 4 is devoted to the Factorization Method. We derive factorizations of the data operator $\mathcal{M}$ and a related auxiliary operator $\widetilde{\mathcal{M}}$ and a characterization of the unknown region Ω by $\widetilde{\mathcal{M}}$. In Section 5 we show how to compute $\widetilde{\mathcal{M}}$ from the known data operator $\mathcal{M}$.

2. Formulations of the direct and inverse scattering problems. In this section we study the direct scattering problem for a medium which consists of two layers $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3 : x_3 > 0\}$ and $\mathbf{R}^3_- = \{x \in \mathbf{R}^3 : x_3 < 0\}$ which are separated by $\mathbf{R}^3_0 = \{x \in \mathbf{R}^3 : x_3 = 0\}$ and a bounded region Ω for which we assume that the boundary $\partial\Omega$ is sufficiently smooth and such that $\overline{\Omega} \subset \mathbf{R}^3_-$ and $\mathbf{R}^3_- \setminus \overline{\Omega}$ is connected. The media are distinguished by their electric and magnetic properties. Let $\varepsilon_\pm$ and $\mu_\pm$ be the (constant) electric permittivity and magnetic permeability, respectively, of the layers $\mathbf{R}^3_\pm$. We require that $\varepsilon_\pm$ as well as $\mu_\pm$ are positive real numbers. The region Ω is described by having permittivity $\varepsilon_-\varepsilon_r$ where $\varepsilon_r \in L^\infty(\Omega)$ can be complex valued to allow the region Ω to be conducting. We extend ε_r by 1 into $\mathbf{R}^3 \setminus \overline{\Omega}$. Further assumptions on ε_r will be added below. We set

$$\varepsilon(y) = \begin{cases} \varepsilon_+ & y \in \mathbf{R}^3_+, \\ \varepsilon_- & y \in \mathbf{R}^3_-, \end{cases} \qquad \mu(y) = \begin{cases} \mu_+ & y \in \mathbf{R}^3_+, \\ \mu_- & y \in \mathbf{R}^3_-. \end{cases}$$

A magnetic dipole of position $x \in \mathbf{R}^3_+$ and moment $p \in \mathbf{C}^3$ produces an electric field $E = E(y,x,p)$ and magnetic field $H = H(y,x,p)$ which satisfy the system (see Stratton [**24**]):

$$\text{(1a)} \qquad \operatorname{curl}_y E(y,x,p) = i\omega\mu(y)\big[H(y,x,p) + \delta_x(y)p\big], y \in \mathbf{R}^3,$$

$$\text{(1b)} \qquad \operatorname{curl}_y H(y,x,p) = -i\omega\varepsilon(y)E(y,x,p),\ y \in \mathbf{R}^3.$$

Here, δ_x denotes the delta distribution with support x. Eliminating the electric field yields

$$(2) \qquad \mathrm{curl}_y \frac{1}{\varepsilon(y)} \mathrm{curl}_y H(y,x,p) - \omega^2 \mu(y) H(y,x,p) = \omega^2 \mu(y) \delta_x(y) p.$$

The solution has to be understood in the distributional sense and can be formulated as a transmission problem in the form

$$\begin{aligned} &\mathrm{curl}_y^2 H(y,x,p) - \omega^2 \varepsilon_+ \mu_+ H(y,x,p) = \omega^2 \varepsilon_+ \mu_+ \delta_x(y) p, \quad y \in \mathbf{R}^3_+, \\ &\mathrm{curl}_y^2 H(y,x,p) - \omega^2 \varepsilon_- \mu_- H(y,x,p) = 0, \quad y \in \mathbf{R}^3_-, \end{aligned}$$

and the tangential components of $H(\cdot,x,p)$ and $(1/\varepsilon)\mathrm{curl}_y H(\cdot,x,p)$ are continuous at $\mathbf{R}^3_0$. The system (1) and equation (2) are complemented, respectively, by the well known Silver-Müller radiation conditions

$$(3) \qquad \int_{|y|=R} \left| \frac{y}{|y|} \times H(y) + (\varepsilon/\mu)^{1/2} E(y) \right|^2 ds(y) \longrightarrow 0, \quad R \to \infty,$$

and

$$(4) \quad \int_{|y|=R} \left| \mathrm{curl} H(y) \times \frac{y}{|y|} - \frac{i}{\omega\,(\varepsilon\mu)^{1/2}} H(y) \right|^2 ds(y) \longrightarrow 0, \quad R \to \infty.$$

We define the (magnetic) Green's tensor $\mathbf{G}(x,y) \in \mathbf{C}^{3\times 3}$ by the solution of

$$(5) \qquad \mathrm{curl}_y \frac{1}{\varepsilon(y)} \mathrm{curl}_y \mathbf{G}(x,y) - \omega^2 \mu(y) \mathbf{G}(x,y) = \frac{1}{\varepsilon(y)}\, \delta_x\, I,$$

i.e.,

$$\mathrm{curl}_y^2 \mathbf{G}(x,y) - k^2(y)\, \mathbf{G}(x,y) = \delta_x(y)\, I, \quad y \in \mathbf{R}^3 \setminus \mathbf{R}^3_0,$$

where $k^2(y) = \omega^2 \varepsilon(y) \mu(y)$, $y \in \mathbf{R}^3$ and I denotes the identity. As usual, we mean the curl of a matrix to be taken column wise. Furthermore, the tangential components of (the columns of) $\mathbf{G}(x,\cdot)$ and $(1/\varepsilon)\mathrm{curl}_y \mathbf{G}(x,\cdot)$ are continuous at $\mathbf{R}^3_0$. We note that $\mathbf{G}(x,y)^\top = \mathbf{G}(y,x)$ for all x, y. For a mathematical derivation of the Green's tensor we refer to the appendix of [**7**].

Then the magnetic dipole at x has the form

$$H(y,x,p) = k_+^2 \mathbf{G}(x,y)p \quad \text{and} \quad E(y,x,p) = -\frac{k_+^2}{\omega\varepsilon(y)} \operatorname{curl}_y \mathbf{G}(x,y)p$$

where $k_+^2 = \omega^2 \varepsilon_+ \mu_+$. In this paper we will always work with the magnetic field H only. Therefore, let the incident field H^i be a superposition of magnetic dipoles at location $y \in \Gamma_i$ with dipole moment $\varphi(y)^2$, i.e.,

$$(6) \qquad H^i(x) = k_+^2 \int_{\Gamma_i} \mathbf{G}(y,x)\,\varphi(y)\,ds(y), \quad x \in \mathbf{R}^3 \setminus (\mathbf{R}_0^3 \cup \Gamma_i),$$

where $\Gamma_i \subset \mathbf{R}_+^3$ is a smooth surface. Then H^i produces the scattered field

$$H^s(x) = k_+^2 \int_{\Gamma_i} H^s(x,y)\,\varphi(y)\,ds(y), \quad x \in \mathbf{R}^3 \setminus \overline{\Omega},$$

where $H^s(\cdot,y)$ is the scattered field corresponding to the incident field $\mathbf{G}(y,\cdot)$ (column by column). The total fields $H = H^i + H^s$ and $E = E^i + E^s$ satisfy the Maxwell system (column by column)

$$\begin{aligned} \operatorname{curl} E(x) &= i\omega\mu(x)\,H(x), \quad x \in \mathbf{R}^3 \setminus \Gamma_i, \\ \operatorname{curl} H(x) &= -i\omega\varepsilon(x)\varepsilon_r(x)\,E(x), \quad x \in \mathbf{R}^3 \setminus \Gamma_i. \end{aligned}$$

The *inverse problem* is to recover the support $\overline{\Omega}$ of $1 - \varepsilon_r$ from the knowledge of $H^s(x,y)$ for all $x \in \Gamma_s$ and $y \in \Gamma_i$ where also $\Gamma_s \subset \mathbf{R}_+^3$ is some measurement surface. This is equivalent to recover the support from the knowledge of the integral operator $\mathcal{M} : L^2(\Gamma_i, \mathbf{C}^3) \to L^2(\Gamma_s, \mathbf{C}^3)$ defined by

$$(7) \qquad (\mathcal{M}\varphi)(x) = k_+^2 \int_{\Gamma_i} H^s(x,y)\,\varphi(y)\,ds(y), \quad x \in \Gamma_s.$$

In this paper we assume the knowledge of all three components of $H^s(y,x)$ for all polarizations of the incident field. At the end of the paper we make some remarks for the case where only the tangential components are given.

The study of this integral operator $\mathcal{M}$ is the essential topic of the factorization method.

Recall that the incident field H^i from (6) solves

$$\operatorname{curl}\frac{1}{\varepsilon}\operatorname{curl}H^i-\omega^2\mu H^i=0\ \text{ in }\ \mathbf{R}^3\setminus(\mathbf{R}^3_0\cup\Gamma_i),\tag{8}$$

while the total field $H=H^i+H^s$ satisfies

$$\operatorname{curl}\frac{1}{\varepsilon\varepsilon_r}\operatorname{curl}H-\omega^2\mu H=0\ \text{ in }\ \mathbf{R}^3\setminus(\mathbf{R}^3_0\cup\Gamma_i).\tag{9}$$

Subtracting both equations yields

$$\operatorname{curl}\frac{1}{\varepsilon\varepsilon_r}\operatorname{curl}H^s-\omega^2\mu\,H^s=\operatorname{curl}\left[\frac{1}{\varepsilon}\left(1-\frac{1}{\varepsilon_r}\right)\operatorname{curl}H^i\right]\tag{10}$$

in $\mathbf{R}^3\setminus(\mathbf{R}^3_0\cup\Gamma_i)$. Furthermore, the requirement that the tangential components of H and E are continuous on the interfaces translate to analogous conditions for the scattered field H^s, see below.

For the factorization method it will be necessary to allow more general source terms on the righthand side of (10). This is the subject of the next section.

3. Solution of the direct problem by an integral equation approach. In this section, we make the following weak *assumptions* on the data. Let Ω be open and bounded with $\overline{\Omega}\subset\mathbf{R}^3_-$ and $\partial\Omega\in C^2$. Furthermore, let $\mu_\pm,\varepsilon_\pm\in\mathbf{R}_{>0}$ and $\varepsilon_r\in L^\infty(\Omega)$ such that $\operatorname{Re}\varepsilon_r\geq 0$ and $\operatorname{Im}\varepsilon_r\geq 0$ and $\operatorname{Re}\varepsilon_r+\operatorname{Im}\varepsilon_r\geq c_0$ on Ω for some $c_0>0$. Then, in particular, $1/\varepsilon_r\in L^\infty(\Omega)$.

Under these assumptions we will consider the following problem: Given $f\in L^2(\Omega,\mathbf{C}^3)$, determine $v\in H_{\mathrm{loc}}(\operatorname{curl},\mathbf{R}^3)$ with

$$\operatorname{curl}\frac{1}{\varepsilon\varepsilon_r}\operatorname{curl}v-\omega^2\mu\,v=\operatorname{curl}\left(\frac{1}{\varepsilon}f\right)\ \text{ in }\ \mathbf{R}^3,\tag{11}$$

and v satisfies the Silver-Müller radiation condition (4), i.e.

$$\int_{|x|=R}\left|\operatorname{curl}v(x)\times\frac{x}{|x|}-ik(x)\,v(x)\right|^2ds(x)\longrightarrow 0,\quad R\to\infty.\tag{12}$$

As usual, the space $H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$ is defined by

$$H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3) = \big\{ v : \mathbf{R}^3 \to \mathbf{C}^3 : v|_B \in H(\mathrm{curl}, B) \text{ for all balls } B \big\},$$

where, for any open set $D \subset \mathbf{R}^3$, the space $H(\mathrm{curl}, D)$ is defined as the completion of $C^1(\overline{D}, \mathbf{C}^3)$ with respect to the norm

$$\|v\|_{H(\mathrm{curl},D)} := \big(\|v\|^2_{L^2(D)} + \|\mathrm{curl}\, v\|^2_{L^2(D)} \big)^{1/2}.$$

The solution $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$ of (11) has to be understood in the following variational sense.

$$\text{(13)} \quad \iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon \varepsilon_r} \mathrm{curl}\, v \cdot \mathrm{curl}\, \psi - \omega^2 \mu\, v \cdot \psi \right] dx = \frac{1}{\varepsilon_-} \iint_{\Omega} f \cdot \mathrm{curl}\, \psi \, dx$$

for all $\psi \in H(\mathrm{curl}, \mathbf{R}^3)$ with compact support.

The radiation condition (12) makes sense due to the following regularity result [**25**] by Weber. Let $R > 0$ be such that $\overline{\Omega}$ is contained in the open ball $B_R(0) = \big\{ x \in \mathbf{R}^3 : |x| < R \big\}$ centered at 0 with radius R and set $G = \mathbf{R}^3 \setminus \overline{B_R(0)}$. A careful reading of the proofs of Theorems 2.2 and 2.9 in [**25**] yields that a boundary condition on ∂G is not necessary if one is only interested in interior regularity results in G. Therefore, with this modification Theorem 2.9 in [**25**] yields that $v \in H^\ell(M \cap G_\pm)$ for all $\ell \in \mathbf{N}$ and all open and bounded regions M such that $\overline{M} \subset G$. Here, $G_\pm = \mathbf{R}^3_\pm \setminus \overline{B_R(0)}$. Sobolev's imbedding theorem even yields that $v \in C^\ell(\overline{G_\pm})$ for all $\ell \in \mathbf{N}$.

We will show below that for smooth data, the solution v of (13) satisfies the following transmission problem.

$$\text{(14a)} \qquad \mathrm{curl}\, \frac{1}{\varepsilon \varepsilon_r} \mathrm{curl}\, v - \omega^2 \mu\, v = \frac{1}{\varepsilon} \mathrm{curl}\, f \quad \text{in } \mathbf{R}^3 \setminus (\mathbf{R}^3_0 \cup \partial\Omega),$$

$$\text{(14b)} \qquad \nu \times (v_+ - v_-) = 0 \quad \text{on } \mathbf{R}^3_0,$$

$$\nu \times \left(\frac{1}{\varepsilon_+} \mathrm{curl} v_+ - \frac{1}{\varepsilon_-} \mathrm{curl} v_- \right) = 0 \quad \text{on } \mathbf{R}^3_0,$$

$$\text{(14c)} \qquad \nu \times (v_+ - v_-) = 0 \quad \text{on } \partial\Omega,$$

$$\nu \times \left(\mathrm{curl} v_+ - \frac{1}{\varepsilon_r} \mathrm{curl} v_- \right) = f \times \nu \quad \text{on } \partial\Omega.$$

Before we show equivalence of the variational formulation to an integral equation we prove the following crucial lemma.

Lemma 3.1. *Consider the volume potential*

$$v(x) = \iint_\Omega \left[\mathrm{curl}_y \mathbf{G}(x,y)\right]^\top f(y)\,dy, \quad x \in \mathbf{R}^3. \tag{15}$$

(a) *For* $f \in C^{1,\alpha}(\Omega, \mathbf{C}^3)$ *the volume potential* $v \in C^2(\mathbf{R}^3 \setminus (\mathbf{R}^3_0 \cup \partial\Omega), \mathbf{C}^3)$ *is the unique classical solution of the transmission problem*

$$\mathrm{curl}\,\frac{1}{\varepsilon}\mathrm{curl}\,v - \omega^2\mu\,v = \frac{1}{\varepsilon}\mathrm{curl}\,f \quad \text{in } \mathbf{R}^3 \setminus (\mathbf{R}^3_0 \cup \partial\Omega), \tag{16a}$$

$$\nu \times (v_+ - v_-) = 0 \quad \text{on } \mathbf{R}^3_0, \tag{16b}$$

$$\nu \times \left(\frac{1}{\varepsilon_+}\mathrm{curl}\,v_+ - \frac{1}{\varepsilon_-}\mathrm{curl}\,v_-\right) = 0 \quad \text{on } \mathbf{R}^3_0,$$

$$\nu \times (v_+ - v_-) = 0 \quad \text{on } \partial\Omega, \tag{16c}$$

$$\nu \times (\mathrm{curl}\,v_+ - \mathrm{curl}\,v_-) = f \times \nu \quad \text{on } \partial\Omega,$$

and v *satisfies the radiation condition* (12). *Here,* $v_\pm$ *denotes the traces of* v *from the outside* (+) *or the inside* (−), *respectively, of* Ω. *Furthermore, in this formulation we extended* f *by zero outside of* $\overline{\Omega}$.

(b) *For* $f \in L^2(\Omega, \mathbf{C}^3)$ *the volume potential is the unique radiating variational solution of* (16a), (16b), (16c), *i.e.,* $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$ *solves*

$$\iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon}\mathrm{curl}\,v \cdot \mathrm{curl}\,\psi - \omega^2\mu\,v \cdot \psi\right] dx = \frac{1}{\varepsilon_-}\iint_\Omega f \cdot \mathrm{curl}\,\psi\,dx \tag{17}$$

for all $\psi \in H(\mathrm{curl}, \mathbf{R}^3)$ *with compact support and satisfies the radiation condition* (12).

(c) *The restriction of* v *to* Ω *defines a bounded operator from* $L^2(\Omega, \mathbf{C}^3)$ *into* $H(\mathrm{curl}, \Omega)$.

Proof. From the construction of the Green's tensor $\mathbf{G}$ in [**7**] it is directly seen that, even for $f \in L^2(\Omega, \mathbf{C}^3)$, the volume potential v is a classical solution of

$$\mathrm{curl}\,\frac{1}{\varepsilon}\mathrm{curl}\,v - \omega^2\mu\,v = 0$$

in $\mathbf{R}^3\backslash(\mathbf{R}^3_0\cup\overline{\Omega})$ satisfying the transmission conditions (16b). Therefore, it is sufficient to consider the problem in $\mathbf{R}^3_-$. Since $\mathbf{G}$ differs in $\mathbf{R}^3_-$ from the Green's tensor $\hat{\mathbf{G}}$ for the homogeneous space with $\varepsilon_+ = \varepsilon_-$ and $\mu_+ = \mu_-$ only by a smooth dyadic, it is sufficient to consider the volume potential

$$(18)\qquad \hat{v}(x) = \iint_\Omega \big[\mathrm{curl}_y \widehat{\mathbf{G}}(x,y)\big]^\top f(y)\,dy, \quad x \in \mathbf{R}^3,$$

where

$$(19)\qquad \widehat{\mathbf{G}}(x,y) = \Phi_-(x,y)\,I + \frac{1}{k_-^2}\,\nabla\mathrm{div}\,\big[\Phi_-(x,y)\,I\big]$$

is the Green's tensor for the homogeneous space with $\varepsilon_+ = \varepsilon_-$ and $\mu_+ = \mu_-$. Here, $k_- = \omega(\varepsilon_-\mu)^{1/2}$ and

$$\Phi_-(x,y) = \frac{\exp(ik_-|x-y|)}{4\pi|x-y|}, \quad x \neq y.$$

(a) Let $f \in C^{1,\alpha}(\Omega, \mathbf{C}^3)$ and $j \in \{1,2,3\}$. For $x \notin \partial\Omega$ we have[3]

$$\begin{aligned} \hat{v}_j(x) &= \iint_\Omega \big[\mathrm{curl}_y\big(\Phi_-(x,y)e_j\big)\big]\cdot f(y)\,dy \\ &= \iint_\Omega \Phi_-(x,y)\,e_j\cdot \mathrm{curl} f(y)\,dy \\ &\quad + \int_{\partial\Omega} \Phi_-(x,y)\,\big(e_j \times f(y)\big)\cdot \nu(y)\,ds(y), \end{aligned}$$

i.e.,

$$\hat{v}(x) = \iint_\Omega \Phi_-(x,y)\,\mathrm{curl}\,f(y)\,dy + \int_{\partial\Omega} \Phi_-(x,y)\,\big[f(y)\times\nu(y)\big]\,ds(y).$$

From the classical jump conditions of the volume- and single layer potential with Hölder continuous densities, cf. [**6**], we conclude that $\hat{v}$ is continuous in $\mathbf{R}^3$ and $\nu \times (\mathrm{curl}\,\hat{v}_+ - \mathrm{curl}\,\hat{v}_-) = f \times \nu$ on $\partial\Omega$ and $\Delta\hat{v} + k_-^2\,\hat{v} = -\mathrm{curl}\,f$ in Ω. Furthermore, the divergence of $\hat{v}$ vanishes also in Ω which is seen from the form

$$(20)\qquad \begin{aligned} \hat{v}(x) &= \iint_\Omega \mathrm{curl}_y\big(\Phi_-(x,y)I\big)f(y)\,dy \\ &= -\mathrm{curl}\iint_\Omega \Phi_-(x,y)\,f(y)\,dy. \end{aligned}$$

Therefore, $\hat{v}$ satisfies (16a) for $\varepsilon \equiv \varepsilon_-$ and $\mu_+ = \mu_-$. We have therefore shown that the volume potential (15) is a classical solution of (16a), (16b), (16c) and (12). It remains to show uniqueness which we will prove later in Theorem 3.4 for a more general case.

(b), (c) Let now $f \in L^2(\Omega, \mathbf{C}^3)$. Then the volume potentials $\hat{v}$ and v are well defined and functions in $H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$. It is again sufficient to show this for $\hat{v}$. Since the volume potential

$$x \longmapsto \iint_\Omega \Phi_-(x,y)\, f(y)\, dy, \quad x \in \mathbf{R}^3,$$

is in $H^2_{\mathrm{loc}}(\mathbf{R}^3, \mathbf{C}^3)$, cf. [**19**], we note from (20) that $\hat{v} \in H^1_{\mathrm{loc}}(\mathbf{R}^3, \mathbf{C}^3) \subset H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$ and $f \mapsto \hat{v}|_\Omega$ is bounded from $L^2(\Omega, \mathbf{C}^3)$ into $H(\mathrm{curl}, \Omega)$. Therefore, since for smooth f the potential $\hat{v}$ solves the variational equation (17) a denseness argument yields this also for $f \in L^2(\Omega, \mathbf{C}^3)$. For uniqueness we refer again to Theorem 3.4 below. □

Now we go back to the transmission problem (13) and write it in the form

$$\iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon} \mathrm{curl}\, v \cdot \mathrm{curl}\, \psi - \omega^2 \mu\, v \cdot \psi\right] dx = \frac{1}{\varepsilon_-} \iint_\Omega \big[f + m\, \mathrm{curl}\, v\big] \cdot \mathrm{curl}\, \psi\, dx$$

for all $\psi \in H(\mathrm{curl}, \mathbf{R}^3)$ with compact support. Here we have set $m = 1 - 1/\varepsilon_r$. By Lemma 3.1 this equation is equivalent to

$$v(x) = \iint_\Omega \big[\mathrm{curl}_y \mathbf{G}(x,y)\big]^\top \big[f(y) + m(y)\, \mathrm{curl}\, v(y)\big]\, dy, \quad x \in \mathbf{R}^3. \tag{21}$$

Therefore, as a corollary of Lemma 3.1, we have:

Lemma 3.2. *Define the integral operator* $A : L^2(\Omega, \mathbf{C}^3) \to H(\mathrm{curl}, \Omega)$ *by*

$$(Af)(x) = \iint_\Omega \big[\mathrm{curl}_y \mathbf{G}(x,y)\big]^\top f(y)\, dy, \quad x \in \Omega. \tag{22}$$

(a) *If* $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$ *solves* (13), *then the restriction* $v|_\Omega \in H(\mathrm{curl}, \Omega)$ *solves the equation*

$$v - A(m\, \mathrm{curl}\, v) = Af. \tag{23}$$

(b) *If* $v \in H(\mathrm{curl}, \Omega)$ *solves the equation* (23)*, then the extension of* v *by the righthand side of* (21) *solves* (13)*.*

We note that A is well defined by part (c) of Lemma 3.1.

It is the aim of the next lemma to prove that the mapping $v \mapsto v - A(m\,\mathrm{curl}\,v)$ is a compact perturbation of an isomorphism. We introduce the operator $A_i : L^2(\Omega, \mathbf{C}^3) \to H(\mathrm{curl}, \Omega)$ by

$$(A_i f)(x) = -\mathrm{curl} \iint_\Omega \widehat{\Phi}(x,y)\, f(y)\, dy, \quad x \in \Omega, \tag{24}$$

where

$$\widehat{\Phi}(x,y) = \frac{\exp(i\hat{k}|x-y|)}{4\pi|x-y|}, \quad x \neq y,$$

and $\hat{k} = i(\varepsilon_- \mu_-)^{1/2}$. We note that A_i is the operator A for $\varepsilon_+ = \varepsilon_-$ and $\mu_+ = \mu_-$ and $\omega = i$, compare (20).

Lemma 3.3. (a) *The operator* $A - A_i$ *is compact from* $L^2(\Omega, \mathbf{C}^3)$ *into* $H(\mathrm{curl}, \Omega)$*.*

(b) *The operator* $v \mapsto v - A_i(m\,\mathrm{curl}\,v)$ *is a bounded isomorphism from* $H(\mathrm{curl}, \Omega)$ *onto itself.*

Proof. Part (a) follows from the smoothness of $\mathrm{curl}\,\big[\mathbf{G} - \widehat{\Phi} I\big]$.

(b) We consider the equation $v - A_i(m\,\mathrm{curl}\,v) = g$ for any $g \in H(\mathrm{curl}, \Omega)$, i.e., $v - g = A_i(m\,\mathrm{curl}\,v)$. The assertion of Lemma 3.2 holds also for A_i. Therefore, $v \in H(\mathrm{curl}, \Omega)$ is a solution of $v - g = A_i(m\,\mathrm{curl}\,v)$ if, and only if, there exists an extension w of $v - g$ into $\mathbf{R}^3$ which solves

$$\begin{aligned}\iint_{\mathbf{R}^3} \big[\mathrm{curl}\,w \cdot \mathrm{curl}\,\psi + \varepsilon_- \mu_-\, w \cdot \psi\big]\, dx &= \iint_\Omega m\,\mathrm{curl}\,v \cdot \mathrm{curl}\,\psi\, dx \\ &= \iint_\Omega m\,\mathrm{curl}\,w \cdot \mathrm{curl}\,\psi\, dx \\ &\quad + \iint_\Omega m\,\mathrm{curl}\,g \cdot \mathrm{curl}\,\psi\, dx\end{aligned}$$

for all $\psi \in H(\mathrm{curl}, \mathbf{R}^3)$ with compact support, i.e.,

$$\iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon_r} \mathrm{curl}\, w \cdot \mathrm{curl}\, \psi + \varepsilon_- \mu_- \, w \cdot \psi \right] dx = \iint_\Omega m \,\mathrm{curl}\, g \cdot \mathrm{curl}\, \psi \, dx \tag{25}$$

for all $\psi \in H(\mathrm{curl}, \mathbf{R}^3)$. Note that, by Lemma 3.1 and the fact that $\widehat{\Phi}(x, y)$ decays exponentially for $y \in \Omega$ we conclude that $w \in H(\mathrm{curl}, \mathbf{R}^3)$ and this equation holds for all $\psi \in H(\mathrm{curl}, \mathbf{R}^3)$.

The lefthand side defines a bilinear form $a(w, \psi)$ on $H(\mathrm{curl}, \mathbf{R}^3) \times H(\mathrm{curl}, \mathbf{R}^3)$ with

$$\mathrm{Re}\left[e^{\pi i/4} a(w, \overline{w})\right] \geq \frac{1}{\sqrt{2}} \min\left\{ \frac{c_0}{\|\varepsilon_r\|_\infty^2}, \, \varepsilon_- \mu_-, \, 1 \right\} \|w\|^2_{H(\mathrm{curl}, \mathbf{R}^3)}$$

for all $w \in H(\mathrm{curl}, \mathbf{R}^3)$. Indeed, by the assumptions given at the beginning of this section, we have

$$\mathrm{Re}\left[e^{\pi i/4} \frac{1}{\varepsilon_r} \right] = \frac{\mathrm{Re}\, \varepsilon_r + \mathrm{Im}\, \varepsilon_r}{\sqrt{2}\, |\varepsilon_r|^2} \geq \frac{c_0}{\sqrt{2}\, \|\varepsilon_r\|_\infty^2}.$$

Therefore, equation (25) has a unique solution for all $g \in H(\mathrm{curl}, \mathbf{R}^3)$. From this the assertion of part (b) follows easily. Indeed, for $g = 0$ we conclude that the corresponding w has to vanish in $\mathbf{R}^3$, thus also v since w is an extension of $v - g$. For given $g \in H(\mathrm{curl}, \Omega)$ we determine $w \in H(\mathrm{curl}, \mathbf{R}^3)$ as the solution of (25). Then $v := w|_\Omega + g$ solves $v - g = A_i(m \,\mathrm{curl}\, v)$. This ends the proof. □

Combining Lemmas 3.1 and 3.3 yields the first part of the following theorem.

Theorem 3.4. *Let $\mu, \varepsilon, \varepsilon_r \in L^\infty(\mathbf{R}^3)$ satisfy the assumptions formulated at the beginning of this section.*

(a) *Then* (13) *satisfies the Fredholm alternative*[4], *i.e., there exists a unique radiating solution $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$ of* (13) *for every $f \in L^2(\Omega, \mathbf{C}^3)$ provided uniqueness holds. In this case, for any compact set*

B containing $\overline{\Omega}$ *in its interior there exists a constant* $c > 0$ (*depending only on* B, ω, μ, ε *and* ε_r) *such that*

$$\|v\|_{H(\mathrm{curl},B)} \le c\|f\|_{L^2(\Omega,\mathbf{C}^3)} \ \text{ for all } \ f \in L^2(\Omega, \mathbf{C}^3).$$

The restriction of v *to* Ω *is the unique solution of the integral equation*

$$v(x) = \int_\Omega \big[\mathrm{curl}_y \mathbf{G}(x,y)\big]^\top \big[f(y) + m(y)\,\mathrm{curl}\, v(y)\big]\, dy, \quad x \in \Omega. \tag{26}$$

(b) *Let, in addition,* $\mathrm{Im}\,\varepsilon > 0$ *almost everywhere on* Ω *or* $\varepsilon_r \in C^{1,\alpha}(\Omega)$. *Then uniqueness holds, i.e., also existence by part* (a).

Proof. It suffices to prove uniqueness. We follow [**7**] and assume that v is a solution of the homogeneous problem, i.e., of (13) for $f = 0$. We set $w(x) = \overline{v(x)}\phi(|x|)$ in (13) where $\phi \in C^\infty(\mathbf{R})$ is some mollifier with $\phi(x) = 1$ for $|x| \le R$ and $\phi(x) = 0$ for $|x| \ge 2R$. Then, by Green's theorem (note that v is smooth for $R < |x| < 2R$) and the boundary conditions $\phi = 1$ for $|x| = R$ and $\phi = 0$ for $|x| = 2R$,

$$\begin{aligned} 0 &= \iint_{|x|<R} \left[\frac{1}{\varepsilon\varepsilon_r}|\mathrm{curl}\, v|^2 - \omega^2\mu\,|v|^2\right]\, dx \\ &\quad + \iint_{R<|x|<2R} \left[\frac{1}{\varepsilon}\,\mathrm{curl}\, v \cdot \mathrm{curl}\,(\overline{v}\phi) - \omega^2\mu\,|v|^2\phi\right]\, dx \\ &= \iint_{|x|<R} \left[\frac{1}{\varepsilon\varepsilon_r}|\mathrm{curl}\, v|^2 - \omega^2\mu\,|v|^2\right]\, dx \\ &\quad - \int_{|x|=R} \frac{1}{\varepsilon}\,(\mathrm{curl}\, v \times \nu) \cdot \overline{v}\, ds. \end{aligned} \tag{27}$$

Taking the imaginary part yields

$$\mathrm{Im} \int_{|x|=R} \frac{1}{\varepsilon}\,(\mathrm{curl}\, v \times \nu) \cdot \overline{v}\, ds \le 0$$

since $\operatorname{Im}(1/\varepsilon_r) \leq 0$ and ε, μ are real and positive. From this and the binomial theorem we estimate

$$\begin{aligned}\int_{|x|=R} \frac{1}{\varepsilon(x)\,k(x)} \left|\operatorname{curl} v(x) \times \frac{x}{|x|} - ik(x)\,v(x)\right|^2 ds(x) \\ = \int_{|x|=R} \frac{1}{\varepsilon k} \left[|\operatorname{curl} v|^2 + k^2|v|^2\right] ds \\ - 2\operatorname{Im}\int_{|x|=R} \left(\frac{1}{\varepsilon}\operatorname{curl} v \times \nu\right)\cdot \overline{v}\, ds \\ \geq \int_{|x|=R} \frac{1}{\varepsilon k}\left[|\operatorname{curl} v|^2 + k^2|v|^2\right] ds.\end{aligned}$$

The radiation condition yields that $\int_{|x|=R}|v|^2\,ds$ and $\int_{|x|=R}|\operatorname{curl} v|^2\,ds$ tend to zero as R tends to infinity. Now we proceed exactly as in [**7**] where it has been shown that all components v_j, $j = 1,2,3$, of v satisfy scalar Helmholtz equations in $G_\pm = \mathbf{R}^3_\pm \setminus \overline{B_R(0)}$ and transmission conditions on $\mathbf{R}^3_0 \setminus \overline{B_R(0)}$. Furthermore, they satisfy the scalar Sommerfeld radiation condition and $\int_{|x|=R}\left[|v_j|^2 + |\nabla v_j|^2\right] ds \to 0$. In [**18**] Kristensson has proven a uniqueness result which can be formulated (by slight modifications of his arguments) as a type of Rellich lemma, i.e., it yields that v vanishes outside of $B_R(0)$. The unique continuation principle (applied separately in $\mathbf{R}^3_+$ and $\mathbf{R}^3_-$) implies that v vanishes in $\mathbf{R}^3 \setminus \Omega$.

If $\partial\Omega \in C^2$ and $\varepsilon_r \in C^{1,\alpha}(\overline{\Omega})$ we extend ε_r to a $C^{1,\alpha}-$ function into a (exterior) neighborhood of $\partial\Omega$ and apply the unique continuation principle again, see [**6**], which yields that v also vanishes in Ω.

If $\operatorname{Im}\varepsilon_r > 0$ on Ω we go back to (27) which takes the form

$$\iint_\Omega \left[\frac{1}{\varepsilon_-\varepsilon_r}|\operatorname{curl} v|^2 - \omega^2\mu\,|v|^2\right] dx = 0. \tag{28}$$

Taking the imaginary part of this equation yields $\operatorname{curl} v = 0$ in Ω and thus also $v = 0$ in Ω. □

4. The factorization method. In this section, we strengthen the *assumptions* on the data. Let Ω be open and bounded with $\overline{\Omega} \subset \mathbf{R}^3_-$ and $\partial\Omega \in C^2$ such that $\mathbf{R}^3_- \setminus \overline{\Omega}$ is connected. Furthermore, let $\varepsilon_\pm \in \mathbf{R}_{>0}$ and

$\varepsilon_r \in L^\infty(\mathbf{R}^3)$ such that $\operatorname{Re}\varepsilon_r \geq 0$ and $\operatorname{Im}\varepsilon_r \geq c_0$ on Ω for some $c_0 > 0$ and $\varepsilon_r = 1$ on $\mathbf{R}^3 \setminus \Omega$. We assume that the medium is nonmagnetic, i.e., $\mu_+ = \mu_-$. Finally, let $\Gamma_s \subset \{x \in \mathbf{R}^3_+ : x_3 = h\}$ be part of a plane with nonempty relative interior (for some given $h > 0$).

By Theorem 3.4 problem (11), i.e., (13), and (12) has a unique radiating solution for every $f \in L^2(\Omega, \mathbf{C}^3)$.

The incident field (6) motivates the definition of the integral operator $\mathcal{H}_\Gamma : L^2(\Gamma, \mathbf{C}^3) \to L^2(\Omega, \mathbf{C}^3)$ by

$$(\mathcal{H}_\Gamma \varphi)(x) = k_+^2 \operatorname{curl} \int_\Gamma \mathbf{G}(y,x)\, \varphi(y)\, ds(y), \quad x \in \Omega, \tag{29}$$

for $\Gamma = \Gamma_i$. However, the definition allows general surfaces $\Gamma \subset \mathbf{R}^3_+$. Comparing (10) and (11) and the definition of $\mathcal{M}$ we observe that $\mathcal{M} = \mathbf{G}\mathcal{H}_{\Gamma_i}$ where $\mathbf{G} : L^2(\Omega, \mathbf{C}^3) \to L^2(\Gamma_s, \mathbf{C}^3)$ is defined by $\mathbf{G}f = v|_{\Gamma_s}$ and v solves (11) for f replaced by $(1 - 1/\varepsilon_r)f$, i.e., in variational form

$$\iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon\varepsilon_r} \operatorname{curl} v \cdot \operatorname{curl} \psi - \omega^2 \mu\, v \cdot \psi \right] dx = \frac{1}{\varepsilon_-} \iint_\Omega m\, f \cdot \operatorname{curl} \psi\, dx \tag{30}$$

for all $\psi \in H(\operatorname{curl}, \mathbf{R}^3)$ with compact support. Again, we have set $m = 1 - 1/\varepsilon_r$.

The transpose $\mathcal{H}_\Gamma^t$ of $\mathcal{H}_\Gamma$ with respect to the bilinear forms

$$\langle \psi, \varphi \rangle = \int_\Gamma \psi \cdot \varphi\, ds \quad \text{and} \quad \langle f, g \rangle = \iint_\Omega f \cdot g\, dx,$$

respectively, is given by

$$(\mathcal{H}_\Gamma^t g)(x) = k_+^2 \iint_\Omega \left[\operatorname{curl}_y \mathbf{G}(x,y) \right]^\top g(y)\, dy, \quad x \in \Gamma. \tag{31}$$

Therefore, $\mathcal{H}_\Gamma^t g$ is the volume potential

$$w(x) = k_+^2 \iint_\Omega \left[\operatorname{curl}_y \mathbf{G}(x,y) \right]^\top g(y)\, dy, \quad x \in \mathbf{R}^3 \setminus (\mathbf{R}^3_0 \cup \partial\Omega),$$

evaluated on Γ. We have seen in Lemma 3.1 that $w \in H_{\text{loc}}(\operatorname{curl}, \mathbf{R}^3)$ solves (16a)–(16c), i.e., in variational form,

$$\iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon} \operatorname{curl} w \cdot \operatorname{curl} \psi - \omega^2 \mu\, w \cdot \psi \right] dx = \frac{k_+^2}{\varepsilon_-} \iint_\Omega g \cdot \operatorname{curl} \psi\, dx$$

for all $\psi \in H(\mathrm{curl}, \mathbf{R}^3)$ with compact support. For any $f \in L^2(\Omega, \mathbf{C}^3)$ we now set $g = (m/k_+^2)(\mathrm{curl}\, v + f)$ where $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$ solves (30). Writing the latter as

$$\begin{aligned}\iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon}\mathrm{curl}\, v \cdot \mathrm{curl}\, \psi - \omega^2 \mu\, v \cdot \psi\right] dx \\ = \frac{1}{\varepsilon_-} \iint_\Omega m\,(\mathrm{curl}\, v + f) \cdot \mathrm{curl}\, \psi\, dx \\ = \frac{k_+^2}{\varepsilon_-} \iint_\Omega g \cdot \mathrm{curl}\, \psi\, dx\end{aligned}$$

we observe that $\mathcal{H}_{\Gamma_s}^t\big((m/k_+^2)(\mathrm{curl}\, v + f)\big) = \mathbf{G} f$. Substituting this form of $\mathbf{G}$ into $\mathcal{M} = \mathbf{G}\mathcal{H}_{\Gamma_i}$ we arrive at the following factorization of $\mathcal{M}$:

$$\mathcal{M} = \mathcal{H}_{\Gamma_s}^t \mathcal{T} \mathcal{H}_{\Gamma_i} \tag{32}$$

where the operator $\mathcal{T} : L^2(\Omega, \mathbf{C}^3) \to L^2(\Omega, \mathbf{C}^3)$ is defined by

$$\mathcal{T} f = \frac{m}{k_+^2}(\mathrm{curl}\, v|_\Omega + f) \tag{33}$$

and v is the radiating solution of (30).

The following property will be crucial in the application of the factorization method.

Theorem 4.1. *Let $\mathcal{T} : L^2(\Omega, \mathbf{C}^3) \to L^2(\Omega, \mathbf{C}^3)$ be defined by* (33) *where v is the radiating solution of* (30). *Then* $\mathrm{Im}\, \mathcal{T} = (\mathcal{T} - \mathcal{T}^*)/(2i)$ *is coercive, i.e., there exists $c > 0$ such that*

$$\mathrm{Im}\, (\mathcal{T} f, f)_{L^2(\Omega)} \geq c \|f\|_{L^2(\Omega)}^2, \quad \text{for all } f \in L^2(\Omega, \mathbf{C}^3). \tag{34}$$

Proof. For $f \in L^2(\Omega, \mathbf{C}^3)$ we write $Tf = (m/k_+^2)\, g$, where $g = \mathrm{curl}\, v + f$. Therefore, since $f = g - \mathrm{curl}\, v$,

$$(\mathcal{T} f, f)_{L^2(\Omega)} = \frac{1}{k_+^2} \iint_\Omega m\, |g|^2\, dx - \frac{1}{k_+^2} \iint_\Omega m\, g \cdot \mathrm{curl}\, \overline{v}\, dx. \tag{35}$$

We study the second integral and write (30) in the form

$$\begin{aligned}(36)\quad \iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon}\operatorname{curl} v\cdot\operatorname{curl}\psi - \omega^2\mu\, v\cdot\psi\right] dx &= \frac{1}{\varepsilon_-}\iint_\Omega m\,(\operatorname{curl} v + f)\cdot\operatorname{curl}\psi\, dx\\ &= \frac{1}{\varepsilon_-}\iint_\Omega m\, g\cdot\operatorname{curl}\psi\, dx.\end{aligned}$$

We set $\psi = \overline{v}\phi$ where $\phi \in C^\infty(\mathbf{R}^3)$ is a mollifier with $\phi(x) = 1$ for $|x| \le R$ and $\phi(x) = 0$ for $|x| \ge 2R$. Then, by Green's theorem (note again that v is smooth for $R < |x| < 2R$) and the boundary conditions $\phi = 1$ for $|x| = R$ and $\phi = 0$ for $|x| = 2R$,

$$\begin{aligned}\frac{1}{\varepsilon_-}\iint_\Omega m\, g\cdot\operatorname{curl}\overline{v}\, dx &= \iint_{|x|<R}\left[\frac{1}{\varepsilon}|\operatorname{curl} v|^2 - \omega^2\mu\,|v|^2\right] dx\\ &\quad + \iint_{R<|x|<2R}\left[\frac{1}{\varepsilon}\operatorname{curl} v\cdot\operatorname{curl}(\overline{v}\phi) - \omega^2\mu\,|v|^2\phi\right] dx\\ &= \iint_{|x|<R}\left[\frac{1}{\varepsilon}|\operatorname{curl} v|^2 - \omega^2\mu\,|v|^2\right] dx\\ &\quad - \int_{|x|=R}\left[\frac{1}{\varepsilon}(\operatorname{curl} v\times\nu)\cdot\overline{v}\right] ds.\end{aligned}$$

Taking the imaginary part yields

$$\begin{aligned}\frac{1}{\varepsilon_-}\operatorname{Im}\iint_\Omega m\, g\cdot\operatorname{curl}\overline{v}\, dx &= -\operatorname{Im}\int_{|x|=R}\left[\frac{1}{\varepsilon}(\operatorname{curl} v\times\nu)\cdot\overline{v}\right] ds\\ &= \int_{|x|=R}\frac{1}{2k\,\varepsilon}\left|\operatorname{curl} v\times\nu - ik\, v\right|^2 ds\\ &\quad - \int_{|x|=R}\frac{1}{2k\,\varepsilon}\left[|\operatorname{curl} v|^2 + k^2|v|^2\right] ds.\end{aligned}$$

Letting R tend to infinity yields

$$\operatorname{Im}\iint_\Omega m\, g\cdot\operatorname{curl}\overline{v}\, dx \le 0$$

where we have used the radiation condition (12). Substituting this into (35) yields

$$\begin{aligned}\operatorname{Im}(\mathcal{T}f,f)_{L^2(\Omega)} &\geq \frac{1}{k_+^2}\iint_\Omega \operatorname{Im} m\,|g|^2\,dx\\ &\geq \frac{c_0}{k_+^2\|\varepsilon_r\|_\infty^2}\,\|g\|_{L^2(\Omega)}^2\\ &= \frac{c_0}{k_+^2\|\varepsilon_r\|_\infty^2}\,\|\operatorname{curl} v+f\|_{L^2(\Omega)}^2\end{aligned}$$

since $\operatorname{Im} m = \operatorname{Im}\varepsilon_r/|\varepsilon_r|^2 \geq c_0/\|\varepsilon_r\|_\infty^2$. To finish the proof we have to show that there exists a constant $c > 0$ with

$$\|\operatorname{curl} v + f\|_{L^2(\Omega)} \geq c\,\|f\|_{L^2(\Omega)} \text{ for all } L^2(\Omega, \mathbf{C}^3).$$

This follows by standard arguments. Indeed, if this were not the case, there exists a sequence $f_n \in L^2(\Omega, \mathbf{C}^3)$ with $\|f_n\|_{L^2(\Omega)} = 1$ and $\|\operatorname{curl} v_n + f_n\|_{L^2(\Omega)} \to 0$ as n tends to infinity. v_n solves (30) for f_n, which is the weak form of (16a)–(16c) for f replaced by $m(\operatorname{curl} v_n + f_n)$, compare (17) and (36). Since the latter converges to zero in $L^2(\Omega, \mathbf{C}^3)$ Lemma 3.1 yields that $v_n|_\Omega$ converges to zero in $H(\operatorname{curl}, \Omega)$. Therefore, also f_n converges to zero in $L^2(\Omega, \mathbf{C}^3)$ which is a contradiction. □

The following theorem is a basic ingredient of the Factorization Method and characterizes the unknown region Ω by the range $\mathcal{R}(\mathcal{H}_{\Gamma_s}^t)$ of $\mathcal{H}_{\Gamma_s}^t$.

Theorem 4.2. *For any $z \in \mathbf{R}^3$ and fixed $p \in \mathbf{C}^3$ we define $\Psi_z \in L^2(\Gamma_s, \mathbf{C}^3)$ by*

$$\Psi_z(x) = k_+^2\,\mathbf{G}(z,x)p, \quad \text{for } x \in \Gamma_s. \tag{37}$$

Then $z \in \Omega$ if and only if $\Psi_z \in \mathcal{R}(\mathcal{H}_{\Gamma_s}^t)$.

Proof. First, let $z \in \Omega$. Choose $\rho > 0$ such that $B[z,\rho] \subset \Omega$ and a mollifier $\phi \in C^\infty(\mathbf{R}^3)$ with $\phi(x) = 1$ for $|x-z| \geq \rho$ and $\phi(x) = 0$ for $|x - z| \leq \rho/2$ and define

$$v(x) = \varepsilon_+ \operatorname{curl}\left[\frac{1}{\varepsilon}\operatorname{curl}\big(\phi(x)\,\mathbf{G}(z,x)p\big)\right], \quad \text{for } x \in \mathbf{R}^3.$$

We note that

$$v(x) = \varepsilon_+ \operatorname{curl} \left[\frac{1}{\varepsilon} \operatorname{curl}\big(\mathbf{G}(z,x)p\big)\right] = k_+^2 \mathbf{G}(z,x)p$$

for $|x-z| \geq \rho$ and, in particular, $v|_{\Gamma_s} = \Psi_z$. By Green's theorem we conclude that

$$\begin{aligned}
&\iint_{\mathbf{R}^3} \left[\frac{1}{\varepsilon}\operatorname{curl} v \cdot \operatorname{curl}\psi - \omega^2 \mu\, v \cdot \psi\right] dx \\
&\quad = \iint_{\mathbf{R}^3} \frac{1}{\varepsilon(x)} \big[\operatorname{curl} v(x) - k_+^2 \operatorname{curl}\big(\phi(x)\, \mathbf{G}(z,x)p\big)\big] \cdot \operatorname{curl}\psi(x)\, dx \\
&\quad = \iint_{\mathbf{R}^3} \frac{1}{\varepsilon}\, f \cdot \operatorname{curl}\psi\, dx
\end{aligned}$$

with

$$f(x) = \operatorname{curl}\big[v(x) - k_+^2 \phi(x)\, \mathbf{G}(z,x)p\big], \quad x \in \mathbf{R}^3.$$

We note that f vanishes for $|x-z| \geq \rho$, and thus outside of Ω, which yields the variational form (17) of

$$v(x) = \iint_\Omega \big[\operatorname{curl}_y \mathbf{G}(x,y)\big]^\top f(y)\, dy, \quad x \in \mathbf{R}^3 \setminus (\mathbf{R}_0^3 \cup \partial\Omega),$$

by Lemma 3.1. In particular, we have that $\Psi_z = v|_{\Gamma_s} = \mathcal{H}_{\Gamma_s}^t f$.

Let now $z \notin \Omega$ and assume, on the contrary, that there exists $f \in L^2(\Omega, \mathbf{C}^3)$ with $\Psi_z = \mathcal{H}_{\Gamma_s}^t f$. From the assumption on Γ_s and the analyticity of Ψ_z and $\mathcal{H}_{\Gamma_s}^t f$ we conclude that $k_+^2\, \mathbf{G}(z,x)p$ and

$$v(x) := \iint_\Omega \big[\operatorname{curl}_y \mathbf{G}(x,y)\big]^\top f(y)\, dy$$

coincide for $x \in \mathbf{R}^3$ with $x_3 = h$. Since both functions satisfy the radiation condition they coincide for $x_3 > h$ and, by unique continuation, on $\mathbf{R}^3 \setminus \big(\Omega \cup \{z\}\big)$. This is a contradiction since $v \in H(\operatorname{curl}, B)$ for any ball containing z by Lemma 3.1 but $\big(\mathbf{G}(z,\cdot)p\big)|_B \notin L^2(B, \mathbf{C}^3)$ by the strong singularity at z. □

This result characterizes the unknown region Ω by the range of the (still unknown) operator $\mathcal{H}_{\Gamma_s}^t$ which is linked to the known operator

$\mathcal{M}$ through the factorization (32). Unfortunately, it is not known to us how to express the range of $\mathcal{H}^t_{\Gamma_s}$ by $\mathcal{M}$. Therefore, we introduce the auxiliary operators $\widetilde{\mathcal{H}}_{\Gamma_s} : L^2(\Gamma_s, \mathbf{C}^3) \to L^2(\Omega, \mathbf{C}^3)$ and $\widetilde{\mathcal{M}} : L^2(\Gamma_s, \mathbf{C}^3) \to L^2(\Gamma_s, \mathbf{C}^3)$ by

$$(\widetilde{\mathcal{H}}_{\Gamma_s}\varphi)(x) = k_+^2 \operatorname{curl} \int_{\Gamma_s} \overline{\mathbf{G}(y,x)}\, \varphi(y)\, ds(y), \quad x \in \Omega, \tag{38}$$

and

$$(\widetilde{\mathcal{M}}\varphi)(x) = k_+^2 \int_{\Gamma_s} \widetilde{H}^s(x,y)\, \varphi(y)\, ds(y), \quad x \in \Gamma_s, \tag{39}$$

respectively, where $\overline{\mathbf{G}(y,x)}$ denotes the complex conjugate of $\mathbf{G}(y,x)$ and $\widetilde{H}^s(\cdot,y)$ denotes the scattered field corresponding to incident field $\overline{\mathbf{G}(y,\cdot)}$.

Denoting by $\widetilde{\mathcal{H}}^*_{\Gamma_s}$ the L^2-adjoint of $\widetilde{\mathcal{H}}_{\Gamma_s}$ one shows the following factorization by exactly the same arguments as above:

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{H}}^*_{\Gamma_s} \mathcal{T}\, \widetilde{\mathcal{H}}_{\Gamma_s} \tag{40}$$

where $\mathcal{T} : L^2(\Omega, \mathbf{C}^3) \to L^2(\Omega, \mathbf{C}^3)$ has been defined in (33). From this we observe that

$$\operatorname{Im} \widetilde{\mathcal{M}} = \widetilde{\mathcal{H}}^*_{\Gamma_s} (\operatorname{Im} \mathcal{T})\, \widetilde{\mathcal{H}}_{\Gamma_s}$$

where the imaginary part of an operator A is defined as $\operatorname{Im} A = (A - A^*)/(2i)$. In Theorem 4.1 we have shown that $\operatorname{Im} \mathcal{T}$ is coercive. Since it is also self adjoint there exists a self adjoint and boundedly invertible operator $W : L^2(\Omega, \mathbf{C}^3) \to L^2(\Omega, \mathbf{C}^3)$ with $W^2 = \operatorname{Im} \mathcal{T}$ and thus

$$\operatorname{Im} \widetilde{\mathcal{M}} = \left[W\, \widetilde{\mathcal{H}}_{\Gamma_s}\right]^* \left[W\, \widetilde{\mathcal{H}}_{\Gamma_s}\right]. \tag{41}$$

Using a singular system of $W\, \widetilde{\mathcal{H}}_{\Gamma_s}$ it is easily seen, see [**14**, Theorem 4.1] that the ranges of $\left[W\, \widetilde{\mathcal{H}}_{\Gamma_s}\right]^*$ and $(\operatorname{Im} \widetilde{\mathcal{M}})^{1/2}$ coincide. Since W is an isomorphism from $L^2(\Omega, \mathbf{C}^3)$ onto itself and $\widetilde{\mathcal{H}}^*_{\Gamma_s} = \mathcal{H}^t_{\Gamma_s}$ we can combine this result with Theorem 4.2 and arrive at the following theorem.

Theorem 4.3. *For any $z \in \mathbf{R}^3$ and fixed $p \in \mathbf{C}^3$ we define $\Psi_z \in L^2(\Gamma_s, \mathbf{C}^3)$ by*

$$\Psi_z(x) = k_+^2\, \mathbf{G}(z,x)p, \quad \textit{for } x \in \Gamma_s. \tag{42}$$

Then $z \in \Omega$ *if and only if* $\Psi_z \in \mathcal{R}\big((\operatorname{Im}\widetilde{\mathcal{M}})^{1/2}\big)$.

The drawback of this result is that it is formulated with respect to the operator $\widetilde{\mathcal{M}}$ rather than $\mathcal{M}$ itself. In the next section we will express $\widetilde{\mathcal{M}}$ by $\mathcal{M}$.

5. Approximation of the auxiliary operator. In addition to the *assumptions* of Section 4 we assume that also $\Gamma_i \subset \{x \in \mathbf{R}^3_+ : x_3 = \hat{h}\}$ is part of a plane with nonempty relative interior (for some $\hat{h} > 0$). Furthermore, we assume that we know an open and bounded region $R \subset \mathbf{R}^3_-$ with smooth boundary ∂R such that $\overline{\Omega} \subset R$ and $\mathbf{R}^3_- \setminus \overline{R}$ is connected and k_-^2 is not an eigenvalue of the problem

$$\begin{aligned} \operatorname{curl}^2 v - k_-^2 v = 0 \ \text{ in } \ R, \\ \nu \times \operatorname{curl} v = 0 \ \text{ on } \ \partial R. \end{aligned}$$

The auxiliary data $\widetilde{H}^s(\cdot, y)$ belong to the, physically meaningless, incident fields $\overline{\mathbf{G}(y,\cdot)}$. It is the aim of this section to approximate the corresponding operator $\widetilde{\mathcal{M}}$ by known quantities. In particular, we will construct a family $P_\delta : L^2(\Gamma_s, \mathbf{C}^3) \to L^2(\Gamma_i, \mathbf{C}^3)$ of bounded operators with

$$\lim_{\delta \to 0} \mathcal{M} P_\delta \varphi = \widetilde{\mathcal{M}} \varphi \ \text{ in } \ L^2(\Gamma_s, \mathbf{C}^3)$$

for all $\varphi \in L^2(\Gamma_s, \mathbf{C}^3)$. To do this we introduce the operators $\mathcal{V}_{\Gamma_i} : L^2(\Gamma_i, \mathbf{C}^3) \to L^2_t(\partial R)$ and $\widetilde{\mathcal{V}}_{\Gamma_s} : L^2(\Gamma_s, \mathbf{C}^3) \to L^2_t(\partial R)$ by, respectively,

$$\begin{aligned} (\mathcal{V}_{\Gamma_i}\varphi)(x) = \nu(x) \times \operatorname{curl} \int_{\Gamma_i} \mathbf{G}(y,x)\, \varphi(y)\, ds(y) \\ \times \nu(x), \quad x \in \partial R, \end{aligned} \tag{43a}$$

$$\begin{aligned} (\widetilde{\mathcal{V}}_{\Gamma_s}\varphi)(x) = \nu(x) \times \operatorname{curl} \int_{\Gamma_s} \overline{\mathbf{G}(y,x)}\, \varphi(y)\, ds(y) \\ \times \nu(x), \quad x \in \partial R. \end{aligned} \tag{43b}$$

Here, $L^2_t(\partial R) = \big\{ f \in L^2(\partial R, \mathbf{C}^3) : \nu \cdot f = 0 \text{ on } \partial R \big\}$ denotes the space of tangential vectors fields on ∂R.

Lemma 5.1. *The ranges* $\mathcal{R}\big(\mathcal{V}_{\Gamma_i}\big)$ *and* $\mathcal{R}\big(\widetilde{\mathcal{V}}_{\Gamma_s}\big)$ *are dense in* $L^2_t(\partial R)$.

Proof. It is sufficient to show that the nullspace $\mathcal{N}(\mathcal{V}_{\Gamma_i}^*)$ of $\mathcal{V}_{\Gamma_i}^*$ is trivial. The proof for $\mathcal{R}(\widetilde{\mathcal{V}}_{\Gamma_s})$ follows then by the same arguments.

Therefore, let $f \in \mathcal{N}(\mathcal{V}_{\Gamma_i}^*)$ and define $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3)$ by

$$v(x) = \int_{\partial R} \big[\mathrm{curl}_y \mathbf{G}(x,y)\big]^\top \overline{f(y)}\, ds(y), \quad x \in \mathbf{R}^3.$$

Then $\overline{v}|_{\Gamma_i} = \mathcal{V}_{\Gamma_i}^* f = 0$. Therefore, by analytic continuation on $x_3 = \hat{h}$, uniqueness of the problem for $x_3 > \hat{h}$, and unique continuation one concludes that v vanishes in $\mathbf{R}^3 \setminus \overline{\Omega}$. The jump conditions are the same as for the homogeneous space, i.e., for

$$\hat{v}(x) = \int_{\partial R} \big[\mathrm{curl}_y \widehat{\mathbf{G}}(x,y)\big]^\top \overline{f(y)}\, ds(y), \quad x \in \mathbf{R}^3,$$

with $\widehat{\mathbf{G}}(x,y)$ from (19). Writing this again as

$$\hat{v}(x) = \int_{\partial R} \big[\mathrm{curl}_y \big(\Phi_-(x,y) I\big)\big] \overline{f(y)}\, ds(y) = \mathrm{curl} \int_{\partial R} \Phi_-(x,y)\, \overline{f(y)}\, ds(y),$$

we note that, see [**6**] for smooth f,

$$\nu \times (v|_+ - v|_-) = -\overline{f} \quad \text{and} \quad \nu \times (\mathrm{curl}\, v|_+ - \mathrm{curl}\, v|_-) = 0 \quad \text{on} \quad \partial R.$$

In particular, $\nu \times \mathrm{curl}\, v|_- = 0$ on ∂R and thus $v = 0$ in R. This yields that $\overline{f} = \nu \times v|_- = 0$ on ∂R. ◻

We now define $P_\delta : L^2(\Gamma_s, \mathbf{C}^3) \to L^2(\Gamma_i, \mathbf{C}^3)$ by

$$P_\delta = \big[\delta\, I + \mathcal{V}_{\Gamma_i}^* \mathcal{V}_{\Gamma_i}\big]^{-1} \mathcal{V}_{\Gamma_i}^* \tilde{\mathcal{V}}_{\Gamma_s} \quad \text{for} \quad \delta > 0. \tag{44}$$

We note that $\varphi_\delta = P_\delta \varphi$ is the Tikhonov regularization of the equation

$$\mathcal{V}_{\Gamma_i} \psi = \widetilde{\mathcal{V}}_{\Gamma_s} \varphi,$$

i.e., the solution of

$$\big[\delta\, I + \mathcal{V}_{\Gamma_i}^* \mathcal{V}_{\Gamma_i}\big] \varphi_\delta = \mathcal{V}_{\Gamma_i}^* \widetilde{\mathcal{V}}_{\Gamma_s} \varphi.$$

From the theory of integral equations of the first kind, see e.g., [**11**], we observe that $\mathcal{V}_{\Gamma_i} P_\delta \varphi$ converges to $\widetilde{\mathcal{V}}_{\Gamma_s}\varphi$ in $L^2_t(\partial R)$ for every $\varphi \in L^2(\Gamma_s, \mathbf{C}^3)$ as δ tends to zero although $P_\delta\varphi$ itself will not converge since the equation $\mathcal{V}_{\Gamma_i}\psi = \widetilde{\mathcal{V}}_{\Gamma_s}\varphi$ is not solvable in general as is easily seen.

Since both

$$x \longmapsto k_+^2 \int_{\Gamma_i} \mathbf{G}(x,y)\,(P_\delta\varphi)(y)\,ds(y) \text{ and } x \longmapsto k_+^2 \int_{\Gamma_s} \overline{\mathbf{G}(x,y)}\,\varphi(y)\,ds(y)$$

solve $\operatorname{curl}^2 v - k_-^2 v = 0$ in R the continuous dependence result yields that $\mathcal{H}_{\Gamma_i} P_\delta\varphi$ converges to $\widetilde{\mathcal{H}}_{\Gamma_s}\varphi$ in $L^2(\Omega, \mathbf{C}^3)$ as δ tends to zero. Therefore,

$$\mathcal{M} P_\delta \varphi = \mathcal{H}^t_{\Gamma_s} \mathcal{T} \mathcal{H}_{\Gamma_i} P_\delta \varphi \longrightarrow \mathcal{H}^t_{\Gamma_s} \mathcal{T} \widetilde{\mathcal{H}}_{\Gamma_s} \varphi = \widetilde{\mathcal{M}}\varphi$$

since $\mathcal{H}^t_{\Gamma_s} = \widetilde{\mathcal{H}}^*_{\Gamma_s}$. Therefore, we derived the following final result.

Theorem 5.2. *Let the family of operators* $P_\delta : L^2(\Gamma_s, \mathbf{C}^3) \rightarrow L^2(\Gamma_i, \mathbf{C}^3)$ *be defined by* (44) *where* $\mathcal{V}_{\Gamma_i}$ *and* $\widetilde{\mathcal{V}}_{\Gamma_s}$ *are given by* (43a) *and* (43b)*, respectively. For any* $z \in \mathbf{R}^3$ *and fixed* $p \in \mathbf{C}^3$ *we define* $\Psi_z \in L^2(\Gamma_s, \mathbf{C}^3)$ *by*

$$\Psi_z(x) = k_+^2\,\mathbf{G}(z,x)p, \quad \text{for } x \in \Gamma_s. \tag{45}$$

Then $z \in \Omega$ *if and only if* $\Psi_z \in \mathcal{R}\big((\operatorname{Im}\widetilde{\mathcal{M}})^{1/2}\big)$ *where* $\widetilde{\mathcal{M}}\varphi = \lim_{\delta\to 0} \mathcal{M} P_\delta \varphi$ *for* $\varphi \in L^2(\Gamma_s, \mathbf{C}^3)$.

We finish this paper with some remarks.

First we note that from the pointwise convergence $\widetilde{\mathcal{M}}\varphi = \lim_{\delta\to 0} \times \mathcal{M} P_\delta \varphi$ for all $\varphi \in L^2(\Gamma_s, \mathbf{C}^3)$ we cannot conclude that also $\operatorname{Im}(\mathcal{M} P_\delta)$ converges pointwise. To overcome this problem we can choose some arbitrary compact operator K from $L^2(\Gamma_s, \mathbf{C}^3)$ into itself which is one-to-one with dense range. Then $K^*\mathcal{M}P_\delta K$ converges to $K^*\widetilde{\mathcal{M}}K$ with respect to the operator norm and, therefore, also $\operatorname{Im}(K^*\widetilde{\mathcal{M}}P_\delta K)$ converges. Now we can argue as before in Theorem 5.2 which yields that for any fixed $p \in \mathbf{C}^3$ a given point $z \in \mathbf{R}^3$ belongs to Ω if, and only if, $K\Psi_z \in \mathcal{R}\big((\operatorname{Im}\widehat{\mathcal{M}})^{1/2}\big)$ where $\widehat{\mathcal{M}} = \lim_{\delta\to 0} K^*\mathcal{M}P_\delta K$.

Second, for many inverse scattering problems where the data consists of near field measurements corresponding to point sources as incident fields the corresponding data operator $\mathcal{M}$ can only be factorized in the form (32) rather than in the form (40). We refer to [**1, 8, 22**] for some examples. In none of theses cases the computation of the operator $\widetilde{\mathcal{M}}$ by $\mathcal{M}P_\delta$ has been carried out numerically. A careful numerical investigation is subject of future research.

As a third remark we observe that the analysis also works for the case where only the tangential components of $\mathcal{M}\varphi(x)$ are measured and only tangential polarizations $\varphi(x)$ are used for the incident fields. One has to replace the spaces $L^2(\Gamma_i, \mathbf{C}^3)$ and $L^2(\Gamma_s, \mathbf{C}^3)$ by the spaces $L_t^2(\Gamma_i)$ and $L_t^2(\Gamma_s)$, respectively, of tangential fields. The function Ψ_z, defined in (37), has to be replaced by its tangential component on Γ_s. Then the assertions of Theorem 4.2 and also Theorems 4.3 and 5.2 still hold.

ENDNOTES

1. "HuMin/MD—Metal detectors for humanitarian demining—Development potentials in data analysis methodology and measurement."

2. Note that we interchange the roles of x and y in the following.

3. The vector e_j denotes the jth coordinate unit vector in $\mathbf{R}^3$.

4. Actually, we show only one part of it.

REFERENCES

1. T. Arens and A. Kirsch, *The factorization method in inverse scattering from periodic structures*, Inverse Problems **19** (2003), 1195–1211.

2. F. Collino, M'B. Fares and H. Haddar, *Numerical and analytical studies of the linear sampling method in electromagnetic inverse scattering problems*, Inverse Problems **19** (2003), 1279–1298.

3. D. Colton, H. Haddar and P. Monk, *The linear sampling method for solving the electromagnetic inverse scattering problem*, SIAM J. Sci. Comput. **24** (2002), 719–731.

4. D. Colton, H. Haddar and M. Piana, *The linear sampling method in inverse* *electromagnetic scattering theory*, Inverse Problems **19** (2003), S105–S137.

5. D. Colton and A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Problems **12** (1996), 383–393.

6. D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering problems*, 2nd edition, Springer-Verlag, New York, 1998.

7. P.-M. Cutzach and C. Hazard, *Existence, uniqueness and analyticity properties for electromagnetic scattering in a two-layered medium*, Math. Meth. Appl. Sci. **21** (1998), 433–461.

8. B. Gebauer, M. Hanke, A. Kirsch, W. Muniz and C. Schneider, *A sampling method for detecting buried objects using electromagnetic scattering*, Inverse Problems **21** (2005), 2035–2050.

9. H. Haddar, *The interior transmission problem for anisotropic maxwell's equations and its applications to the inverse problem*, Math. Meth. Appl. Sci. **27** (2004), 2111–2129.

10. H. Haddar and P. Monk, *The linear sampling method for solving the electromagnetic inverse medium problem*, Inverse Problems **18** (2002), 891–906.

11. A. Kirsch, *Introduction to the mathematical theory of inverse problems*, Springer-Verlag, New York, 1998.

12. ———, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems **14** (1998), 1489–1512.

13. ———, *Factorization of the far field operator for the inhomogeneous medium case and an application in inverse scattering theory*, Inverse Problems **15** (1999), 413–429.

14. ———, *The MUSIC-algorithm and the factorization method in inverse scattering theory for inhomogeneous media*, Inverse Problems **18** (2002), 1025–1040.

15. ———, *the factorization method for Maxwell's equations*, Inverse Problems **20** (2004), S117–S134.

16. A Kirsch and P. Monk, *A finite element/spectral method for approximating the time harmonic Maxwell system in* $\mathbf{R}^n$, SIAM J. Appl. Math. **55** (1995), 1324–1344.

17. R. Kress, *Electromagnetic wave scattering*: *Theory* in *Scattering*, R. Pike and P. Sabatier, eds., Academic Press, London, 2002.

18. G. Kristensson, *A uniqueness theorem for Helmholtz equation*: *Penetrable media with an infinite interface*, SIAM J. Math. Anal. **11** (1980), 1104–1117.

19. W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.

20. P. Monk, *Finite element methods for Maxwell's equations*, Oxford Science Publications, Oxford, 2003.

21. J.-C. Nédélec, *Acoustic and electromagnetic equations*, Springer, New York, 2001.

22. M. Orispää, *On point sources and near field measurements in inverse acoustic obstacle scattering*, PhD thesis, University of Oulu, Finland, 2002.

23. R. Potthast, *A survey on sampling and probe methods for inverse problems*, Inverse Problems **22** (2006), R1–R47.

24. J.A. Stratton, *Electromagnetic theory*, McGraw-Hill, New York, 1941.

25. C. Weber, *Regularity theorems for Maxwell's equations*, Math. Meth. Appl. Sci. **3** (1981), 523–536.

University of Karlsruhe, Department of Mathematics, 76128 Karlsruhe, Germany
Email address: kirsch@math.uni-karlsruhe.de