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SECOND ORDER LINEAR VOLTERRA EQUATIONS GOVERNED BY A SINE FAMILY

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ABSTRACT. Let A be a closed linear operator in a Banach space X. This paper is concerned with the second order linear Volterra equation in X when A is the generator of a sine family on X.

1. Introduction. In this paper we study the second order linear Volterra equation in a Banach space X with norm $\|\cdot\|$

$$(\mathrm{SE}^f) \quad \begin{cases} u''(t) = Au(t) + \int_0^t B(t-s)u(s)\,ds + f(t) & \text{for } t \in [0,T] \\ u(0) = x \quad \text{and} \quad u'(0) = y. \end{cases}$$

Many authors considered $(\mathrm{SE}^{\mathrm{f}})$ in the case where A generates a cosine family on X (see [**4**], [**10**] and [**16**]).

It is, however, well known that the Laplacian Δ on the space $L^p(\mathbf{R}^N)$ does not generate a cosine family when $p \neq 2$ and $N > 1$ (see [**9**]).

As a generalization of cosine families, the theory of sine families (for the definition, see Section 2 below) was initiated by Arendt and Kellermann [**2**] to investigate the wave equation on the spaces like $L^p(\mathbf{R}^2)$ or $L^p(\mathbf{R}^3)$ $(1 \leq p < \infty)$ (see also Hieber [**6**], Kéyantuo [**8**], Rhandi [**13**] and Serizawa [**15**]).

The purpose of this paper is to study (SE) when A is the generator of a sine family on X.

To solve (SE^f) we consider the integral equation

$$(\mathrm{SE}_1) \quad u(t) = tx + A\int_0^t \int_0^s u(r)\,dr\,ds + \int_0^t B(t-s) \int_0^s \int_0^r u(\eta)\,d\eta\,dr\,ds$$

and construct the strongly continuous family $\{R(t) : t \geq 0\} \subset B(X)$ which gives the solution of (SE_1):

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(r1) For all $x \in X$, $R(\cdot)x \in C([0,\infty) : X)$.

(r2) For all $x \in X$, $\int_0^{\cdot} \int_0^s R(r)x\,dr\,ds \in C([0,\infty) : Y)$.

(r3) $R(t)x - tx = A\int_0^t \int_0^s R(r)x\,dr\,ds + \int_0^t B(t-s)\int_0^s \int_0^r R(\eta)x\,d\eta\,dr\,ds$ for all $x \in X$ and $t \geq 0$.

(r4) $R(t)x - tx = \int_0^t \int_0^s R(r)Ax\,dr\,ds + \int_0^t \int_0^s \int_0^r R(r-\eta)B(\eta)x\,d\eta\,dr\,ds$ for all $x \in Y$ and $t \geq 0$.

Here denote by Y the Banach space $D(A)$ endowed with the graph norm of A and by $B(X)$ the set of all bounded linear operators on X. We call $\{R(t) : t \geq 0\}$ a *solution family* for (SE_1).

In the previous paper [**11**], the author studied (SE^f) when A satisfies the cosine resolvent condition without assuming the density of $D(A)$ in X, i.e., A generates a locally Lipschitz continuous sine family on X, and investigated the solution family $\{R(t) : t \geq 0\}$ for (SE_1) and proved the following:

A solution family for (SE_1) is unique if it exists and the solution u of (SE^f) is then given by

$$u(t) = \frac{d}{dt}(R(t)x + R^{(1)}(t)y + (R^{(1)} * f)(t)),$$

where $R^{(1)}(t)z = \int_0^t R(s)z\,ds$ for $t \geq 0$ and $z \in X$, and "$*$" denotes the convolution. Moreover, in the case $\rho(A)$ (the resolvent set of A) $\neq \phi$, there exists a unique classical solution u of (SE^f) if and only if $u_{[1]} \in C^3([0,T] : X)$ where $u_{[1]}$ is defined by

$$u_{[1]}(t) = R(t)x + R^{[1]}(t)y + (R^{[1]} * f)(t) \tag{1.1}$$

for $t \in [0,T]$. In this case, $u = u'_{[1]}$.

In the present paper we aim to construct a solution family $\{R(t) : t \geq 0\}$ for (SE_1) assuming that A is the generator of a sine family on X and that the appropriate conditions for a family $\{B(t) : t \geq 0\}$ of bounded linear operators from Y into X. Our approach to (SE^f) is different from [**3**] where the Laplace transform technique was used to study first-order Volterra equations for generators of integrated semigroups. The result obtained can be applied to the wave equation with the memory term:

$$\begin{cases} u_{tt}(t,x) = \Delta u(t,x) \\ \quad + \int_0^t b(t-s)\Delta u(s,x)\,ds + f(t,x), & (t,x) \in [0,T] \times \mathbf{R}^N \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) & x \in \mathbf{R}^N \end{cases}$$

on the spaces $L^p(\mathbf{R}^N)$, $N = 2, 3$.

2. Main results. First we recall the theory of sine families. Let A be an operator in X for which (2.1) below holds for some strongly continuous, exponentially bounded operator family $\{S(t) : t \geq 0\} \subset B(X)$ satisfying $\|S(t)\| \leq Me^{\omega t}$:

$$(\lambda^2 - A)^{-1}x = \int_0^\infty e^{-\lambda t} S(t)x\, dt \tag{2.1}$$

for all $x \in X$ and $\lambda > \omega$. Then $\{S(t) : t \geq 0\}$ is called a *sine family* on X and A its *generator*. The following properties of sine families are well known [**2**, Lemmas 1.4, 1.5 and 1.7] and are used later in our discussion.

Proposition 2.1. *Let A be the generator of a sine family $\{S(t) : t \geq 0\}$ on X. Then the following hold*:

(i) *For every $x \in D(A)$ we have $S(t)x \in D(A)$, $AS(t)x = S(t)Ax$ and*

$$S(t)x = tx + \int_0^t \int_0^s S(r)Ax\, dr\, ds \quad \text{for } t \geq 0. \tag{2.2}$$

(ii) *For every $x \in X$ we have $\int_0^t \int_0^s S(r)x\, dr\, ds \in D(A)$ and*

$$A \int_0^t \int_0^s S(r)x\, dr\, ds = S(t)x - tx \quad \text{for } t \geq 0. \tag{2.3}$$

(iii) *Let $f \in L^1(0, T : X)$ and put $v(t) = (S * f)(t)$ for $t \in [0, T]$. Then $\int_0^{\cdot} \int_0^s v(r)\, dr\, ds \in D(A)$ and*

$$A \int_0^t \int_0^s v(r)\, dr\, ds = v(t) - \int_0^t \int_0^s f(r)\, dr\, ds \quad \text{for } t \in [0, T]. \tag{2.4}$$

(iv) *Let $f \in L^1(0, T : X)$. If $u \in C([0, T] : X)$ satisfies $u(t) = A \int_0^t \int_0^s u(r)\, dr\, ds + \int_0^t \int_0^s f(r)\, dr\, ds$ for $t \in [0, T]$, then $u(t) = (S * f)(t)$ for $t \in [0, T]$.*

We turn to the second order Volterra equation (SE^f).

Suppose the following conditions for the operator A in X and the family $\{B(t) : t \geq 0\}$ of bounded linear operators from Y into X:

(H1) A linear operator A in X is the generator of a sine family $\{S(t) : t \geq 0\}$ on X and densely defined in X.

(H2) For $x \in Y$, the function $B(\cdot)x$ is strongly measurable and there exists a function $b \in L^1_{\mathrm{loc}}(\mathbf{R}^+ : \mathbf{R}^+)$ such that

$$\|B(t)x\| \leq b(t)\|x\|_Y \quad \text{for a.e. } t \geq 0. \tag{2.5}$$

(H3) For $t \geq 0$, $\sup\{\| \int_0^t B(t-s)S(s)x\, ds\| : x \in Y, \|x\| \leq 1\} < \infty$.

Here recall the property of a solution family $\{R(t) : t \geq 0\}$ of (SE_1) proved in [**11**].

Lemma 2.2. *Let $f \in L^1(0,T : X)$. Then we have $\int_0^{\cdot} \int_0^s (R^{[1]} * f)(r)\, dr\, ds \in C([0,T] : Y)$ and*

$$\begin{aligned} A\int_0^t \int_0^s (R^{[1]} * f)(r)\, dr\, ds &= (R^{[1]} * f)(t) - \int_0^t \int_0^s \int_0^r f(\eta)\, d\eta\, dr\, ds \\ &\quad - \left(B * \int_0^{\cdot} \int_0^s (R^{[1]} * f)(r)\, dr\, ds\right)(t) \end{aligned} \tag{2.6}$$

for $t \in [0,T]$.

Note that this lemma holds good if (r1), (r2) and (r3) are satisfied (see the proof of [**11**, Lemma 2.5]).

Now we are in a position to state the main result in this paper:

Theorem 2.3. *Suppose* (H1)–(H3). *Then there exists a unique solution family $\{R(t) : t \geq 0\}$ in $B(X)$ for* (SE_1).

Proof. Set

$$U(t)y = (B * S(\cdot)y)(t) \quad \text{for } t \geq 0 \text{ and } y \in Y.$$

Since Y is dense in X, the assumption (H3) shows that $U(t)$ can be extended to a bounded linear operator on X and denote it by the same symbol $U(t)$ for $t \geq 0$. Then $U(\cdot)x \in C([0,\infty):X)$ for $x \in X$.

For convenience we use the abbreviation: if $\{V_i(t) : t \geq 0\}$, $1 \leq i \leq 4$, are strongly continuous families in $B(X)$, the equation $V_1 = V_2 + V_3 * V_4$ means that

$$V_1(t)x = V_2(t)x + \int_0^t V_3(t-s)V_4(s)x\,ds \quad \text{for } t \geq 0 \text{ and } x \in X.$$

Let R_U be the resolvent kernel of U, i.e.,

$$R_U = U + U * R_U = U + R_U * U. \tag{2.7}$$

Define a strongly continuous family $\{R(t) : t \geq 0\}$ in $B(X)$ by

$$R = S + S * R_U. \tag{2.8}$$

Noting (2.3) and (2.4), in view of (2.8) we see that $\int_0^{\cdot} \int_0^s R(r)x\,dr\,ds \in D(A)$ and

$$\begin{aligned} A\int_0^t \int_0^s R(r)x\,dr\,ds &= A\int_0^t \int_0^s S(r)x\,dr\,ds \\ &\quad + A\int_0^t \int_0^s (S * R_U)(r)x\,dr\,ds \\ &= S(t)x - tx + (S * R_U)(t)x \\ &\quad - \int_0^t \int_0^s R_U(r)\,dr\,ds \\ &= R(t)x - tx \\ &\quad - \int_0^t \int_0^s R_U(r)x\,dr\,ds \end{aligned} \tag{2.9}$$

for $x \in X$ and $t \geq 0$, which proves (r2).

We also have, by (2.7) and (2.8),

$$\begin{aligned} R * U &= (S + S * R_U) * U \\ &= S * U + S * (R_U - U) \\ &= S * R_U \end{aligned}$$

and this together with (2.8) yields

$$R = S + R * U. \tag{2.10}$$

Put $S^{[2]}(t)x = \int_0^t \int_0^s S(r)x\,dr\,ds$ for $t \geq 0$ and $x \in X$. For $y \in Y$, we have

$$\begin{aligned}\int_0^t \int_0^s U(r)y\,dr\,ds &= \int_0^t \int_0^s (B * S(\cdot)y)\,dr\,ds \\ &= \left(B * \int_0^{\cdot} \int_0^s S(r)y\,dr\,ds\right)(t) \\ &= (B * S^{[2]})(t)y.\end{aligned}$$

From the density of Y in X, we deduce

$$U^{[2]} = B * S^{[2]}, \tag{2.11}$$

where we put $U^{[2]}(t)x = \int_0^t \int_0^s U(r)x\,dr\,ds$ for $t \geq 0$ and $x \in X$.

Let $R^{[2]}(t)x = \int_0^t \int_0^s R(r)x\,dr\,ds$ for $t \geq 0$ and $x \in X$. Then the integration of (2.10) gives

$$R^{[2]} = S^{[2]} + R^{[2]} * U \tag{2.12}$$

and so by (r2) and (2.3) we find $R^{[2]} * U \in C([0,\infty) : Y)$. Convolving B to the equation (2.12) from the left-hand side, we have by (2.11)

$$\begin{aligned}B * R^{[2]} &= B * S^{[2]} + B * R^{[2]} * U \\ &= U^{[2]} + B * R^{[2]} * U.\end{aligned} \tag{2.13}$$

On the other hand, integrating (2.7) twice and setting $R_U^{[2]}(t)x = \int_0^t \int_0^s R_U(r)x\,dr\,ds$ for $t \geq 0$ and $x \in X$, we have

$$R_U^{[2]} = U^{[2]} + R_U^{[2]} * U.$$

Combining this with (2.13) we have $R_U^{[2]} = B * R^{[2]}$, which implies with (2.9) that (r3) is satisfied.

To prove (r4), let $x \in Y$ and put

$$y(t) = tx + \int_0^t \int_0^s R(r)Ax\,dr\,ds + \int_0^t \int_0^s \int_0^r R(r-w)B(w)x\,dw\,dr\,ds$$

for $t \geq 0$.

Then by Fubini's theorem we have $y(t) - tx = \int_0^t \int_0^s R(s-r)(Ax + \int_0^r B(w)x\,dw)\,dr\,ds = \int_0^t R^{[1]}(t-s)(Ax + \int_0^s B(r)x\,dr)\,ds$. So the equation (2.6) in Lemma 2.2 with $f(t) = Ax + \int_0^t B(s)x\,ds$ gives

$$\begin{aligned} y(t) - tx &= (R^{[1]} * f)(t) \\ &= A\int_0^t \int_0^s (y(r) - rx)\,dr\,ds \\ &\quad + \int_0^t B(t-s)\int_0^s \int_0^r (y(\eta) - \eta x)\,d\eta\,dr\,ds \\ &\quad + \int_0^t \int_0^s \int_0^r \left(Ax + \int_0^\eta B(\xi)x\,d\xi\right) d\eta\,dr\,ds \\ &= A\int_0^t \int_0^s y(r)\,dr\,ds \\ &\quad + \int_0^t B(t-s)\int_0^s \int_0^r y(\eta)\,d\eta\,dr\,ds \end{aligned}$$

for $t \geq 0$. Then putting $z(t) = y(t) - R(t)x$ and $v(t) = \int_0^t \int_0^s z(r)\,dr\,ds$ for $t \geq 0$, and using the closedness of A, we have

$$v(t) = A\int_0^t \int_0^s v(r)\,dr\,ds + \int_0^t \int_0^s (B * v)(r)\,dr\,ds$$

for $t \geq 0$. Since A is a generator of a sine family $\{S(t) : t \geq 0\}$ on X, we have from Proposition 2.1 (iii) and (iv) that $v = S * B * v$ and $v \in C([0,\infty) : Y)$. Hence $B * v = B * S * B * v = U * B * v$. The estimation of this equality gives that for $t \in [0, T]$,

$$\begin{aligned} \|(B * v)(t)\| &\leq \int_0^t \|U(t-s)\| \|(B * v)(s)\|\,ds \\ &\leq \sup\{\|U(r)\| : r \in [0, T]\} \int_0^t \|(B * v)(s)\|\,ds, \end{aligned}$$

which implies by Gronwall's inequality that $B * v = 0$. Thus we have $v = S * B * v = 0$ and so $z = 0$. This proves (r4). □

Next we consider the sufficient condition for (H3) to be satisfied in the special case where $B(t) = b(t)A$. Then we obtain the following theorem.

Theorem 2.4. *Suppose* (H1)*, and* $b \in AC_{\text{loc}}(\mathbf{R}^+ : \mathbf{R}^+)$*,* $b' \in BV_{\text{loc}}(\mathbf{R}^+ : \mathbf{R}^+)$ *and* $b(0) = 0$*. Then the condition* (H3) *is satisfied.*

Proof. Let $x \in Y$. Integrating by parts, and noting that $b(0) = 0$ and (2.3), we have

$$\begin{aligned}\int_0^t b(t-s)AS(s)x\,ds &= \int_0^t b'(t-s)\int_0^s AS(r)x\,dr\,ds \\ &= b'(0)(S(t)x - tx) \\ &\quad + \int_0^t dc(t-s)(S(s)x - sx),\end{aligned}$$

where we put $c = b'$ and the second term in the above equation denotes the Stieltjes integral. This implies (H3). □

To prove the existence and uniqueness of classical solutions of (SE^f), we use the next result proved in [**11**]:

Theorem 2.5 [**11**, Theorem 2.3]. *Suppose that the solution family* $\{R(t) : t \geq 0\}$ *for* (SE_1) *exists and that* $\rho(A) \neq \varnothing$*. Then there exists a unique classical solution* u *of* (SE^f) *if and only if the function* $u_{[1]}$ *defined by* (1.1) *in Section* 1 *is of class* C^3*. In this case,* $u = u'_{[1]}$*.*

By virtue of Theorem 2.5 we obtain the following:

Theorem 2.6. *Suppose that the assumptions of Theorem* 2.4 *are satisfied. If* $x \in D(A^2)$*,* $y \in D(A)$ *and* $f \in C([0,T] : Y)$*, then there exists a unique classical solution* u *of* (SE^f) *and* u *satisfies*

$$\|u(t)\| \leq C\left(\|x\| + \|y\| + \int_0^t (1 + b(s))\|Ax\|\,ds + \int_0^t \|f(s)\|\,ds\right) \tag{2.14}$$

for $t \in [0,T]$*, where* C *is a constant independent of* x, y *and* f*.*

Proof. Theorems 2.3 and 2.4 show that the solution family $\{R(t) : t \geq 0\}$ for (SE_1) exists. We shall show $u_{[1]} \in C^3([0,T] : X)$. By using the property (r4) we differentiate (1.1) to get

$$\begin{aligned} u'_{[1]}(t) &= x + R^{[1]}(t)Ax + (R^{[1]} * b(\cdot)Ax)(t) \\ &\quad + R(t)y + (R * f)(t); \end{aligned} \tag{2.15}$$

$$\begin{aligned} u''_{[1]}(t) &= R(t)Ax + (R * b(\cdot)Ax)(t) + y \\ &\quad + R^{[1]}(t)Ay + (R^{[1]} * b(\cdot)Ay)(t) \\ &\quad + (1 * f)(t) + (R^{[1]} * Af)(t) \\ &\quad + (R^{[1]} * bA * f)(t). \end{aligned} \tag{2.16}$$

In view of equation (2.16), from the assumption we get the desired conclusion. The estimation of the equation (2.15) yields the estimate (2.14) of a classical solution u of (SE^f). □

Let $X = L^p(\mathbf{R}^N)$ ($N = 2$ or 3; $1 \leq p < \infty$), and $A = \Delta$ with distributional domain. It is known that A generates a sine family on X (see [**8**, Theorem 3.1]). Thus, Theorem 2.6 gives an operator-theoretical approach to the wave equation with the memory term:

$$\begin{cases} u_{tt}(t,x) = \Delta u(t,x) + \int_0^t b(t-s)\Delta u(s,x)\,ds & \\ \qquad\qquad + f_{(t,x)}, & (t,x) \in [0,T] \times \mathbf{R}^N \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) & x \in \mathbf{R}^N \end{cases}$$

on the space $L^p(\mathbf{R}^N)$.

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