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A NOTE ON SOLUTIONS IN $L^1[0,1]$ TO HAMMERSTEIN INTEGRAL EQUATIONS

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ABSTRACT. In this paper we seek solutions in $L^1[0,1]$ to Hammerstein integral equations. Some general existence principles are derived.

1. Introduction. We present some existence principles for the Hammerstein integral equation

$$(1.1)\qquad y(t) = g(t) + \int_0^1 k(t,s) f(s, y(s))\, ds \quad \text{a.e. } t \in [0,1].$$

Throughout we have $k : [0,1] \times [0,1] \to \mathbf{R}$ and $f : [0,1] \times \mathbf{R} \to \mathbf{R}$. In this paper we are mostly interested in solutions which lie in $L^1[0,1]$ (we will for completeness also discuss the case $L^p[0,1]$, $1 < p < \infty$). Banas [**2, 3**] and Emmanuele [**8, 9**] have examined this type of problem extensively over the last ten years or so. Their analyses rely on the notion of measures of weak noncompactness and on the Schauder fixed point theorem. However, in this paper we will use old compactness results of Riesz and Komogorov to establish some very general existence principles (and theory) for (1.1). Our proofs are elementary and follow classical type arguments. It is worth remarking here also that essentially the same reasoning would establish existence principles for Volterra and even Urysohn integral equations.

To conclude the introduction, we gather together some results which will be used frequently in Section 2. We first state the compactness criteria of Riesz and Kolmogorov, see [**1, 4, 5, 7, 11**].

Theorem 1.1 (Riesz). *Let* $\Omega \subseteq L^p[0,1]$, $1 \le p < \infty$. *If*

(i) Ω *is bounded in* $L^p[0,1]$,

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(ii) $\int_0^1 |u(t+h)-u(t)|^p\,dt$ *as* $h \to 0$ *uniformly with respect to* $u \in \Omega$, *then* Ω *is relatively compact in* $L^p[0,1]$.

Theorem 1.2 (Kolmogorov). *Let* $\Omega \subseteq L^p[0,1]$, $1 \le p < \infty$. *If*

(i) Ω *is bounded in* $L^p[0,1]$,

(ii) $u_h \to u$ (*convergence in* $L^p[0,1]$) *as* $h \to 0$ *uniformly with respect to* $u \in \Omega$, *then* Ω *is relatively compact in* $L^p[0,1]$; *here*

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)\,ds.$$

Remark. In both Theorems 1.1 and 1.2 it is agreed that $u \in \Omega$ is extended to an interval $(a,b) \supseteq [0,1]$ by letting $u(t)=0$ outside $[0,1]$.

Next we state a nonlinear alternative [**6**] of Leray-Schauder type which will be used in Section 2.

Theorem 1.3. *Let* U *be an open subset of a convex set* K *in a Banach space* E. *Assume* $0 \in U$ *and that* $N : \overline{U} \to K$ *is a continuous, compact map. Then either*

(A1) N *has a fixed point in* $\overline{U}$, *or*

(A2) *there exists* $\lambda \in (0,1)$ *and* $u \in \partial U$ *such that* $u = \lambda N u$.

For notational purposes for $1 \le p < \infty$, let $L^p[0,1]$ denote the Banach space of pth-power integrable functions with $\|u\|_p = (\int_0^1 |u|^p\,dt)^{1/p}$. Also, $L^\infty[0,1]$ denotes the Banach space of essentially bounded measurable functions together with the essential supremum norm (denoted by $\|\cdot\|_\infty$).

2. Existence. Throughout this section we assume f is a Carathéodory function; by this, we mean

(C1) for almost every $t \in [0,1]$, the map $z \mapsto f(t,z)$ is continuous;

(C2) for every $z \in \mathbf{R}$ the map $t \mapsto f(t,z)$ is measurable.

We now state and prove the main result in this paper.

Theorem 2.1. *Suppose the following conditions are satisfied*:

$$g \in L^1[0,1] \tag{2.1}$$

$$k : [0,1] \times [0,1] \to \mathbf{R} \text{ is measurable with respect to both variables and } \left\| \int_0^1 |k(t,s)|\,dt \right\|_\infty < \infty \tag{2.2}$$

$$\lim_{h\to 0} \int_0^1 \left\| \frac{1}{h} \int_t^{t+h} |k(x,s) - k(t,s)|\,dx \right\|_\infty dt = 0 \tag{2.3}$$

and

(2.4) $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ *is a Carathéodory function, and there exist* $a \in L^1[0,1]$, *a constant* $b \geq 0$ *with* $|f(t,u)| \leq a(t) + b|u|$ *for almost every* $t \in [0,1]$ *and all* $u \in \mathbf{R}$.

In addition, assume there is a constant M_0, *independent of* λ, *with*

$$\|y\|_1 \neq M_0 \tag{2.5}$$

for any solution y (*here* $y \in L^1[0,1]$) *to*

$$y(t) = \lambda\left(g(t) + \int_0^1 k(t,s) f(s, y(s))\,ds\right) \quad a.e.t \in [0,1] \tag{2.6_λ}$$

for each $\lambda \in (0,1)$. *Then* (1.1) *has at least one solution in* $L^1[0,1]$.

Proof. Define the operators

$$K : L^1[0,1] \longrightarrow L^1[0,1] \quad \text{by} \quad Ku(t) = g(t) + \int_0^1 k(t,s)u(s)\,ds$$

$$K_0 : L^1[0,1] \to L^1[0,1] \quad \text{by} \quad K_0 u(t) = \int_0^1 k(t,s)u(s)\,ds$$

and

$$F : L^1[0,1] \longrightarrow L^1[0,1] \quad \text{by} \quad Fu(t) = f(t, u(t)).$$

A solution to (1.1) is a fixed point of the operator $N = KF : L^1[0,1] \to L^1[0,1]$. Also finding a $y \in L^1[0,1]$ which satisfies $(2.6)_\lambda$ is equivalent to solving the fixed point problem $y = \lambda N y$. A well known result of Krasnoselskii [**12**] implies (since (2.4) holds) that $F : L^1[0,1] \to L^1[0,1]$ is continuous and bounded. We now show $K : L^1[0,1] \to L^1[0,1]$ is continuous and completely continuous. The fact that K is continuous will follow once we show the linear operator $K_0 : L^1[0,1] \to L^1[0,1]$ is continuous. Let $u \in L^1[0,1]$. Then

$$\begin{aligned}\int_0^1 \int_0^1 |k(t,s)|\,|u(s)|\,ds\,dt &= \int_0^1 |u(s)| \int_0^1 |k(t,s)|\,dt\,ds \\ &\le \|u\|_1 \bigg\| \int_0^1 |k(t,s)|\,dt \bigg\|_\infty\end{aligned}$$

and so

$$\|K_0 u\|_1 \le \|u\|_1 \left(\bigg\| \int_0^1 |k(t,s)|\,dt \bigg\|_\infty \right).$$

Hence $K_0 : L^1[0,1] \to L^1[0,1]$ (and so $K : L^1[0,1] \to L^1[0,1]$) is continuous. To show $K : L^1[0,1] \to L^1[0,1]$ is completely continuous, we apply Theorem 1.2. Let Ω be a bounded subset of $L^1[0,1]$, i.e., there exists M with $\|v\|_1 \le M$ for all $v \in \Omega$. Let $u \in \Omega$. Then

$$\begin{aligned}\|Ku\|_1 &= \int_0^1 \left| g(t) + \int_0^1 k(t,s)u(s)\,ds \right| dt \\ &\le \int_0^1 |g(t)|\,dt + \int_0^1 \int_0^1 |k(t,s)|\,|u(s)|\,ds \\ &\le \int_0^1 |g(t)|\,dt + \|u\|_1 \bigg\| \int_0^1 |k(t,s)|\,dt \bigg\|_\infty \\ &\le \int_0^1 |g(t)|\,dt + M \bigg\| \int_0^1 |k(t,s)|\,dt \bigg\|_\infty,\end{aligned}$$

and so $K(\Omega)$ is bounded in $L^1[0,1]$. Next we will show $(Ku)_h \to Ku$ (convergence in $L^1[0,1]$) as $h \to 0$, uniformly with respect to $u \in \Omega$.

To see this, notice

$$\begin{aligned}
\int_0^1 |(Ku)_h(t) - Ku(t)|\,dt \\
&= \int_0^1 \left| \frac{1}{h} \int_t^{t+h} Ku(x)\,dx - Ku(t) \right| dt \\
&= \int_0^1 \left| \frac{1}{h} \int_t^{t+h} [Ku(x) - Ku(t)]\,dx \right| dt \\
&\leq \int_0^1 \frac{1}{h} \int_t^{t+h} |g(x) - g(t)|\,dx\,dt \\
&\quad + \int_0^1 \frac{1}{h} \int_t^{t+h} \int_0^1 |k(x,s) - k(t,s)|\,|u(s)|\,ds\,dx\,dt \\
&= \int_0^1 \frac{1}{h} \int_t^{t+h} |g(x) - g(t)|\,dx\,dt \\
&\quad + \int_0^1 \int_0^1 |u(s)| \left(\frac{1}{h} \int_t^{t+h} |k(x,s) - k(t,s)|\,dx \right) ds\,dt \\
&\leq \int_0^1 \frac{1}{h} \int_t^{t+h} |g(x) - g(t)|\,dx\,dt \\
&\quad + \|u\|_1 \int_0^1 \left\| \frac{1}{h} \int_t^{t+h} |k(x,s) - k(t,s)|\,dx \right\|_\infty dt \\
&\leq \int_0^1 \frac{1}{h} \int_t^{t+h} |g(x) - g(t)|\,dx\,dt \\
&\quad + M \int_0^1 \left\| \frac{1}{h} \int_t^{t+h} |k(x,s) - k(t,s)|\,dx \right\|_\infty dt.
\end{aligned}$$

From [**13**, Chapter 4], since $g \in L^1[0,1]$, we have

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} |g(x) - g(t)|\,dx = 0 \quad \text{for a.e. } t \in [0,1]$$

and so

$$\lim_{h \to 0} \int_0^1 \left(\frac{1}{h} \int_t^{t+h} |g(x) - g(t)|\,dx \right) dt = 0, \tag{2.7}$$

since

$$\int_0^1 \frac{1}{h} \int_t^{t+h} |g(x) - g(t)|\, dx\, dt = \int_0^h \frac{1}{h} \int_0^1 |g(t+x) - g(t)|\, dt\, dx$$
$$\leq \sup_{0<s<h} \|g(\cdot + s) - g(\cdot)\|_1.$$

Now (2.3) and (2.7) imply

$$(Ku)_h \longrightarrow Ku \text{ (convergences in } L^1[0,1]) \text{ as } h \longrightarrow 0,$$

uniformly with respect to $u \in \Omega$. Consequently, $K : L^1[0,1] \to L^1[0,1]$ is completely continuous. As a result, $N = KF : L^1[0,1] \to L^1[0,1]$ is continuous and completely continuous. Set

$$U = \{u \in L^1[0,1] : \|u\|_1 < M_0\}, \qquad K = E = L^1[0,1].$$

Now Theorem 1.3 implies that N has a fixed point (notice (A2) in Theorem 1.3 does not occur since (2.5) holds). □

Next we discuss the case when solutions to (1.1) lie in $L^p[0,1]$, $p \geq 1$.

Theorem 2.2. *Let $1 \leq p < \infty$ and $1 < q < \infty$. Suppose the following conditions are satisfied*:

(2.8) $$g \in L^p[0,1],$$

(2.9) *$k : [0,1] \times [0,1] \to \mathbf{R}$ is measurable with respect to both variables*

(2.10) $$\int_0^1 \left(\int_0^1 |k(t,s)|^r\, ds \right)^{p/r} dt < \infty;$$

here r is the conjugate of q and

(2.11) *$f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function and there exists $a \in L^q[0,1]$, a constant $b \geq 0$ with $|f(t,u)| \leq a(t) + b|u|^{p/q}$ for almost every $t \in [0,1]$ and all $u \in \mathbf{R}$.*

In addition, assume there is a constant M_0, independent of λ, with

$$\|y\|_p \neq M_0 \tag{2.12}$$

for any solution y (here $y \in L^p[0,1]$) to $(2.6)_\lambda$ for each $\lambda \in (0,1)$. Then (1.1) has at least one solution in $L^p[0,1]$.

Remark. The case $q = 1$ is discussed in Theorem 2.3. Of course, the case $p = q = 1$ was already discussed in Theorem 2.1.

Proof. Let $K, K_0 : L^q[0,1] \to L^p[0,1]$ and $F : L^p[0,1] \to L^q[0,1]$ be given as in Theorem 2.1. A solution to (1.1) is a fixed point of the operator $N = KF : L^p[0,1] \to L^p[0,1]$. A result of Krasnoselskii [**12**] implies $F : L^p[0,1] \to L^q[0,1]$ is continuous and bounded. Essentially the same reasoning as in Theorem 2.1 establishes the result once we show $K : L^q[0,1] \to L^p[0,1]$ is continuous and completely continuous. Let $u \in L^q[0,1]$. Then

$$\begin{aligned}\int_0^1 \left(\int_0^1 |k(t,s)|\,|u(s)|\,ds \right)^p dt &\\ \leq \int_0^1 \left(\int_0^1 |u(s)|^q\,ds \right)^{p/q} \left(\int_0^1 |k(t,s)|^r\,ds \right)^{p/r} dt &\\ = \|u\|_q^p \int_0^1 \left(\int_0^1 |k(t,s)|^r\,ds \right)^{p/r} dt &\end{aligned}$$

and so

$$\|K_0 u\|_p \leq \|u\|_q \left(\int_0^1 \left(\int_0^1 |k(t,s)|^r\,ds \right)^{p/r} dt \right)^{1/p}.$$

Thus $K : L^q[0,1] \to L^p[0,1]$ is continuous. To show $K : L^q[0,1] \to L^p[0,1]$ is completely continuous, we apply Theorem 1.1. Let Ω be a bounded subset $L^q[0,1]$, i.e., there exists M with $\|v\|_q \leq M$ for all

$v \in \Omega$. Let $u \in \Omega$. Then

$$\begin{aligned}\|Ku\|_p^p &= \int_0^1 \left| g(t) + \int_0^1 k(t,s)u(s)\,ds \right|^p dt \\ &\le 2^{p-1} \int_0^1 |g(t)|^p\,dt \\ &\quad + 2^{p-1} \int_0^1 \left(\int_0^1 |k(t,s)|\,|u(s)|\,ds \right)^p dt \\ &\le 2^{p-1} \int_0^1 |g(t)|^p\,dt \\ &\quad + 2^{p-1} \|u\|_q^p \int_0^1 \left(\int_0^1 |k(t,s)|^r\,ds \right)^{p/r} dt,\end{aligned}$$

and so $K(\Omega)$ is bounded in $L^p[0,1]$. Next we show

$$\int_0^1 |Ku(t+h) - Ku(t)|^p\,dt \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0,$$

uniformly with respect to $u \in \Omega$. To see this, notice

$$\begin{aligned}&\int_0^1 |Ku(t+h) - Ku(t)|^p\,dt \\ &\qquad \le 2^{p-1} \int_0^1 |g(t+h) - g(t)|^p\,dt \\ &\qquad\quad + 2^{p-1} \int_0^1 \left(\int_0^1 |k(t+h,s) - k(t,s)| |u(s)|\,ds \right)^p dt \\ &\qquad \le 2^{p-1} \int_0^1 |g(t+h) - g(t)|^p\,dt \\ &\qquad\quad + 2^{p-1} M^p \int_0^1 \left(\int_0^1 |k(t+h,s) - k(t,s)|^r\,ds \right)^{p/r} dt \\ &\qquad \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0,\end{aligned}$$

see [**14**, Chapter 12]. Consequently, $K : L^q[0,1] \to L^p[0,1]$ is completely continuous. □

Theorem 2.3. *Let $1 < p < \infty$. Suppose the following conditions are satisfied:*

$$g \in L^p[0,1] \tag{2.13}$$

(2.14) $k : [0,1] \times [0,1] \to \mathbf{R}$ *is measurable with respect to both variables and* $\| \int_0^1 |k(t,s)|^p \, dt \|_\infty < \infty$,

$$\lim_{h\to 0} \left\| \int_0^1 |k(t+h,s) - k(t,s)|^p \, dt \right\|_\infty = 0, \tag{2.15}$$

and

(2.16) $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ *is a Carathéodory function and there exists* $a \in L^1[0,1]$*, a constant* $b \geq 0$ *with* $|f(t,u)| \leq a(t) + b|u|^p$ *for almost every* $t \in [0,1]$ *and all* $u \in \mathbf{R}$.

In addition, assume there is a constant M_0*, independent of* λ*, with*

$$\|y\|_p \neq M_0 \tag{2.17}$$

for any solution y *to* $(2.6)_\lambda$ *for each* $\lambda \in (0,1)$*. Then* (1.1) *has at least one solution in* $L^p[0,1]$.

Proof. Let $K, K_0 : L^1[0,1] \to L^p[0,1]$ and $F : L^p[0,1] \to L^1[0,1]$ be given as in Theorem 2.1. Essentially the same reasoning as in Theorem 2.1 establishes the result once we show $K : L^1[0,1] \to L^p[0,1]$ is continuous and completely continuous. Let $u \in L^1[0,1]$, and let m be the conjugate of p. Then

$$\begin{aligned}
\int_0^1 \left(\int_0^1 |k(t,s)| \, |u(s)| \, ds \right)^p dt
&= \int_0^1 \left(\int_0^1 |k(t,s)| \, |u(s)|^{1/p} |u(s)|^{1/m} \, ds \right)^p dt \\
&\leq \int_0^1 \int_0^1 |k(t,s)|^p |u(s)| \, ds \left(\int_0^1 |u(s)| \, ds \right)^{p/m} dt \\
&= \|u\|_1^{p/m} \int_0^1 \int_0^1 |k(t,s)|^p |u(s)| \, ds \, dt \\
&= \|u\|_1^{p/m} \int_0^1 |u(s)| \int_0^1 |k(t,s)|^p \, dt \, ds \\
&\leq \|u\|_1^{p/m} \|u\|_1 \left\| \int_0^1 |k(t,s)|^p \, dt \right\|_\infty \\
&= \|u\|_1^p \left\| \int_0^1 |k(t,s)|^p \, dt \right\|_\infty ,
\end{aligned}$$

and so

$$\|K_0 u\|_p \leq \|u\|_1 \left(\left\| \int_0^1 |k(t,s)|^p \, dt \right\|_\infty \right)^{1/p}.$$

Thus, $K : L^1[0,1] \to L^p[0,1]$ is continuous. To show $K : L^1[0,1] \to L^p[0,1]$ is completely continuous, we apply Theorem 1.1. Let Ω be a bounded subset of $L^1[0,1]$, i.e., there exists M with $\|v\|_1 \leq M$ for all $v \in \Omega$. Let $u \in \Omega$. Then

$$\begin{aligned}
\|Ku\|_p^p &\leq 2^{p-1} \int_0^1 |g(t)|^p \, dt \\
&\quad + 2^{p-1} \int_0^1 \left(\int_0^1 |k(t,s)| \, |u(s)| \, ds \right)^p dt \\
&\leq 2^{p-1} \int_0^1 |g(t)|^p \, dt \\
&\quad + 2^{p-1} \|u\|_1^p \left(\left\| \int_0^1 |k(t,s)|^p \, dt \right\|_\infty \right) \\
&\leq 2^{p-1} \int_0^1 |g(t)|^p \, dt \\
&\quad + 2^{p-1} M^p \left(\left\| \int_0^1 |k(t,s)|^p \, dt \right\|_\infty \right)
\end{aligned}$$

so $K(\Omega)$ is bounded in $L^p[0,1]$. Also

$$\begin{aligned}
\int_0^1 &|Ku(t+h) - Ku(t)|^p \, dt \\
&\leq 2^{p-1} \int_0^1 |g(t+h) - g(t)|^p \, dt \\
&\quad + 2^{p-1} \int_0^1 \left(\int_0^1 |k(t+h,s) - k(t,s)| \, |u(s)| \, ds \right)^p dt \\
&\leq 2^{p-1} \int_0^1 |g(t+h) - g(t)|^p \, dt \\
&\quad + 2^{p-1} M^p \left(\left\| \int_0^1 |k(t+h,s) - k(t,s)|^p \, dt \right\|_\infty \right).
\end{aligned}$$

Thus

$$\int_0^1 |Ku(t+h) - Ku(t)|^p \, dt \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0,$$

uniformly with respect to $u \in \Omega$. Consequently $K : L^1[0,1] \to L^p[0,1]$ is completely continuous. □

We can now use these existence principles to establish existence theory for (1.1). For completeness we give *one* application of Theorem 2.2.

Theorem 2.4. *Let* $1 \le p < \infty$ *and* $1 < q < \infty$. *Suppose* (2.8)–(2.11) *hold. In addition, assume*

(2.18)

$$\sup_{x\in[0,\infty]} \left(\frac{x}{2^{p-1}\int_0^1 |g|^p\,dt + 2^{2(p-1)}[(\int_0^1 a^q\,dt)^{p/q} + b^p x^{p/q}]\int_0^1(\int_0^1 |k(t,s)|^r\,ds)^{p/r}\,dt} \right) > 1$$

holds. Then (1.1) *has at least one solution in* $L^p[0,1]$.

Remark. If $q > p$, then (2.18) is satisfied since the lefthand side of (2.18) would be infinity.

Remark. If $q \le p$, then (2.18) is again satisfied if b is sufficiently small.

Proof. Let $M_1 > 0$ satisfy

(2.19)

$$\frac{M_1}{2^{p-1}\int_0^1 |g|^p\,dt + 2^{2(p-1)}[(\int_0^1 a^q\,dt)^{p/q} + b^p M_1^{p/q}]\int_0^1(\int_0^1 |k(t,s)|^r\,ds)^{p/r}\,dt} > 1.$$

Let $y \in L^p[0,1]$ be any solution of $(2.6)_\lambda$ for $0 < \lambda < 1$. Then, for almost every $t \in [0,1]$, we have

$$|y(t)|^p \le 2^{p-1}|g(t)|^p + 2^{p-1}\left(\int_0^1 |k(t,s)|[a(s) + b|y(s)|^{p/q}]\,ds\right)^p$$

and so

$$\begin{aligned}\int_0^1 |y|^p\,dt &\le 2^{p-1}\int_0^1 |g|^p\,dt \\ &\qquad + 2^{p-1}\Bigg[2^{p-1}\int_0^1 \left(\int_0^1 |k(t,s)|a(s)\,ds\right)^p dt \\ &\qquad\qquad + 2^{p-1}\int_0^1 \left(\int_0^1 |k(t,s)|b|y(s)|^{p/q}\,ds\right)^p dt\Bigg] \\ &\le 2^{p-1}\int_0^1 |g|^p\,dt \\ &\qquad + 2^{2(p-1)}\left(\int_0^1 a^q\,ds\right)^{p/q}\int_0^1\left(\int_0^1 |k(t,s)|^r\,ds\right)^{p/r} dt \\ &\qquad + 2^{2(p-1)}b^p\left(\int_0^1 |y|^p\,ds\right)^{p/q}\int_0^1\left(\int_0^1 |k(t,s)|^r\,ds\right)^{p/r} dt.\end{aligned}$$

Thus

(2.20)

$$\frac{\int_0^1 |y|^p\,dt}{2^{p-1}\int_0^1 |g|^p\,dt + 2^{2(p-1)}[(\int_0^1 a^q\,dt)^{p/q} + b^p(\int_0^1 |y|^p\,dt)^{p/q}]\int_0^1(\int_0^1 |k(t,s)|^r\,ds)^{p/r}\,dt} \le 1.$$

Let $M_0 = M_1^{1/p}$. Suppose $\|y\|_p = M_0$. Then (2.20) yields

$$\frac{M_1}{2^{p-1}\int_0^1 |g|^p\,dt + 2^{2(p-1)}[(\int_0^1 a^q\,dt)^{p/q} + b^p M_1^{p/q}]\int_0^1(\int_0^1 |k(t,s)|^r\,ds)^{p/r}\,dt} \le 1.$$

This contradicts (2.19). Thus, any solution y of $(2.6)_\lambda$ satisfies $\|y\|_p \ne M_0$. Theorem 2.2 implies that (1.1) has a solution $y \in L^p[0,1]$. □

Remark. It is worth noticing in the proof of Theorem 2.4 that we showed any solution of $(2.6)_\lambda$ satisfies $\|y\|_p \ne M_0$. We do *not* claim that any solution of $(2.6)_\lambda$ satisfies $\|y\|_p \le M_0$.

Finally we would like to remark that we could also obtain new results for the general operator equation

$$y(t) = Ny(t) \quad \text{a.e.} \quad t \in [0,1]. \tag{2.21}$$

For completeness we state the analogue of Theorem 2.1 in this situation (the proof is immediate).

Theorem 2.4. *Suppose*

$$N : L^1[0,1] \longrightarrow L^1[0,1] \tag{2.22}$$

is continuous and completely continuous. In addition, assume there is a constant M_0*, independent of* λ*, with*

$$\|y\|_1 \neq M_0$$

for any solution y *(here* $y \in L^1[0,1]$*) to*

$$y(t) = \lambda Ny(t) \quad a.e. \quad t \in [0,1]$$

for each $\lambda \in (0,1)$*. Then* (2.21) *has at least one solution in* $L^1[0,1]$.

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REFERENCES

1. R.A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

2. J. Banas, *Integrable solutions of Hammerstein and Urysohn integral equations*, J. Australian Math. Soc. **46** (1989) 61–68.

3. J. Banas and W.G. El-Sayed, *Measures of noncompactness and solvability of an integral equation in the class of functions of locally bounded variation*, J. Math. Anal. Appl. **167** (1992), 133–151.

4. H. Brezis, *Analyse fonctionnelle, theorie et applications*, Masson, Paris, 1993.

5. C. Corduneanu, *Integral equations and applications*, Cambridge Univ. Press, New York, 1990.

6. J. Dugundji and A. Granas, *Fixed point theory*, Monografie Mathematyczne, PWN, Warsaw, 1982.

7. N. Dunford and J.T. Schwartz, *Linear operators*, Interscience/Wiley, New York, 1958.

8. G. Emmanuele, *Integrable solutions of Hammerstein integral equations*, Appl. Anal. **50** (1993), 277–284.

9. ———, *Integrable solutions of a functional-integral equation*, J. Integral Equations Appl. **4** (1992), 89–94.

10. G. Gripenberg, S.O. Londen and O. Staffans, *Volterra integral and functional equations*, Cambridge Univ. Press, New York, 1990.

11. L.V. Kantorovich and G.P. Akilov, *Functional analysis*, Pergamon, Oxford, 1982.

12. M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Macmillan Co., New York, 1964.

13. C. Swartz, *Measure, integration and function spaces*, World Scientific, Singapore, 1994.

14. A. Torchinsky, *Real variables*, Addison Wesley, Redwood City, 1988.

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