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A STEFAN/MULLINS-SEKERKA TYPE PROBLEM WITH MEMORY

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ABSTRACT. Existence is proven for the system

(0.1)

$$a_1 u_t + a_2 w_t = \int_{-\infty}^{t} k(t-s)\Delta u(s)\,ds \quad (x,t) \in \Omega \times (0,T)$$

$$a_3 w_t = \Delta\mu \quad (x,t) \in \Omega \times (0,T)$$

$$\mu + 2u \in \partial\Gamma(w) \quad (x,t) \in \Omega \times (0,T)$$

where

$$\Gamma(w) := \begin{cases} \int_\Omega |\nabla w| < \infty & |w| \leq 1 \text{ a.e.} \\ \infty & \text{otherwise} \end{cases}$$

for arbitrary $T > 0$ on a smooth bounded domain $\Omega \subset \mathbf{R}^n$, $n = 1, 2$, or 3 via the inclusion of a relaxation dynamic, for initial data $(u, w) \in L^2(\Omega) \times BV(\Omega)$ and for "prehistory" $u_h \in L^2(\mathbf{R}^-; H^2(\Omega))$. Here u denotes the temperature field, w a conserved phase variable, and μ the chemical potential. Neumann boundary conditions are assumed for μ and the heat flux or normal derivative of the temperature field is prescribed. The kernel k is assumed to be of positive type. The system (0.1) represents a coupled Stefan/Mullins-Sekerka type problem which has recently been derived by formal asymptotics from a formulation of the conserved phase-field equations which allows for memory effects in the temperature field, [**7**].

1. Introduction. In this paper we study existence and uniqueness for the following Stefan/Mullins-Sekerka problem in which memory

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effects have been incorporated into the temperature field:

$$
\begin{aligned}
a_1 u_t + a_2 w_t &= \int_{-\infty}^{t} k(t-s)\Delta u(s)\,ds \quad (x,t) \in \Omega \times (0,T) \\
a_3 w_t &= \Delta \mu \quad (x,t) \in \Omega \times (0,T) \\
\mu + 2u &\in \partial\Gamma(w) \quad (x,t) \in \Omega \times (0,T) \\
\mathbf{n}\cdot\nabla u = b, &\qquad \mathbf{n}\cdot\nabla\mu = 0 \quad (x,t) \in \partial\Omega \times (0,T) \\
u(x,0) = u_0(x), &\qquad w(x,0) = w_0(x) \quad x \in \Omega \\
\text{and} \quad u(x,t) &= u_h(x,t) \quad (x,t) \in \Omega \times \mathbf{R}^-
\end{aligned} \tag{1.1}
$$

where

$$
\Gamma(w) := \begin{cases} \int_\Omega |\nabla w| < \infty & |w| \le 1 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise.} \end{cases}
$$

The domain $\Omega \subset \mathbf{R}^n$, $n = 1,2,3$ is taken to be bounded, and its boundary $\partial\Omega$ is assumed to be smooth. The notation $\mathbf{R}^- := (-\infty, 0)$ is used. Here $u(x,t)$ denotes the temperature field, $w(x,t)$ a conserved phase variable, and $\mu(x,t)$ the chemical potential. By the definition of Γ, $w(x,t)$ is constrained to lie in the interval $[-1,1]$: two limiting phases are indicated in the regions in which $w(x,t) = \pm 1$, and "mushy regions" or diffuse regions correspond to regions in which $|w(x,t)| < 1$. Equation (1.1c) is a generalization of the Gibbs-Thomson relation, see [**13**]. The coefficients a_1, a_2 and a_3 are assumed to be positive. A kernel is said to be of *positive type* if

$$
Q(v,T;k) := \int_0^T \left\langle v(t), \int_0^t k(t-s)v(s)\,ds \right\rangle dt \ge 0, \tag{1.2}
$$

for all $T > 0$ and $v \in L^2_{\text{loc}}(\mathbf{R}^+; L^2(\Omega))$ where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\Omega)$ and $\mathbf{R}^+ = [0,\infty)$, and *strongly positive* if there exists $\eta > 0$ such that

$$
Q(v,T;k) \ge \eta Q(v,T;e)
$$

for all $v \in L^2_{\text{loc}}(\mathbf{R}^+; L^2(\Omega))$ and $T > 0$. We shall assume throughout that k is of positive type. Off hand, (1.2) does not suffice to control either $\|v\|_{L^2(\Omega)}$ or $\|v\|_{L^2(0,T;L^2(\Omega))}$; however, if

(a) $\quad k : \mathbf{R}^+ \to \mathbf{R}$ is continuous and of positive type, then for each $v \in L^1_{\mathrm{loc}}(\mathbf{R}^+; L^2(\Omega))$ and $T > 0$,

$$\|(k * v)(T)\|^2_{L^2(\Omega)} \le 2k(0)Q(v, T; k),$$

and if

(b) $k, k' \in L^1(\mathbf{R}^+)$ and k is strongly positive,

then

$$\|k * v\|^2_{L^2(0,T;L^2(\Omega))} < c_k Q(v, T; k),$$

where $c_k > 0$ is a constant which depends only on $\|k\|_{L^1}$, $\|k'\|_{L^1}$ and η. See [**11**]. By the differential inclusion it is implied that $\mu + 2u$ is a subgradient of Γ at $w \in X$, i.e.,

$$\int_Q (\mu + 2u)(\eta - w)\, dx\, dt + \int_Q |\nabla w|\, dx\, dt \le \int_Q |\nabla \eta|\, dx\, dt \qquad \text{(1.3)}$$
$$|w| \le 1 \quad \text{a.e. in } Q$$

for all $\eta \in X$, where X is taken to be the Banach space $L^1(0, T; BV(\Omega))$ and $Q := \Omega \times (0, T)$.

It was demonstrated in [**7**], where the following system of conserved phase-field equations with memory

$$\begin{gathered} u_t + \frac{l}{2}\phi_t = \int_{-\infty}^{t} k(t-s)\Delta u(s)\, ds \quad (x,t) \in \Omega \times \mathbf{R}^+ \\ \tau\phi_t = -\xi^2\Delta(\xi^2\Delta\phi + \alpha^{-1}(\phi - \phi^3) + 2u) \quad (x,t) \in \Omega \times \mathbf{R}^+ \\ \mathbf{n}\cdot\nabla u = b, \qquad \mathbf{n}\cdot\nabla\phi = \mathbf{n}\cdot\nabla\Delta\phi = 0 \quad (x,t) \in \partial\Omega \times \mathbf{R}^+ \\ u(x,0) = u_0(x), \qquad \phi(x,0) = \phi_0(x) \quad x \in \Omega \\ \text{and} \\ u(x,t) = u_h(x,t) \quad (x,t) \in \Omega \times \mathbf{R}^- \end{gathered} \qquad \text{(1.4)}$$

was studied, that the system (1.1) could be obtained by formal asymptotics under the assumption that

$$\xi^2 = \varepsilon\bar{\xi}^2, \qquad \tau = \varepsilon\bar{\tau}, \qquad \alpha = \varepsilon, \qquad (\bar{\xi}, \bar{\tau} = O(1)),$$

where $0 < \varepsilon \ll 1$ and for initial conditions such that the domain Ω is initially partitioned into large domains dominated by one of two phases

$\phi = \pm 1$ which are separated by slowly varying smooth thin interfaces whose curvature is assumed to be large.

As noted in [**7**], the memory system (1.1) obtained in this particular asymptotic limit contains an "unresolved" chemical potential, i.e., in the classical Stefan problem formulation the chemical potential has evolved to a uniform equilibrium, and as such no longer appears explicitly. The present formulation may be considered to correspond to an "earlier" quasi-steady state time scale in which these effects have yet to resolve themselves. These additional effects will be seen here to be stabilizing.

Although memory effects have been considered in numerous formulations in the past, e.g., [**3, 4, 5, 9, 12, 13**], to our knowledge this particular formulation has not been considered previously. For example, in [**9**] hyperbolic effects are considered both in the internal energy and in the latent heat. Similarly, in [**3**] memory effects are taken into account in the internal energy and in the heat flux in a model for two-phase Stefan problems with memory. Our approach is based on a fixed point argument and the solution of a differential inclusion by a time discretization method. The treatment of the differential inclusion we present follows roughly the treatment of Visintin [**12, 13**]; however, relaxational chemical potential terms which we mentioned above do not appear in these papers, though the Gibbs-Thomson curvature effects accounted for in the latter of these papers [**13**] is included in the present formulation. The energy balance equation is resolved via the application and weak limit of a resolvent formula.

Existence of a weak solution is proven under the assumptions that $(u_0, w_0) \in L^2(\Omega) \times BV(\Omega)$, $|w_0| \le 1$ almost everywhere in Ω, $u_h \in L^2(\mathbf{R}^-; H^2(\Omega))$, $\int_{-\infty}^0 k(t-s)\Delta u_h(s)\,ds \in L^2_{\rm loc}(\mathbf{R}^+; L^2(\Omega))$ and $b \in H^1_{\rm loc}(\mathbf{R}^+; L^2(\partial\Omega))$, $\int_{\partial\Omega} b\,ds = 0$, via the inclusion of a "relaxation dynamic" [**12, 13**], i.e., existence is proven for the original system by first proving existence for a relaxed system of equations which in the present case is given by

$$
\begin{gathered}
a_1 u_t + a_2 w_t = \int_{-\infty}^{t} k(t-s)\Delta u(s)\,ds \quad (x,t) \in \Omega \times (0,T) \\
a_3 w_t = \Delta\mu \quad (x,t) \in \Omega \times (0,T) \\
\mu + 2u - a_4 w_t \in \partial\Gamma(w) \quad (x,t) \in \Omega \times (0,T) \\
\mathbf{n}\cdot\nabla u = b, \qquad \mathbf{n}\cdot\nabla\mu = 0 \quad (x,t) \in \partial\Omega \times (0,T) \\
u(x,0) = u_0(x), \qquad w(x,0) = w_0(x) \quad x \in \Omega.
\end{gathered} \tag{1.5}
$$

More explicitly, the relaxed problem may be formulated as:

Problem $\mathcal{P}_{a_4}$. *Under the assumption that k is a kernel of positive type such that $k \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$, $(u_0, w_0) \in L^2(\Omega) \times BV(\Omega)$, $u_h \in L^2(\mathbf{R}^-; H^2(\Omega))$, $\int_{-\infty}^0 k(t-s)\Delta u_h(s)\,ds \in L^2_{\mathrm{loc}}(\mathbf{R}^+; L^2(\Omega))$, $|w_0| \le 1$ almost everywhere in Ω and $b \in H^1_{\mathrm{loc}}(\mathbf{R}^+; L^2(\partial\Omega))$, $\int_{\partial\Omega} b\,ds = 0$, for $T > 0$ find $u \in C([0,T]; H^{-1}(\Omega))$, $w \in C([0,T]; L^2(\Omega))$, such that $u \in L^2(0,T; L^2(\Omega))$, $u_t \in L^2(0,T; H^{-2}(\Omega))$, $w \in L^\infty(0,T; BV(\Omega))$, $w_t \in L^2(0,T; H^{-1}(\Omega))$, $a_4^{1/2} w_t \in L^2(0,T; L^2(\Omega))$, $\mu \in L^2(0,T; H^1(\Omega))$, which satisfy*

$$
\begin{aligned}
&-\int_0^T \langle a_1 u + a_2 w, \xi_t\rangle\,dt - \langle a_1 u_0 + a_2 w_0, \xi(0)\rangle \\
&\qquad = \int_0^T \Big\langle\Big\langle \int_0^t k(t-s) b(s)\,ds, \xi \Big\rangle\Big\rangle\,dt \\
&\qquad\quad + \int_0^T \Big\langle \int_0^t k(t-s) u(s)\,ds, \Delta\xi \Big\rangle\,dt \\
&\qquad\quad + \int_0^T \Big\langle \int_{-\infty}^t k(t-s) u_h(s)\,ds, \Delta\xi \Big\rangle\,dt, \\
&-a_3 \int_0^T \langle w, \varphi_t\rangle\,dt - a_3\langle w_0, \varphi(0)\rangle + \int_0^T \langle\nabla\mu, \nabla\varphi\rangle\,dt = 0, \\
&\int_Q (\mu + 2u - a_4 w_t)(\eta - w)\,dx\,dt + \int_Q |\nabla w|\,dx\,dt \\
&\qquad \le \int_Q |\nabla\eta|\,dx\,dt, \qquad |w| \le 1 \ \text{a.e. in } Q,
\end{aligned} \tag{1.6}
$$

for any $\xi \in H^1([0,T[\,;H^2(\Omega))$, $\varphi \in H^1([0,T[\,;H^1(\Omega))$ *and* $\eta \in L^1(0,T;BV(\Omega))$, $|\eta| \le 1$ *almost everywhere in* Q, *where* $Q = \Omega\times(0,T)$. *If* k *satisfies* (a) *or* (b), *find* u *which satisfies additionally* $k * u \in L^\infty(0,T;H^1(\Omega))$ *or* $k * u \in L^2(0,T;H^1(\Omega))$, *respectively, and* (1.6a) *holds in the stronger sense, i.e.,*

$$\begin{aligned} -\int_0^T \langle a_1 u + a_2 w, \xi_t\rangle\, dt - \langle a_1 u_0 + a_2 w_0, \xi(0)\rangle \\ + \int_0^T \Big\langle \nabla\Big(\int_0^t k(t-s)u(s)\,ds\Big), \nabla\xi\Big\rangle\, dt \\ = \int_0^T \Big\langle\Big\langle\Big(\int_0^t k(t-s)b(s)\,ds\Big), \xi\Big\rangle\Big\rangle\, dt \\ + \int_0^T \Big\langle \int_{-\infty}^0 k(t-s)\Delta u_h(s)\,ds, \xi\Big\rangle\, dt, \end{aligned}$$

for any $\xi \in H^1([0,T[\,;H^1(\Omega))$, *where* $\langle\langle\cdot,\cdot\rangle\rangle$ *denotes the inner product in* $L^2(\partial\Omega)$ *and* $u_t \in L^2(0,T;H^{-1}(\Omega))$.

We remark here that to guarantee that $\int_{-\infty}^0 k(t-s)\Delta u_h(s)\,ds \in L^2_{\text{loc}}(\mathbf{R}^+;L^2(\Omega))$, it suffices to assume that $k \in L^1(\mathbf{R}^+)$ in addition to the assumption $u_h \in L^2(\mathbf{R}^-;H^2(\Omega))$. A solution to the system (1.1), that is to Problem $\mathcal{P}_0$, is then obtained by examining the solutions to Problem $\mathcal{P}_{a_4}$ in the limit $a_4 \to 0$.

We introduce the following definition:

Definition 1. A *weak solution* (u,w) of (1.5a) (a solution in the sense of Problem $\mathcal{P}_0$) will be said to be moreover a *weak-mild solution* of the energy balance equation (1.5a) if

$$\begin{aligned} \int_0^T \Big\langle a_1 u + a_2 w - a_1 u_0 - a_2 w_0 - \int_0^t \int_{-\infty}^0 k(s-s')\Delta u_h(s')\,ds'\,ds \\ - \Delta(\alpha * u), \xi\Big\rangle\, dt = 0, \end{aligned}$$

for every $\xi \in L^2(0,T;L^2(\Omega))$ where $\alpha(t) = \int_0^t k(s)\,ds$, and if u is *weakly continuous* in $L^2(\Omega)$ with respect to time.

The main result of this paper can be summarized in the following theorem.

Theorem 1. *There exists a global solution* (u, w) *to Problem* $\mathcal{P}_0$. *Moreover,* (u, w) *constitutes a weak-mild solution of* (1.5a) (*in the sense of Definition* 1 *above*).

We outline below the plan of the paper. In Section 2 by making an appropriate substitution, a formulation with homogeneous boundary conditions and where "prehistory" effects have been separated out is obtained. Next, by solving for the chemical potential μ in terms of w, a system consisting of two equations results. The resultant system is then solved by first freezing the temperature field and demonstrating in Section 3 that for any $T > 0$ the resultant differential inclusion for w has a solution, and by then returning in Section 4 to prove local existence of a weak solution for the full system of equations by a fixed point argument. Within this formulation, the chemical potential μ is found only up to a spatially constant function. This, however, is reasonable as μ is a "potential." For simplicity, we shall set

$$\int_\Omega \mu \, dx = 0. \tag{1.7}$$

In Section 5 it is first demonstrated that the solution obtained in Section 4 may be extended to exist globally and is unique. Afterwards, the limit $a_4 \to 0$ is considered and the solution to (1.1) indicated in Theorem 1 is obtained.

2. The reduced formulation.

Let us introduce the notation

$$k * u := \int_0^t k(t-s)u(s)\, ds,$$

in terms of which we may rewrite the convolution as

$$\int_{-\infty}^t k(t-s)\Delta u(s)\, ds = k * \Delta u + y_0, \tag{2.1}$$

where $y_0 \equiv \int_{-\infty}^0 k(t-s)\Delta u_h(s)\, ds$. Here y_0 represents "prehistory" effects. We assume that $y_0 \in L^2_{\text{loc}}(\mathbf{R}^+; L^2(\Omega))$ (and that $u_h \in$

$L^2(\mathbf{R}^-; H^2(\Omega)))$. In order to obtain a homogeneous formulation, let $\bar{u}$ be the solution to the time-dependent boundary value problem

$$\begin{aligned} 0 &= \Delta \bar{u} \quad (x,t) \in \Omega \times (0,T) \\ \mathbf{n} \cdot \nabla \bar{u} &= b \quad (x,t) \in \partial\Omega \times (0,T). \end{aligned}$$

It follows from standard elliptic theory that if $b \in H^1_{\mathrm{loc}}(\mathbf{R}^+; L^2(\partial\Omega))$ and $\int_{\partial\Omega} b\, ds = 0$, then $\bar{u} \in H^1_{\mathrm{loc}}(\mathbf{R}^+; H^1(\Omega))$. (Actually $\bar{u} \in H^1_{\mathrm{loc}}(\mathbf{R}^+; H^{3/2}(\Omega))$, but this additional regularity will not be needed here.) Thus, defining $v \equiv u - \bar{u}$ and employing the notation of (2.1), the system (1.5) may be written as

$$\begin{aligned} a_1 v_t + a_2 w_t &= k * \Delta v + y_1 \quad (x,t) \in \Omega \times (0,T) \\ a_3 w_t &= \Delta \mu \quad (x,t) \in \Omega \times (0,T) \\ 2v + \mu - a_4 w_t + y_2 &\in \partial\Gamma(w) \quad (x,t) \in \Omega \times (0,T) \\ \mathbf{n} \cdot \nabla v = 0, \qquad \mathbf{n} \cdot \nabla \mu &= 0 \quad (x,t) \in \partial\Omega \times (0,T) \\ v(x,0) = u_0(x) - \bar{u}(x,0), \qquad w(x,0) &= w_0(x) \quad x \in \Omega \end{aligned} \tag{2.2}$$

where $y_1 = y_0 - a_1 \bar{u}_t \in L^2_{\mathrm{loc}}(\mathbf{R}^+; L^2(\Omega))$ and $y_2 \equiv 2\bar{u} \in L^2_{\mathrm{loc}}(\mathbf{R}^+; H^1(\Omega))$.

Next we express μ in terms of the temperature field u. To this end, we set

$$\dot{H}^s(\Omega) := \left\{ z \in H^s(\Omega) \Big| \int_\Omega z\, dx = 0 \right\},$$

for any $s \in \mathbf{R}$, noting that according to this definition $\dot{L}^2(\Omega) = \dot{H}^0(\Omega)$, and we define $\mathcal{N}$ to be the inverse of minus the Laplacian with Neumann boundary conditions, i.e., let $\mathcal{N} : \dot{H}^s(\Omega) \to \dot{H}^{s+2}(\Omega)$ be defined for $s \in \mathbf{R}$ by the following.

Set $z = \mathcal{N} w$ if u and w satisfy

$$-\Delta z = w, \quad x \in \Omega, \qquad \mathbf{n} \cdot \nabla z = 0, \quad x \in \partial\Omega \quad \text{and} \quad \int_\Omega z\, dx = 0.$$

Note that it follows from (2.2b) that $\int_\Omega w_t\, dx = 0$, hence in terms of the above definition it follows now from (2.2) that if $w_t \in L^2(0,T; \dot{H}^s(\Omega))$, then

$$\mu = -a_3 \mathcal{N} w_t \in L^2(0,T; \dot{H}^{s+2}(\Omega)). \tag{2.3}$$

Substituting (2.3) into (2.2b) yields

$$
\begin{aligned}
&a_1 v_t + a_2 w_t = k * \Delta v + y_1 \quad (x,t) \in \Omega \times (0,T) \\
&2v - a_3 \mathcal{N} w_t - a_4 w_t + y_2 \in \partial\Gamma(w) \quad (x,t) \in \Omega \times (0,T) \\
&\int_\Omega w\,dx = \int_\Omega w_0\,dx \quad t \in (0,T) \\
&\mathbf{n} \cdot \nabla v = 0 \quad (x,t) \in \partial\Omega \times (0,T) \\
&v(x,0) = v_0(x), \qquad w(x,0) = w_0(x) \quad x \in \Omega,
\end{aligned} \tag{2.4}
$$

where $v_0(x) = u_0(x) - \bar{u}(x,0)$. The integral constraint (2.4c) reflects the integral constraint $\int_\Omega w_t\,dx = 0$ noted above. More precisely, the reduced relaxed formulation may be stated as

Problem $\tilde{\mathcal{P}}_{a_4}$. *Under the assumption that k is a kernel of positive type, $k \in L^1_{\text{loc}}(\mathbf{R}^+)$, $(v_0, w_0) \in L^2(\Omega) \times BV(\Omega)$, $y_1 \in L^2_{\text{loc}}(\mathbf{R}^+; L^2(\Omega))$, $y_2 \in L^2_{\text{loc}}(\mathbf{R}^+; H^1(\Omega))$, $|w_0| \leq 1$ almost everywhere in Ω, and $b \in H^1_{\text{loc}}(\mathbf{R}^+; L^2(\partial\Omega))$ such that $\int_{\partial\Omega} b\,ds = 0$, for $T > 0$ find $v \in C([0,T]; H^{-1}(\Omega))$, $w \in C([0,T]; L^2(\Omega))$ such that $v \in L^2(0,T; L^2(\Omega))$, $v_t \in L^2(0,T; H^{-2}(\Omega))$, $w \in L^\infty(0,T; BV(\Omega))$, $w_t \in L^2(0,T; H^{-1}(\Omega))$, $a_4^{1/2} w_t \in L^2(0,T; L^2(\Omega))$, which satisfies*

$$
\begin{aligned}
&-\int_0^T \langle a_1 v + a_2 w, \xi_t \rangle\,dt - \langle a_1 v_0 + a_2 w_0, \xi(0) \rangle \\
&\qquad = \int_0^T \langle k * v, \Delta\xi \rangle\,dt + \int_0^T \langle y_1, \xi \rangle\,dt, \\
&\int_Q (2v - a_3 \mathcal{N} w_t - a_4 w_t + y_2)(\eta - w)\,dx\,dt \\
&\qquad + \int_Q |\nabla w|\,dx\,dt \leq \int_Q |\nabla \eta|\,dx\,dt, \\
&\int_\Omega w\,dx = \int_\Omega w_0\,dx \quad t \in (0,T), \\
&|w| \leq 1 \quad \text{a.e. in } Q,
\end{aligned} \tag{2.5}
$$

for any $\xi \in H^1([0,T[\,; H^2(\Omega))$, and $\eta \in L^1(0,T; BV(\Omega))$, $\int_\Omega \eta\,dx = 0$, $|\eta| \leq 1$ almost everywhere in Q, where $Q = \Omega \times (0,T)$. If k satisfies (a) or (b), find u such that additionally $k * v \in L^\infty(0,T; H^1(\Omega))$ or

$k * v \in L^2(0,T;H^1(\Omega))$, respectively, and (1.6a) holds in the stronger sense, i.e.,

$$-\int_0^T \langle a_1 v + a_2 w, \xi_t\rangle\,dt - \langle a_1 v_0 + a_2 w_0, \xi(0)\rangle \\ + \int_0^T \langle \nabla(k*v), \nabla\xi\rangle\,dt = \int_0^T \langle y_1, \xi\rangle\,dt,$$

for any $\xi \in H^1([0,T[\,;H^1(\Omega))$, where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the inner product in $L^2(\partial\Omega)$.

The equivalence of Problems $\tilde{\mathcal{P}}_{a_4}$ and $\mathcal{P}_{a_4}$ follows from the well-posedness of the inverse operator $\mathcal{N}$ in $\dot{L}^2(\Omega)$. It is the relaxed formulation Problem $\tilde{\mathcal{P}}_{a_4}$ which we solve locally in Sections 3 and 4. In Section 5, it is demonstrated that solutions to Problem $\tilde{\mathcal{P}}_{a_4}$ (and hence $\mathcal{P}_{a_4}$) may be extended to exist globally. The limit $a_4 \to 0$ is then taken employing estimates obtained in Sections 3 and 4, producing by this method the solution described in Theorem 1.

3. Solution of the differential inclusion. In order to solve the system (2.4), we "freeze" the perturbed temperature field v in the following sense. Let $T > 0$ be arbitrary and fixed. Then set

$$K_R = \{v \in L^2(0,T;L^2(\Omega)) \mid \|v\|_{L^2(0,T;L^2(\Omega))} \le R\}$$

where $R > 0$ is arbitrary, and let v in (2.4b) be fixed as an element of K_R. Thus, the system which we wish to solve is

$$\begin{aligned} &2v - a_3\mathcal{N}w_t - a_4 w_t + y_2 \in \partial\Gamma(w) \quad (x,t) \in \Omega\times(0,T) \\ &\int_\Omega w\,dx = \int_\Omega w_0\,dx \quad t \in (0,T) \\ &w(x,0) = w_0(x) \quad x \in \Omega \end{aligned} \tag{3.1}$$

in $L^1(0,T;BV(\Omega))$ where $v \in K_R$ and $y_2 \in L^2(0,T;H^1(\Omega))$ are assumed to be known, though only the $L^2(0,T;L^2(\Omega))$ regularity of y_2 will be used. More precisely, we wish to solve Problem $\tilde{\mathcal{P}}'_{a_4}$.

Problem $\tilde{\mathcal{P}}'_{a_4}$. *Let k be a kernel of positive type, $k \in L^1_{\rm loc}(\mathbf{R}^+)$, $w_0 \in BV(\Omega)$, $|w_0| \le 1$ almost everywhere in Ω, $v, y_2 \in L^2(0,T;L^2(\Omega))$,* *find* $w \in C([0,T];L^2(\Omega))$ *such that* $w \in L^\infty(0,T;BV(\Omega))$, $w_t \in L^2(0,T;H^{-1}(\Omega))$ *and* $a_4^{1/2} w_t \in L^2(0,T;L^2(\Omega))$, *which satisfies*

$$\int_Q (2v - a_3 \mathcal{N} w_t - a_4 w_t + y_2)(\eta - w)\,dx\,dt + \int_Q |\nabla w|\,dx\,dt \le \int_Q |\nabla \eta|\,dx\,dt,$$
$$\int_\Omega w\,dx = \int_\Omega w_0\,dx \quad t \in (0,T),$$
$$|w| \le 1 \quad a.e.\ in\ Q,$$

for any $\eta \in L^1(0,T;BV(\Omega))$, *such that* $|\eta| \le 1$ *almost everywhere in* Q *and* $\int_\Omega \eta\,dx = 0$ *for almost every* $t \in (0,T)$.

We turn now to solve the differential inclusion. As in [**12, 13**], we introduce a discretized time variable,

$$0 = t_m^0 < t_m^1 < t_m^2 < \cdots < t_m^m = T,$$

where $t_m^n = nh$, $0 \le n \le m$ and $h = T/m$, and we denote by w_m^n the approximants of the function $w(x,t)$ at the times t_m^n. Thus, we may write

$$(3.2)\quad 2v_m^n - \frac{a_3}{h}\mathcal{N}(w_m^n - w_m^{n-1}) - \frac{a_4}{h}(w_m^n - w_m^{n-1}) + y_{2_m}^n \in \partial\Gamma(w_m^n),$$

where the definition of the differential inclusion is as in (1.3) except now the integrals are understood to be taken over the domain Ω and not over $Q = \Omega \times (0,T)$. Furthermore,

$$v_m^n := \left[\frac{1}{h}\int_{t_m^{n-1}}^{t_m^n} v^2\,dt\right]^{1/2} = \frac{1}{h^{1/2}}\|v\|_{L^2([t_{n-1}^m, t_n^m])},$$

and $y_{2_m}^n$ is defined similarly. Equivalently, defining $\tilde{w}_m^n := w_m^n - w_0$ for $0 \le n \le m$ and $\tilde{\Gamma}(\tilde{w}_m^n) := \Gamma(\tilde{w}_m^n + w_0)$,

$$(3.3)\quad 2v_m^n + h^{-1}(a_3\mathcal{N}\tilde{w}_m^{n-1} + a_4\tilde{w}_m^{n-1}) + y_{2_m}^n \in \partial\tilde{\Gamma}(\tilde{w}_m^n) + h^{-1}(a_3\mathcal{N}\tilde{w}_m^n + a_4\tilde{w}_m^n) := \mathcal{M}(\tilde{w}_m^n).$$

Noting that $\tilde{\Gamma}(w)$ is a convex functional, as is the functional $\psi(w) : \dot{L}^2(\Omega) \to \mathbf{R}$ where

$$\psi \equiv \frac{a_4}{2h}\int_\Omega w^2\,dx + \frac{a_3}{2h}\int_\Omega (\mathcal{N}^{1/2}w)^2\,dx,$$

it follows that if we define

$$\phi : \dot{BV}(\Omega) \longrightarrow R, \qquad \phi := \tilde{\Gamma} + \psi$$

where

$$\dot{BV}(\Omega) := BV(\Omega) \cap \dot{L}^2(\Omega),$$

then ϕ is convex. Then, by the *sum rule* for convex functionals, see, e.g., [**14**, Theorem 47.B],

$$\mathcal{M} = \partial\psi + \partial\tilde{\Gamma} = \partial(\psi + \tilde{\Gamma}) \quad (= \partial\phi)$$

on $\dot{BV}(\Omega)$. Thus, $\mathcal{M}$ is the subdifferential of a convex functional. Since $\dot{BV}(\Omega)$ is a real locally convex space, solutions of the differential inclusion correspond to solutions of the minimization problem:

$$\inf \phi(u)_{u\in \dot{BV}(\Omega)} = \alpha,$$

see [**14**, Proposition 47.12]. But because $\phi(u)$ is weakly coercive and lower semicontinuous on $\dot{BV}(\Omega)$, and since the unit ball in $\dot{BV}(\Omega)$ is weak-star sequentially compact (for a characterization of $BV^*(\Omega)$, see, e.g., [**15**]) and $\dot{BV}(\Omega)$ is the dual of a separable Banach space, it follows from [**14**, Proposition 38.12] that such a minimizer exists. Thus, for every n and m, $1 \le n \le m$, there exists a solution in $\dot{BV}(\Omega)$ to the differential inclusion (3.3).

We proceed now to obtain estimates needed to take the appropriate limits to obtain a solution to Problem $\mathcal{P}_{a_4}$. It follows from (3.3) that, for any $\eta \in \dot{BV}(\Omega)$,

$$\text{(3.4)} \quad \phi(\eta) \ge \phi(\tilde{w}_m^n) + \int_\Omega \left(2v_m^n + \frac{a_3}{h}\mathcal{N}\tilde{w}_m^{n-1} + \frac{a_4}{h}\tilde{w}_m^{n-1} + y_{2_m}^n\right)(\eta - \tilde{w}_m^n).$$

From (3.4) it follows that

$$\tilde{\Gamma}(\eta) \ge \tilde{\Gamma}(\tilde{w}_m^n) + \int_\Omega \Bigg(2v_m^n + \frac{a_3}{h}\mathcal{N}\tilde{w}_m^{n-1} + \frac{a_4}{h}\tilde{w}_m^{n-1} - \frac{a_3}{2h}\mathcal{N}(\eta + \tilde{w}_m^n) - \frac{a_4}{2h}(\eta + \tilde{w}_m^n) + y_{2_m}^n\Bigg)(\eta - \tilde{w}_m^n).$$

Setting $\eta = \tilde{w}_m^{n-1}$ and returning to the original variables, we obtain

$$\begin{aligned}\Gamma(w_m^{n-1}) - \Gamma(w_m^n) \geq &-\int_\Omega (2v_m^n + y_{2_m}^n)(w_m^n - w_m^{n-1}) \\ &+ \frac{a_3}{2h}\int_\Omega [\mathcal{N}^{1/2}(w_m^n - w_m^{n-1})]^2 \\ &+ \frac{a_4}{2h}\int_\Omega (w_m^n - w_m^{n-1})^2.\end{aligned}$$

Summing over n from 1 to $\tilde{m}$, for $\tilde{m} \in \mathbf{Z}^+$, $1 \leq \tilde{m} \leq m$, and cancelling intermediary terms,

$$\begin{aligned}&\frac{a_3}{2h}\sum_{n=1}^{\tilde{m}}\int_\Omega [\mathcal{N}^{1/2}(w_m^n - w_m^{n-1})]^2 \\ &\qquad + \frac{a_4}{2h}\sum_{n=1}^{\tilde{m}}\int_\Omega (w_m^n - w_m^{n-1})^2 + [\Gamma(w_{\tilde{m}}^m) - \Gamma(w_0^m)] \\ &\qquad \leq \sum_{n=1}^{\tilde{m}}\int_\Omega (2v_m^n + y_{2_m}^n)(w_m^n - w_m^{n-1}).\end{aligned}$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned}&\frac{a_3}{2h}\sum_{n=1}^{\tilde{m}}[\mathcal{N}^{1/2}(w_m^n - w_m^{n-1})]^2 \\ &\quad + \frac{a_4}{4h}\sum_{n=1}^{\tilde{m}}(w_m^n - w_m^{n-1})^2 + [\Gamma(w_{\tilde{m}}^m) - \Gamma(w_0^m)] \\ &\qquad \leq \frac{8h}{a_4}\sum_{n=1}^{\tilde{m}}\int_\Omega (v_m^n)^2 + \frac{2h}{a_4}\sum_{n=1}^{\tilde{m}}\int_\Omega (y_{2_m}^n)^2 \\ &\qquad \leq \frac{8h}{a_4}\sum_{n=1}^{m}\int_\Omega (v_m^n)^2 + \frac{2h}{a_4}\sum_{n=1}^{m}\int_\Omega (y_{2_m}^n)^2.\end{aligned}$$

Employing the definitions of v_m^n and $y_{2_m}^n$, we obtain that

$$\begin{aligned} &\frac{a_3}{2h}\sum_{n=1}^{\tilde m}[\mathcal{N}^{1/2}(w_m^n - w_m^{n-1})]^2 \\ &\quad + \frac{a_4}{4h}\sum_{n=1}^{\tilde m}(w_m^n - w_m^{n-1})^2 + [\Gamma(w_{\tilde m}^m) - \Gamma(w_0^m)] \\ &\qquad \le \frac{8}{a_4}\|v\|^2_{L^2(0,T;L^2(\Omega))} + \frac{2}{a_4}\|y_2\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned} \frac{a_4}{4h}\sum_{n=1}^{m}\int_\Omega (w_m^n - w_m^{n-1})^2 \le{}& \frac{8}{a_4}\|v\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \frac{2}{a_4}\|y_2\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \|w_0\|_{BV(\Omega)}. \end{aligned} \tag{3.6}$$

Denoting by $\{w_m\}$ the linear time-interpolants of the functions $\{w_m^n\}$ obtained above, (3.6) yields

$$\begin{aligned} \|w_{m_t}\|_{L^2(0,T;L^2(\Omega))} \le \frac{\mathcal{C}}{a_4}\cdot [&\|v\|_{L^2(0,T;L^2(\Omega))} + \|y_2\|_{L^2(0,T;L^2(\Omega))} \\ &+ a_4^{1/2}\|w_0\|^{1/2}_{BV(\Omega)}], \end{aligned} \tag{3.7}$$

where $\mathcal{C}$ is a constant which is independent of a_i, $i = 1, \dots, 4$. Since, for every m and n, $1 \le n \le m$, $w_m^n - w_m^{n-1} \in \dot{BV}(\Omega)$, it follows that

$$\int_\Omega w_{m_t}\,dx = 0. \tag{3.8}$$

Moreover, by construction

$$\|w_m\|_{L^\infty(0,T;L^\infty(\Omega))} \le 1. \tag{3.9}$$

Returning, from (3.5) and (3.9),

$$\begin{aligned} \|w_m\|_{L^\infty(0,T;BV(\Omega))} \le{}& [\|w_0\|_{BV(\Omega)} + |\Omega|] \\ &+ \frac{1}{a_4}[8\|v\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad + 2\|y_2\|^2_{L^2(0,T;L^2(\Omega))}]. \end{aligned} \tag{3.10}$$

Therefore, there exists a subsequence denoted here again by $\{w_m\}$ and a function w such that

$$w_{mt} \rightharpoonup w_t \quad \text{weakly in } L^2(0,T;\dot{L}^2(\Omega)), \tag{3.11}$$

and

$$w_m - w_0 \rightharpoonup w - w_0 \quad \text{weakly}^* \text{ in } L^\infty(0,T;\dot{BV}). \tag{3.12}$$

From the compact embedding of $BV(\Omega)$ in $L^2(\Omega)$, it follows from (3.7) and (3.10) that

$$w_m \longrightarrow w \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.13}$$

From (3.7)–(3.8) and the embedding of $\dot{L}^2(\Omega)$ in $\dot{H}^{-2}(\Omega)$

$$\mathcal{N}w_{mt} \rightharpoonup \mathcal{N}w_t \quad \text{weakly in } L^2(0,T;\dot{L}^2(\Omega)). \tag{3.14}$$

Note by construction that

$$\int_\Omega w_t\,dx = 0, \qquad \int_\Omega w\,dx = \int_\Omega w_0\,dx \quad \text{a.e. } t \in (0,T).$$

Integrating (3.2) from 0 to T, it follows that, for each w_m and for any $\eta \in L^1(0,T;\dot{BV}(\Omega))$,

$$\int_Q (2v_m - a_3\mathcal{N}w_{m_t} - a_4 w_{m_t} + y_{2_m})(\eta - w_m)\,dx\,dt + \int_Q |\nabla w_m|\,dx\,dt \le \int_Q |\nabla \eta|\,dx\,dt.$$

By lower semicontinuity,

$$\int_Q |\nabla w|\,dx\,dt \le \liminf_{m\to\infty} \int_Q |\nabla w_m|\,dx\,dt.$$

By the definition of v_m and y_{2_m}, $2v_m + y_{2_m}$ converges strongly in $L^2(0,T;L^2(\Omega))$. Since $\eta \in L^1(0,T;\dot{BV}(\Omega))$, it follows from the definition of $\dot{BV}(\Omega)$ that $\eta \in L^2(0,T;L^2(\Omega))$. Therefore, by (3.13), $\eta - w_m$ converges strongly in $L^2(0,T;L^2(\Omega))$. Hence,

$$\int_Q (2v_m + y_{2_m})(\eta - w_m)\,dx\,dt \longrightarrow \int_Q (2v + y_2)(\eta - w)\,dx\,dt.$$

By (3.7)–(3.8), we obtain that a subsequence $a_3\mathcal{N}w_{m_t}+a_4w_{m_t}$ is weakly convergent in $L^2(0,T;\dot{L}^2(\Omega))$, and hence using the strong convergence of $\eta - w_m$ in $L^2(0,T;L^2(\Omega))$ noted above we find that

(3.15)
$$\int_Q (a_3\mathcal{N}w_{m_t} + a_4w_{m_t})w_m\,dx\,dt \longrightarrow \int_Q (a_3\mathcal{N}w_t + a_4w_t)w_m\,dx\,dt.$$

Thus,

$$\int_Q (2v - a_3\mathcal{N}w_t - a_4w_t + y_2)(\eta - w)\,dx\,dt + \int_Q |\nabla w|\,dx\,dt \leq \int_Q |\nabla\eta|\,dx\,dt$$

for all $\eta \in L^1(0,T;BV(\Omega))$ such that $|\eta| \leq 1$ almost everywhere in Q, $\int_\Omega \eta\,dx = 0$ for almost every $t \in (0,T)$. From the weak convergence of w_{m_t} in $L^2(0,T;L^2(\Omega))$, and the strong convergence of w_m in $L^2(0,T;L^2(\Omega))$, it follows [**10**] that $w \in C([0,T];L^2(\Omega))$. Hence, w is a solution of the differential inclusion (3.1).

Thus we have proven

Theorem 2. *For any $T > 0$ and $v \in K_R$, there exists a solution to Problem $\tilde{P}'_{a_4}$.*

4. A fixed point argument. In order to complete the proof of existence for Problem $\tilde{P}a_4$; i.e., for the system (2.4), which for convenience we rewrite below as,

$$\begin{gathered} a_1v_t + a_2w_t = k * \Delta v + y_1 \quad (x,t) \in \Omega \times (0,T), \\ \mathbf{n}\cdot\nabla v = 0 \quad (x,t) \in \partial\Omega \times (0,T), \\ 2v - a_3\mathcal{N}w_t - a_4w_t + y_2 \in \partial\Gamma(w) \quad (x,t) \in \Omega \times (0,T), \\ \int_\Omega w\,dx = \int_\Omega w_0\,dx \quad t \in (0,T), \\ v(x,0) = v_0(x), \qquad w(x,0) = w_0(x) \quad x \in \Omega, \end{gathered} \tag{4.1}$$

we employ a fixed point argument.

To proceed, we consider the linear Volterra integro-differential equation

$$\begin{aligned} &\psi_t = \kappa * \Delta\psi + f \quad (x,t) \in \Omega \times (0,T), \\ (4.2) \qquad &\mathbf{n} \cdot \nabla\psi = 0 \quad (x,t) \in \partial\Omega \times (0,T), \\ &\psi(x,0) = \psi_0 \quad x \in \Omega, \end{aligned}$$

where $f \in L^1_{\mathrm{loc}}(\mathbf{R}^+; L^2(\Omega))$. Setting $\beta(t) = \int_0^t \kappa(s)\,ds$ and $g(t) = \psi_0 + \int_0^t f(s)\,ds$ and letting A denote the Laplacian operator with Neumann boundary conditions, we define

Definition 2. A function $\psi \in C([0,T]; L^2(\Omega))$ is called

(i) a *strong solution* of (4.2) on $[0,T]$ if $\psi \in C([0,T]; H^2(\Omega))$, $\psi_t \in L^2(0,T; L^2(\Omega))$ and ψ satisfies (4.2a,b) almost everywhere on $[0,T]$ and (4.2c),

(ii) a *mild solution* of (4.2) on $[0,T]$ if $\kappa * \psi \in C([0,T]; H^2(\Omega))$ and $\psi(t) = g(t) + A(\beta * \psi)(t)$ on $[0,T]$.

A function $\psi \in L^2(0,T; L^2(\Omega))$ is called

(iii) a *weak-mild solution* of (4.2) on $[0,T]$ if

$$\int_0^T \langle \psi(t) - g(t) - A(\beta * \psi), \xi \rangle \, dt = 0$$

for every $\xi \in L^2(0,T; L^2(\Omega))$, and if ψ is weakly continuous in $L^2(\Omega)$ with respect to time.

Clearly in terms of these definitions, a strong solution constitutes a mild solution, and a mild solution constitutes a weak-mild solution.

Remark 1. A weak-mild solution as defined here does not correspond to a "weak solution" of $\psi = g + A(\beta * \psi)$ as defined in [**8**, Chapter 1].

Remark 2. It is easy to demonstrate that if ψ is a weak-mild solution of (4.2) on $[0,T]$, then on $[0,T]$,

$$-\int_0^T \langle \psi(t), x_t^* \rangle \, dt - \langle \psi(0), x^*(0) \rangle = \int_0^T \langle (\kappa * \psi)(t), \Delta x^* \rangle \, dt + \int_0^T \langle f(t), x^* \rangle \, dt$$

for each $x^* \in H^1([0,T[\,; H^2(\Omega))$, i.e., ψ is a *weak solution* by which we mean a weak solution in the sense employed in the statement of Problem $\tilde{\mathcal{P}}_{a_4}$. Similarly, if ψ is a weak solution of (4.2) and if $\beta * \psi \in L^2(0,T; H^2(\Omega))$, then it is also straightforward to demonstrate that ψ is moreover a weak-mild solution of (4.2).

Remark 3. Clearly, a strong solution constitutes a weak solution. It is also possible to demonstrate that a mild solution constitutes a weak solution by regularizing the initial data and f and passing to the limit.

In terms of existence and uniqueness of solutions to (4.2), it follows from [**8**, Chapter 1]

Proposition 3. *If* $\kappa \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$ *and* κ *is a kernel of positive type, then*

(i) *if* $\psi_0 \in H^2(\Omega)$ *and* $f \in L^1(0,T; H^2(\Omega))$, *there exists a unique strong solution to* (4.2), *and*

(ii) *if* $\psi_0 \in L^2(\Omega)$ *and* $f \in L^1(0,T; L^2(\Omega))$, *there exists a unique mild solution to* (4.2).

Moreover, following [**8**], we define

Definition 3. A family $\{S(t)\}_{t \geq 0} \subset \mathcal{L}(L^2(\Omega))$ of bounded linear operators in $L^2(\Omega)$ is called a "*resolvent*" of

$$\psi(t) = g(t) + A(\beta * \psi)(t)$$

for $t \geq 0$ if

(S1) $S(t)$ is *strongly continuous* on $\mathbf{R}^+$ and $S(0) = I$;

(S2) $S(t)$ *commutes with* A, which means that $S(t)H^2(\Omega) \subset H^2(\Omega)$ and $AS(t)x = S(t)Ax$ for all $x \in H^2(\Omega)$ and $t \geq 0$;

(S3) the resolvent equation holds

$$S(t)x = x + \int_0^t \beta(t-s)AS(s)x\,ds \quad x \in H^2(\Omega), \quad t \geq 0,$$

and obtain

Proposition 4. *Equation* (4.2) *admits a resolvent such that*

$$\|S(t)\|_{\mathcal{L}(L^2(\Omega))} \leq 1.$$

Moreover, for each $\psi_0 \in L^2(\Omega)$ *and* $f \in L^1(0,T;L^2(\Omega))$,

$$\psi(t) = S(t)\psi_0 + \int_0^t S(t-s)f(s)\,ds \tag{4.3}$$

is a mild solution of (4.2). *If* $\psi_0 \in H^2(\Omega)$ *and if* $f \in L^1(0,T;H^2(\Omega))$, *then* (4.3) *yields the unique strong solution of* (4.2).

Let us define

$$\mathcal{F}_v = S(t)v_0 + \int_0^t S(t-s)\left(\frac{1}{a_1}y_1 - \frac{a_2}{a_1}w_t\right)(s)\,ds, \qquad t \in [0,T]. \tag{4.4}$$

From (4.3) and (4.4), it is easy to see that

$$\begin{aligned}\|\mathcal{F}_v\|_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{\sqrt{2}}T\bigg[&\frac{1}{a_1}\|y_1\|_{L^2(0,T;L^2(\Omega))} \\ &+ \frac{a_2}{a_1}\|w_t\|_{L^2(0,T;L^2(\Omega))}\bigg] \\ &+ T^{1/2}[\|v_0\|_{L^2(\Omega)}].\end{aligned}$$

Employing (3.7), it then follows that

$$\begin{aligned}\|\mathcal{F}_v\|_{L^2(0,T;L^2(\Omega))} \leq \mathcal{C}T\bigg[&\frac{a_2}{a_1a_4}\|v\|_{L^2(0,T;L^2(\Omega))} + \frac{1}{a_1}\|y_1\|_{L^2(0,T;L^2(\Omega))} \\ &+ \frac{a_2}{a_1a_4^{1/2}}\|w_0\|_{BV(\Omega)}^{1/2} + \frac{a_2}{a_1a_4}\|y_2\|_{L^2(0,T;L^2(\Omega))}\bigg] \\ &+ T^{1/2}[\|v_0\|_{L^2(\Omega)}],\end{aligned}$$

where $\mathcal{C}$ is a constant which is independent of a_i, $i = 1, \ldots, 4$. Therefore, for any $0 < T < T_0$ where

$$T_0 = \tilde{C}R^2[1 + \|w_0\|_{BV(\Omega)}^{1/2} + \|v_0\|_{L^2(\Omega)} + \|y_1\|_{L^2(0,T;L^2(\Omega))} + \|y_2\|_{L^2(0,T;L^2(\Omega))}]^{-2},$$

and $\tilde{C} = \tilde{C}(a_1, a_2, a_3, a_4)$, $\mathcal{F}_v$ maps K_R into K_R.

Next, for any $v \in K_R$, let $\{v^k\} \in K_R$ be a sequence such that

$$v^k \rightharpoonup v \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

Consider now the solutions $\{w^k\}$ to the "frozen" system (3.1) corresponding to the sequence $\{v^k\}$, which by virtue of (3.7)–(3.10) satisfy

$$\begin{aligned} \|w^k\|_{L^\infty(0,T;BV(\Omega))} &\leq C \\ \|w_t^k\|_{L^2(0,T;\dot{L}^2(\Omega))} &\leq C \\ \|w^k\|_{L^\infty(0,T;L^\infty(\Omega))} &\leq 1, \end{aligned}$$

and

$$\int_\Omega w^k \, dx = \int_\Omega w_0 \, dx.$$

Thus, without loss of generality, taking subsequences,

$$\begin{gathered} w^k \rightharpoonup w \quad \text{weak}^* \text{ in } L^\infty(0,T;\dot{BV}(\Omega)), \\ w_t^k \rightharpoonup w_t \quad \text{weakly in } L^2(0,T;\dot{L}^2(\Omega)), \\ w^k \rightharpoonup w \quad \text{weak}^* \text{ in } L^\infty(0,T;L^\infty(\Omega)), \\ \int_\Omega w^k \, dx \longrightarrow \int_\Omega w \, dx. \end{gathered}$$

With these estimates in hand, we return to find as in the existence proof given in Section 3 that (v, w) satisfy the differential inclusion. From the definition of $\mathcal{F}_v$, and noting that

$$w_t^k \rightharpoonup w_t \quad \text{weakly in } L^2(0,T;\dot{L}^2(\Omega)),$$

we obtain that

$$\mathcal{F}_{v^k} \rightharpoonup \mathcal{F}_v \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

It now follows from the Schauder-Tychonoff fixed point theorem that, for $0 < T < T_0$, $\mathcal{F}_v$ has a fixed point $v \in K_R$ such that (v, w) satisfy Proposition 4 in $L^2(0, T; L^2(\Omega))$, i.e., (v, w) is a mild solution in the sense of (4.3), and (v, w) satisfy (4.1c)–(4.1d) in the sense prescribed in Section 3. Returning, we find by the uniqueness of mild solutions (Proposition 3) that v is, moreover, a mild solution in the sense of Definition 2. Thus $v \in C([0, T]; L^2(\Omega))$ and the initial conditions are satisfied. Since $w \in L^\infty(0, T; BV(\Omega))$ and $w_t \in L^2(0, T; L^2(\Omega))$, it follows also [**10**] that $w \in C([0, T]; L^2(\Omega))$. Since, as we have remarked, a mild solution also constitutes a weak solution, we obtain that (v, w) satisfies the weak formulation (2.5) of Problem $\tilde{\mathcal{P}}_{a_4}$.

Thus, we have proven:

Theorem 5. *For $T > 0$ sufficiently small, there exists a solution (u, w) to Problem $\tilde{\mathcal{P}}_{a_4}$. Moreover, (v, w) constitutes a mild solution of* (1.6a) (*in the sense of Definition* 2).

Returning and combining the estimates obtained for $\bar{u}$ in Section 2, and setting $u = v + \bar{u}$ and solving (2.2b) for μ, we obtain that (u, w) satisfies the weak formulation (1.6) of Problem $\mathcal{P}_{a_4}$.

Theorem 6. *For $T > 0$ sufficiently small, there exists a weak solution (u, w) to Problem $\mathcal{P}_{a_4}$.*

The discussion of a weak-mild formulation of (1.6a) for Problem $\mathcal{P}_{a_4}$ is postponed until after global existence is proved in Section 5.

5. The memory Stefan/Mullins-Sekerka problem. Now that we have proven local existence for the relaxed problem, Problem $\mathcal{P}_{a_4}$, we return to obtain additional estimates in order to obtain first a global solution for Problem $\mathcal{P}_{a_4}$ and then for the unrelaxed problem, Problem $\mathcal{P}_0$, as well. To this end, we consider again the auxiliary system (Problem $\tilde{\mathcal{P}}_{a_4}$)

$$
\begin{aligned}
a_1 v_t + a_2 w_t = k * \Delta v + y_1 \quad (x,t) \in \Omega \times (0,T) \\
2v - a_3 \mathcal{N} w_t - a_4 w_t + y_2 \in \partial\Gamma(w) \quad (x,t) \in \Omega \times (0,T) \\
\int_\Omega w\,dx = \int_\Omega w_0\,dx \quad t \in (0,T) \\
\mathbf{n} \cdot \nabla v = 0 \quad (x,t) \in \partial\Omega \times (0,T) \\
v(x,0) = v_0(x), \qquad w(x,0) = w_0(x) \quad x \in \Omega
\end{aligned} \tag{5.1}
$$

which was solved in Section 4. Here $T \in [T_0, T_{\max})$, where $[0, T_{\max})$ indicates the maximal time interval over which a given solution obtained in Section 4 may be extended to exist. We multiply the first equation by v and the second equation by w_t, and integrate over time and space to obtain

$$
\frac{a_1}{2}\langle v(T), v(T)\rangle + a_2 \int_0^T \langle v, w_t\rangle\,dt \leq \frac{a_1}{2}\langle v(0), v(0)\rangle + \int_0^T \langle v, y_1\rangle\,dt, \tag{5.2}
$$

$$
\begin{aligned}
-\int_0^T \langle 2v, w_t\rangle\,dt + a_3 \int_0^T \langle \mathcal{N}^{1/2} w_t, \mathcal{N}^{1/2} w_t\rangle\,dt \\
+ a_4 \int_0^T \langle w_t, w_t\rangle\,dt - \int_0^T \langle y_2, w_t\rangle\,dt \\
+ \int_\Omega |\nabla w(T)|\,dx \leq \int_\Omega |\nabla w(0)|\,dx.
\end{aligned} \tag{5.3}
$$

Since the solution obtained in Section 4 yielded $v, w_t, y_1 \in L^2(0,T; L^2(\Omega))$, to justify (5.2), v_0, w_t and y_t must first be regularized so that the multiplication and integration may be carried out in terms of a strong solution. For the strong solution, since k is a positive kernel,

$$
Q(\nabla v, T; k) = \int_0^T \int_\Omega \nabla v \cdot \nabla(k * v)\,dx\,dt \geq 0. \tag{5.4}
$$

Passing to the limit, it follows from (5.4) that (5.2) also holds for weak and mild solutions. Equation (5.3) may be justified as in Section 4.

Adding together twice equation (5.2) and a_2 times equation (5.3), and noting that by the Cauchy-Schwartz inequality,

$$\int_0^T \langle v, y_1 \rangle \, dt \leq \frac{1}{2}\|v\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}\|y_1\|^2_{L^2(0,T;L^2(\Omega))}$$

and

$$\int_0^T \langle y_2, w_t \rangle \, dt \leq \frac{1}{2a_3}\|y_2\|^2_{L^2(0,T;H^1(\Omega))} + \frac{a_3}{2}\|\mathcal{N}^{1/2} w_t\|^2_{L^2(0,T;L^2(\Omega))}$$

yields

$$\begin{aligned}(5.5) \quad a_1\|v(T)\|^2_{L^2(\Omega)} &+ \frac{a_2 a_3}{2}\|\mathcal{N}^{1/2} w_t\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ a_2 a_4 \|w_t\|^2_{L^2(0,T;L^2(\Omega))} + a_2 \int_\Omega |\nabla w(T)| \, dx \\ &\leq a_1\|v(0)\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(0,T;L^2(\Omega)} \\ &+ a_2 \int_\Omega |\nabla w(0)| \, dx \\ &+ \|y_1\|^2_{L^2(0,T;L^2(\Omega))} + \frac{a_2}{2a_3}\|y_2\|^2_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

From (5.4) and (5.5), by employing Gronwall's lemma we ascertain now a number of estimates. Let

$$\rho = [a_2\|w_0\|_{BV(\Omega)} + a_1\|v_0\|^2_{L^2(\Omega)} + \|y_1\|^2_{L^2_{\text{loc}}(\mathbf{R}^+;L^2(\Omega))} + \frac{a_2}{2a_3}\|y_2\|_{L^2_{\text{loc}}(\mathbf{R}^+;H^1(\Omega))}]^{1/2}.$$

Then for w we obtain that

$$\text{(i)} \qquad \|w_t\|_{L^2(0,T;H^{-1}(\Omega))} < \sqrt{a_2 a_3/2}\, e^{T/2a_1}\rho,$$

and

$$\text{(ii)} \qquad a_4^{1/2}\|w_t\|_{L^2(0,T;L^2(\Omega))} < \frac{1}{\sqrt{a_2}} e^{T/2a_1}\rho.$$

By the definition of Γ, solutions (v, w) of (5.1) satisfy

$$\|w\|_{L^\infty(0,T;L^2(\Omega))} \leq 1.$$

Therefore, by (5.5),

$$\text{(iii)} \qquad \|w\|_{L^\infty(0,T;BV(\Omega))} < \frac{1}{a_2} e^{T/a_1}\rho^2 + 1.$$

Similarly, for v we obtain from (5.5) that

$$\text{(iv)} \qquad \|v\|_{L^\infty(0,T;L^2(\Omega))} < \frac{1}{\sqrt{a_1}} e^{T/2a_1}\rho,$$

Therefore by (5.1a),

$$\text{(v)} \quad \|a_1 v_t + a_2 w_t\|_{L^2(0,T;H^{-2}(\Omega))} < C_1(\Omega)\Bigg(\|y_1\|_{L^2_{\text{loc}}(\mathbf{R}^+;L^2(\Omega))} + \frac{1}{\sqrt{a_1}} \|k\|_{L^1_{\text{loc}}(\mathbf{R}^+)} e^{T/2a_1}\rho \Bigg).$$

Hence, using (i),

$$\text{(vi)} \quad \|a_1 v_t\|_{L^2(0,T;H^{-2}(\Omega))} < C_2(\Omega)\Bigg(\|y_1\|_{L^2_{\text{loc}}(\mathbf{R}^+;L^2(\Omega))} + \left[\sqrt{a_2^3 a_3} + \frac{1}{\sqrt{a_1}} \|k\|_{L^1_{\text{loc}}(\mathbf{R}^+)} \right] e^{T/2a_1}\rho \Bigg),$$

where $C_i(\Omega), i = 1, 2$ are geometric constants. Moreover, if $\{v_n\}$ is an approximating sequence for v such that $v_n(0) \in H^2(\Omega)$ and $v_n(0) \to v_0$ in $L^2(\Omega)$, it follows by retracing steps (5.2)–(5.5) that

$$\text{(vii)} \qquad \liminf_{n\to\infty} Q(\nabla v_n, T; k) \le \frac{1}{2} e^{T/a_1}\rho.$$

Since the solution we have obtained for Problem $\mathcal{P}_{a_4}$ is a mild solution, it follows that

$$\Delta(\beta * v) + g(t) = v(t),$$

where

$$\begin{aligned} g(t) &= v_0 + \int_0^t \left[\frac{1}{a_1} y_1 - \frac{a_2}{a_1} w_t \right](s)\, ds \\ &= v_0 + \frac{1}{a_1} \int_0^t y_1(s)\, ds - \frac{a_2}{a_1}(w(t) - w_0) \end{aligned}$$

and where $\beta = \frac{1}{a_1}\int_o^t k(s)ds$, which implies by incorporating the Gronwall estimates that
(viii)

$$\|\Delta(\beta * v)\|_{L^2(0,T;L^2(\Omega))} \leq T^{1/2}\left[\|v_0\|_{L^2(\Omega)} + \frac{a_2}{a_1}\|w_0\|_{L^2(\Omega)} + \frac{a_2}{a_1}\right] + \frac{T}{\sqrt{2a_1}}\|y_1\|_{L^2(0,T;L^2(\Omega))} + \frac{1}{\sqrt{a_1}}e^{T/2a_1}\rho.$$

In order to obtain a global extension of a given solution $(\hat{u}, \hat{v})$, let us proceed by contradiction to assume that $T_{\max} < \infty$. Set

$$\tilde{R} \geq \sqrt{3}\|\hat{v}\|_{L^2(0,T_{\max};L^2(\Omega))},$$

and, employing the notation of Section 3, let

$$K_{\tilde{R}} = \{v \in L^2(0, 2T_{\max}; L^2(\Omega)) \mid \|v\|_{L^2(0,2T_{\max};L^2(\Omega))} < \tilde{R}\},$$

and define

$$\tilde{K}_{\tilde{R}} = \{v \in K_{\tilde{R}} \mid v(t) = \bar{v}(t) \text{ for } t \in [0, T_{\max})\}.$$

Following the discussion in Section 3, for any $v \in \tilde{K}_{\tilde{R}}$ it is possible to obtain a solution w of the differential inclusion (5.1a) in the sense of Problem $\tilde{\mathcal{P}}'_{a_4}$, on the interval $[0, 2T_{\max}]$ such that

$$w(t) = \bar{w}(t) \quad \text{for } t \in [0, T_{\max}].$$

Moreover, this solution will satisfy the estimate (3.7) on the interval $[0, 2T_{\max}]$.

To complete the argument, we employ again the fixed point method of Section 4. Let $\bar{T} = T_{\max} + \Delta$ for some $0 < \Delta < T_{\max}$, and let us define

$$\mathcal{F}_v(t) = S(t)v_0 + \int_0^T S(t-s)\left(\frac{1}{a_1}y_1 - \frac{a_2}{a_1}w_t\right)(s)\,ds$$

for $t \in [0, \bar{T}]$, setting $w_t(t) = \bar{w}_t(t)$ and hence $\mathcal{F}_v(t) = \bar{v}(t)$ for $t \in [0, T_{\max})$. To demonstrated that $\mathcal{F} : \tilde{K}_{\tilde{R}} \to \tilde{K}_{\tilde{R}}$, we proceed as follows.

Letting

$$h(s) = \frac{1}{a_1} y_1(s) - \frac{a_2}{a_1} w_t(s),$$

we note that

$$\begin{aligned}
\|\mathcal{F}_v\|^2_{L^2(0,\bar{T};L^2(\Omega))} &= \int_0^{T_{\max}} \|\mathcal{F}_v\|^2_{L^2(\Omega)}\,dt + \int_{T_{\max}}^{\bar{T}} \|\mathcal{F}_v\|^2_{L^2(\Omega)}\,dt \\
&= \|v\|^2_{L^2(0,T_{\max};L^2(\Omega))} \\
&\quad + \int_{T_{\max}}^{\bar{T}} \left\| S(t)v_0 + \int_0^t S(t-s)h(s)\,ds \right\|^2 dt \\
&\le \|v\|^2_{L^2(0,T_{\max};L^2(\Omega))} + 2\Delta \|v_0\|^2_{L^2(\Omega)} \\
&\quad + 2\int_{T_{\max}}^{\bar{T}} \left\| \int_0^t S(t-s)h(s)\,ds \right\|^2 dt.
\end{aligned}$$

To estimate these terms, note that, by construction

$$\|v\|^2_{L^2(0,T_{\max};L^2(\Omega))} \le \tilde{R}^2/3,$$

and by taking Δ sufficiently small,

$$2\Delta \|v_0\|^2_{L^2(\Omega)} \le \tilde{R}^2/3.$$

Estimating the third term, we note that

$$\begin{aligned}
&\int_{T_{\max}}^{\bar{T}} \left\| \int_0^{\bar{T}} S(t-s)h(s)\,ds \right\|^2 dt \\
&\qquad \le 2\Delta\bar{T} \left\{ \frac{1}{a_1^2} \|y_1\|^2_{L^2(0,\bar{T};L^2(\Omega))} + \frac{a_2^2}{a_1^2} \|w_t\|^2_{L^2(0,\bar{T};L^2(\Omega))} \right\}.
\end{aligned}$$

By employing the a priori estimate (3.7), we obtain that

$$\begin{aligned}
&\int_{T_{\max}}^{\bar{T}} \left\| \int_0^{\bar{T}} S(t-s)h(s)\,ds \right\|^2 dt \\
&\qquad \le \Delta\bar{T}C[\|y_1\|^2_{L^2_{\mathrm{loc}}(\mathbf{R}^+;L^2(\Omega))} + \|y_2\|^2_{L^2_{\mathrm{loc}}(\mathbf{R}^+;L^2(\Omega))} \\
&\qquad\qquad + \|v\|^2_{L^2(0,T;L^2(\Omega))} + \|w_0\|_{BV(\Omega)}] \\
&\qquad \le \Delta\bar{T}C[\|y_1\|^2_{L^2_{\mathrm{loc}}(\mathbf{R}^+;L^2(\Omega))} + \|y_2\|^2_{L^2_{\mathrm{loc}}(\mathbf{R}^+;L^2(\Omega))} \\
&\qquad\qquad + \tilde{R}^2 + \|w_0\|_{BV(\Omega)}] \\
&\qquad \le \frac{\tilde{R}^2}{3} \quad (\text{for } \Delta \text{ sufficiently small}),
\end{aligned}$$

where C is a generic constant. Summing the contributions from the three terms, it follows that for Δ sufficiently small,

$$\|\mathcal{F}_v\|_{L^2(0,\bar{T};L^2(\Omega))} \leq \tilde{R}, \tag{5.6}$$

and hence $\mathcal{F}_v$ maps $\tilde{K}_{\tilde{R}}$ into $\tilde{K}_{\tilde{R}}$.

Proceeding as in Section 4 allows us to conclude that $v^k \rightharpoonup w_t$ weakly in $L^2(0,\bar{T};\dot{L}^2(\Omega))$ implies that $\mathcal{F}_{v^k} \rightharpoonup \mathcal{F}_v$ weakly in $L^2(0,\bar{T};L^2(\Omega))$, and the Schauder-Tychonoff fixed point theorem indicates the existence of a solution to Problem $\tilde{\mathcal{P}}_{a_4}$ on the interval $[0, T_{\max} + \Delta)$ in contradiction to the maximality of the interval $[0, T_{\max})$.

We conclude

Theorem 7. *There exists a global solution to Problem* $\tilde{\mathcal{P}}_{a_4}$.

Finally we let $T > 0$ be arbitrary and we take a subsequence of solutions (v_{a_4}, w_{a_4}) to obtain in the limit $a_4 \to 0$ a global solution to the unrelaxed Stefan/Mullins-Sekerka memory problem, Problem $\tilde{\mathcal{P}}_0$. From (i), (iii) and (iv), it follows that a weak solution to (5.1a) is obtained in the limit $a_4 \to 0$. In order to obtain a solution (in the sense of Problem $\tilde{\mathcal{P}}_{a_4}$) to (5.1b) in the limit, we continue now by noting that by lower semicontinuity,

$$\int_Q |\nabla \tilde{w}|\, dx \leq \liminf_{a_4 \to 0} \int_Q |\nabla w_{a_4}|\, dx.$$

With regard to the other terms in (5.1b), from the assumptions on y_2 and from the estimates (i), (ii) and (iv), we get that

$$\begin{aligned} &\|2v_{a_4} - a_3 \mathcal{N} w_{a_4 t} - a_4 w_{a_4 t} + y_2\|_{L^2(0,T;L^2(\Omega))} \\ &\quad \leq \left[\frac{2}{\sqrt{a_1}} + \sqrt{a_2 a_3^2}\, C(\Omega) + \sqrt{a_4/a_2}\right] e^{T/2a_1} \rho + \|y_2\|_{L^2_{\text{loc}}(\mathbf{R}^+;L^2(\Omega))}; \end{aligned} \tag{5.7}$$

hence, $2u_{a_4} - a_3 \mathcal{N} w_{a_4 t} - a_4 w_{a_4 t} + y_2$ contains a subsequence which is weakly convergent in $L^2(0,T;\dot{L}^2(\Omega))$. Since $\eta \in L^1(0,T;\dot{BV}(\Omega))$ it follows from the definition of $\dot{BV}(\Omega)$ that $\eta \in L^2(0,T;L^2(\Omega))$. From (i) and (iii), it follows as in Section 3 that there exists a subsequence

w_{a_4} which is strongly convergent in $L^2(0,T;L^2(\Omega))$. Thus, it is also possible to pass to the limit in the second term in (5.1b).

Since by (iii) and (i), $\{w_{a_4}\}$ is uniformly bounded in $L^\infty(0,T;BV(\Omega))$ and $\{w_{a_4t}\}$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$, it follows [**10**, Corollary 4] taking subsequences, that $w \in C([0,T];L^2(\Omega))$ and $w \in L^\infty(0,T;BV(\Omega))$. Similarly, since $\{v_{a_4}\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $\{v_{a_4t}\}$ is uniformly bounded in $L^2(0,T;H^{-2}(\Omega))$, taking subsequences yields that $u \in C([0,T];H^{-1}(\Omega))$ and v is weakly continuous in $L^2(\Omega)$ with respect to time. From the estimate (viii), we obtain moreover that $\Delta(\beta * v_{a_4})$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$; therefore, we may conclude by Remark 2 in Section 4 that (v,w) constitutes a weak-mild solution of $\tilde{\mathcal{P}}_0$ in the sense of Definition 1. If k satisfies additionally conditions (a) or (b), then it follows from (vii) that $k * v \in L^\infty(0,T;H^1(\Omega))$ or $k * v \in L^2(0,T;H^1(\Omega))$, respectively. In either case, employing (i), we conclude that $v_t \in L^2(0,T;H^{-1}(\Omega))$. In this manner, a solution to Problem $\tilde{\mathcal{P}}_0$ is found. Setting $u = v + \bar{u}$ and $\mu = \mathcal{N} w_t$, a solution to Problem $\mathcal{P}_0$ is also attained. Clearly, as in the discussion in Remark 2, a weak solution (u,w) constitutes a weak-mild solution (in the sense of Definition 1) if $\Delta(\beta * u) \in L^2(0,T;L^2(\Omega))$. Since we have demonstrated that $\Delta(\beta * v) \in L^2(0,T;L^2(\Omega))$ and since by construction $\Delta(\beta * \bar{u}) = 0$ we obtain that (u,w) is a weak-mild solution of (1.5a) (in the sense of Definition 1), and Theorem 1 is proved.

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