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EIGENVALUES OF A FREDHOLM INTEGRAL OPERATOR AND APPLICATIONS TO PROBLEMS OF STATISTICAL INFERENCE

V.K. JANDHYALA AND P.L. JIANG

ABSTRACT. We determine the eigenvalues of a Fredholm integral operator of the second kind. The solution of the eigenvalue problem has applications to finding the distribution function of a stochastic integral. The stochastic integral itself represents the asymptotic form of a statistical test. Also discussed are related results for inference and applications.

1. Introduction. Applications of Fredholm integral operators in the areas of physical sciences and engineering are well known. Their applications to problems of statistical inference are probably less known among researchers in mathematics and other disciplines. The dynamic instability inherent in physical processes can often be statistically modelled by the change-point method. The change-point problem primarily consists of testing for a model with no change in the model parameters against a model where parameter changes occur after a certain unknown point of time. The problem has received wide attention among researchers in statistical inference. Test statistics for the problem have been derived and their distribution theory has been discussed in the literature; see Jandhyala and MacNeill [**2, 3, 4**], MacNeill [**6**] and Nabeya and Tanaka [**7**] and the references therein. Asymptotic distributions of these change-detection test statistics have been shown to involve a variety of Fredholm integral operators. In this article, we consider the problem of solving for the eigenvalues of a specific Fredholm integral operator. The solution has applications to the asymptotic distribution of a change-detection statistic derived in the literature; see Jandhyala and MacNeill [**3**].

The eigenvalue problem of interest is

$$\lambda_p \phi_p(t) = \int_0^1 K_p(s,t)\phi_p(s)\, ds \tag{1}$$

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where the kernel $K_p(s,t)$ is given by

$$K_p(s,t) = \min(s,t) - \sum_{m=0}^{p} (2m+1) g_m(t)\, g_m(s) \tag{2}$$

and

$$g_m(t) = \sum_{v=0}^{[m/2]} \frac{(-1)^v \binom{2m}{m, v, v, m-2v}}{2^{4v} \binom{m-1/2}{v}} \int_0^t \left(x - \frac{1}{2}\right)^{m-2v} dx. \tag{3}$$

This problem has been formulated by MacNeill [**6**] while finding the characteristic function of the stochastic integral given by

$$\int_0^1 B_p^2(t)\, dt. \tag{4}$$

The process $\{B_p(t), t \in [0,1]\}$ is a continuous functional defined on the standard Brownian motion $\{B(t), t \in [0,1]\}$ and is given by

$$\begin{aligned} B_p(t) &= h_p(B(t)) \\ &\equiv B(t) - \sum_{m=0}^{p} (2m+1) \Bigg\{ \sum_{v=0}^{[m/2]} (-1)^v \binom{2m}{m, v, v, m-2v} \\ &\qquad\qquad \times \int_0^t \left(x - \frac{1}{2}\right)^{m-2v} dx \Bigg\} \\ &\quad \times \Bigg\{ \sum_{v=0}^{[m/2]} \frac{(-1)^v \binom{2m}{m, v, v, m-2v}}{2^{4v} \binom{m-1/2}{v}} \Bigg[\left(\frac{1}{2}\right)^{m-2v} B(1) \\ &\qquad\qquad - (m-2v) \int_0^1 \left(x - \frac{1}{2}\right)^{m-2v-1} B(s)\, ds \Bigg] \Bigg\}. \end{aligned}$$

MacNeill [**6**] states that the sequence of eigenvalues $\{\lambda_{p,n}\}_{n=1}^{\infty}$ of (1) are

$$\begin{aligned} \lambda_{p,2n-1} &= \frac{1}{4\, z_{p-1,n}^2} \\ \lambda_{p,2n} &= \frac{1}{4\, z_{p,n}^2}, \qquad n = 1, 2, \dots \end{aligned} \tag{6}$$

where $z_{p,n}$ is the nth positive zero of the pth order spherical Bessel function of the first kind. While MacNeill [**6**] established the validity of the above solution for $p=1$, the proof for a general p was not given. The main objective of this paper is to provide a complete proof of (6) for any general p. Once the eigenvalues $\{\lambda_{p,n}\}_{n=1}^{\infty}$ are identified, the characteristic function of the stochastic integral in (4) may then be obtained as

$$\Phi_p(u) = \prod_{n=1}^{\infty}(1-2iu\lambda_{p,n})^{-1/2}. \tag{7}$$

The distribution function of the stochastic integral $\int_0^1 B_p^2(t)\,dt$ may be obtained by inverting the characteristic function $\Phi_p(u)$ applying Lévy's inversion formula. This function has direct applications to the change-point problem in statistical inference.

We first give the proof of (6) in Section 2. The specific applications of this solution and solutions of other related integral equations for problems of statistical inference are discussed in Section 3.

2. Determination of the eigenvalues. We shall now give the proof of (6). The proof requires several theorems and they are stated and proved in the following. A major difficulty in constructing a proof arises due to the fact that the function $g_m(t)$ as given by (3) has a complicated structure. We, however, take advantage of the following theorem due to Jandhyala and Minogue [**5**]. The theorem provides an alternative exact expression for $g_m(t)$. Needless to say, this expression is simpler in form and proves to be the basis of our proof.

Theorem 1. *For $t\in[0,1]$, we have*

$$g_m(t) = \sum_{k=0}^{m}(-1)^{m+k}\frac{\binom{m+k}{k}\binom{m}{k}t^{k+1}}{k+1}, \qquad m=0,1,2,\ldots. \tag{8}$$

Differentiating the integral equation (1) twice with respect to t, one obtains

$$\lambda_p\phi_p^{(2)}(t)+\phi_p(t) = -\sum_{m=1}^{p}(2m+1)g_m^{(2)}(t)\int_0^1 g_m(s)\phi_p(s)\,ds. \tag{9}$$

The boundary conditions are

$$\phi_p(0) = \phi_p(1) = 0. \tag{10}$$

We shall rewrite the equation (9) as

$$\lambda_p \phi_p^{(2)}(t) + \phi_p(t) = \sum_{m=1}^{p} T_m(t) c_m \tag{11}$$

where

$$c_m = -(2m+1) \binom{m+1}{2} 2 \int_0^1 g_m(s)\phi_p(s)\,ds$$
$$T_m(t) = g_m^{(2)}(t) \Big/ \binom{m+1}{2} 2, \qquad m = 1, 2, \dots, p.$$

Since $T_m(t)$ is a polynomial function of degree $m-1$, the general solution of (11) may be written as

$$\begin{aligned} \phi_p(t) &= a \cos \frac{t}{\sqrt{\lambda_p}} + b \sin \frac{t}{\sqrt{\lambda_p}} \\ &\quad + \sum_{m=1}^{p} T_m(t) c_m + \sum_{m=3}^{p} \tilde{T}_m(t) c_m \end{aligned} \tag{12}$$

where

$$\tilde{T}_m(t) = \sum_{l=1}^{[(m-1)/2]} (-\lambda_p)^l T_m^{(2l)}(t).$$

We first note some properties of $T_m(t)$ and $g_m(t)$ in the following lemma. These properties will be used often later in the proof. The proof of the lemma is for the most part algebraic and is omitted.

Lemma 2. *The functions $g_m(t)$ and $T_m(t)$ given above satisfy the following.*

(i) *For $t \in [0,1]$ and $m \geq 1$,*

$$g_m(t) = -\frac{1}{\binom{m+1}{2} 2} t(1-t) g_m^{(2)}(t).$$

Furthermore, $g_m(t) = -t(1-t)T_m(t)$ *and* $g_m(0) = g_m(1) = 0$.

(ii) *For* $m \geq 1$, $T_m(0) = (-1)^{m+1}$ *and* $T_m(1) = 1$.

(iii) *For* $t \in [0,1]$ *and* $m \geq 2$,

$$g_m^{(1)}(t) = 2(2m-1)g_{m-1}(t) + g_{m-2}^{(1)}(t).$$

The boundary conditions $\phi_p(0) = \phi_p(1) = 0$ together with Lemma 2 (ii) imply that

$$a + c_1 - c_2 + \sum_{m=3}^{p} \{(-1)^{m+1} + \tilde{T}_m(0)\} c_m = 0 \tag{13}$$

and

$$a \cos \frac{1}{\sqrt{\lambda_p}} + b \sin \frac{1}{\sqrt{\lambda_p}} + c_1 + c_2 + \sum_{m=3}^{p} (1 + \tilde{T}_m(1)) c_m = 0. \tag{14}$$

The conditions on c_m, $m = 1, 2, \ldots, p$ are

$$\begin{aligned} -\frac{c_m}{(2m+1)\binom{m+1}{2}2} &= \int_0^1 g_m(s)\phi_p(s)\,ds \\ &= a\int_0^1 g_m(s)\cos\frac{s}{\sqrt{\lambda_p}}\,ds \\ &\quad + b\int_0^1 g_m(s)\sin\frac{s}{\sqrt{\lambda_p}}\,ds \\ &\quad + \sum_{n=1}^{p} c_n \int_0^1 g_m(s)T_n(s)\,ds \\ &\quad + \sum_{n=3}^{p} c_n \int_0^1 g_m(s)\tilde{T}_n(s)\,ds. \end{aligned} \tag{15}$$

In the following, we deal with the respective coefficients of a, b and c_n, $n = 1, 2, \ldots, p$ in the right hand side of (15).

Let

$$a_m = \int_0^1 g_m(s) \cos \frac{s}{\sqrt{\lambda_p}}\, ds$$
$$b_m = \int_0^1 g_m(s) \sin \frac{s}{\sqrt{\lambda_p}}\, ds, \qquad m = 1, \dots, p.$$

Theorem 3. *The following recurrence relations hold.*

(i) *For* $m \geq 1$,

$$\begin{aligned} a_m &= -2(2m-1)\sqrt{\lambda_p} b_{m-1} + a_{m-2} \\ b_m &= 2(2m-1)\sqrt{\lambda_p} a_{m-1} + b_{m-2} \end{aligned} \tag{16}$$

where $a_{-1} = \lambda_p(\cos(1/\sqrt{\lambda_p}) + 1)$, $a_0 = \lambda_p(\cos(1/\sqrt{\lambda_p}) - 1)$ *and* $b_{-1} = b_0 = \lambda_p \sin(1/\sqrt{\lambda_p})$.

(ii) *For* $m \geq -1$,

$$a_m\left(-\sin \frac{1}{\sqrt{\lambda_p}}\right) + b_m\left(\cos \frac{1}{\sqrt{\lambda_p}}\right) = (-1)^m b_m$$
$$a_m\left(\cos \frac{1}{\sqrt{\lambda_p}}\right) + b_m\left(\sin \frac{1}{\sqrt{\lambda_p}}\right) = (-1)^{m+1} a_m.$$

(iii) *For* $m \geq 1$,

$$a_m = -2(2m-1)\sqrt{\lambda_p}\frac{(-1)^m - \cos(1/\sqrt{\lambda_p})}{\sin(1/\sqrt{\lambda_p})} a_{m-1} + a_{m-2} \tag{17}$$

and

$$b_m = 2(2m-1)\sqrt{\lambda_p}\frac{(-1)^{m-1} - \cos(1/\sqrt{\lambda_p})}{-\sin(1/\sqrt{\lambda_p})} b_{m-1} + b_{m-2}.$$

Proof. (i) For $m \geq 3$,

$$a_m + i b_m = \int_0^1 g_m(s) e^{i(s/\sqrt{\lambda_p})}\, ds.$$

Integration by parts together with application of Lemma 2 (iii) implies

$$a_m + ib_m = -\frac{\sqrt{\lambda_p}}{i}\bigg\{2(2m-1)\int_0^1 g_{m-1}(s)e^{i(s/\sqrt{\lambda_p})}\,ds + \int_0^1 g_{m-2}^{(1)}(s)e^{i(s/\sqrt{\lambda_p})}\,ds\bigg\}$$

Further integration gives

$$a_m + ib_m = \{-2(2m-1)\sqrt{\lambda_p}b_{m-1} + a_{m-2}\} + i\{2(2m-1)\sqrt{\lambda_p}a_{m-1} + b_{m-2}\}.$$

The result follows by letting $m \geq 1$ with a_0, a_{-1}, b_0, b_{-1} as defined in the theorem.

The relationships in (ii) follow easily from (i), and (iii) itself follows from (i) and (ii). $\square$

Theorem 4. *Let*

$$c_{m,n} = \int_0^1 g_m(s)T_n(s)\,ds, \qquad m, n = 1, 2, \ldots, p.$$

Then,

$$c_{m,n} = \begin{cases} 0, & n \neq m \\ -\bigg(1/(2m+1)\begin{pmatrix} m+1 \\ 2 \end{pmatrix} 2\bigg), & n = m. \end{cases}$$

Proof. Lemma 2 (i) implies, for $n \neq m$,

$$\begin{aligned} c_{m,n} &= \frac{1}{\binom{m+1}{2}2}\int_0^1 g_m^{(2)}(s)g_n(s)\,ds \\ &= \frac{-1}{\binom{m+1}{2}2}\int_0^1 g_m^{(1)}(s)g_n^{(1)}(s)\,ds \\ &= \frac{1}{\binom{m+1}{2}2}\int_0^1 g_m(s)g_n^{(2)}(s)\,ds \\ &= \frac{\binom{n+1}{2}}{\binom{m+1}{2}}c_{m,n}. \end{aligned}$$

Thus, it follows that for $n \neq m$, $c_{m,n} = 0$. For the case $n = m$, note that

$$\int_0^1 g_m(s)s^k ds = 0 \quad \text{for } k \leq m-2.$$

Through repeated integration by parts and application of Lemma 2 (iii), we can show

$$\int_0^1 g_m(s)s^{m-1}ds = -\frac{(m-1)!m!}{(2m+1)!}.$$

We then have

$$\int_0^1 g_m(s)T_m(s)\,ds = \int_0^1 \; g_m(s)\frac{1}{\binom{m+1}{2}2}\binom{2m}{m}\binom{m}{m}ms^{m-1}\,ds$$

and the result follows. $\square$

Theorem 5. *Let*

$$\tilde{c}_{m,n} = \int_0^1 g_m(s)\tilde{T}_n(s)\,ds,$$
$$m = 1,\dots,p; \quad n = 3,\dots,p.$$

Then

$$\tilde{c}_{m,n} = \begin{cases} 0, & n < m+2 \\ (4(2m+3)/(m+3)(m+2))\lambda_p, & n = m+2 \end{cases}.$$

Proof. Both parts follow easily from the above integral and the details are omitted. □

Applying Theorems 3, 4 and 5 into (15), we obtain

$$a\, a_m + b\, b_m + c_{m+2}\frac{4(2m+3)}{(m+3)(m+2)}\lambda_p + \sum_{n=m+3}^{p} c_n\, \tilde{c}_{m,n} = 0,$$
$$m = 1, 2, \ldots, p.$$

These p equations together with (13) and (14) form a system of $(p+2)$ linear homogeneous equations in $(p+2)$ variables, a, b, and c_m, $m = 1, 2, \ldots, p$. For $\phi_p(t)$ to be a nontrivial solution of the Fredholm equation (1), we must set the determinant of the coefficient matrix of the system of equations to zero. Thus,

$$\left|\begin{array}{cc|cccccc}
1 & 0 & 1 & -1 & 1+\tilde{T}_3(0) & -1+\tilde{T}_4(0) & \cdots & (-1)^p+\tilde{T}_{p-1}(0) & (-1)^{p+1}+\tilde{T}_p(0) \\
\cos\frac{1}{\sqrt{\lambda_p}} & \sin\frac{1}{\sqrt{\lambda_p}} & 1 & 1 & 1+\tilde{T}_3(1) & 1+\tilde{T}_4(1) & \cdots & 1+\tilde{T}_{p-1}(1) & 1+\tilde{T}_p(1) \\
a_1 & b_1 & 0 & 0 & \frac{5}{3}\lambda_p & \tilde{c}_{1,4} & \cdots & \tilde{c}_{1,p-1} & \tilde{c}_{1,p} \\
a_2 & b_2 & 0 & 0 & 0 & \frac{7}{5}\lambda_p & \cdots & \tilde{c}_{2,p-1} & \tilde{c}_{2,p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
a_{p-2} & b_{p-2} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{4(2p-1)}{(p+1)p}\lambda_p \\
\hline
a_{p-1} & b_{p-1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
a_p & b_p & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right| = 0.$$

When p is equal to 0 or 1, the solution in (6) can easily be verified. For $p \geq 2$, the partitioning as above implies

$$\begin{vmatrix} a_{p-1} & b_{p-1} \\ a_p & b_p \end{vmatrix} = 0.$$

It follows from Theorem 3 (ii) that

$$b_{p-1}\, b_p = 0 \tag{18}$$

and also that

$$\sin(1/\sqrt{\lambda p}) \neq 0. \tag{19}$$

Theorem 6. *Let* $(1/2\sqrt{\lambda_p}) = z$. *Then, for* $m \geq -1$,

$$b_m = \begin{cases} (-1)^n (\cos z/2z) j_m(z), & m = 2n \\ (-1)^{n+1} (\sin z/2z) j_m(z), & m = 2n+1 \end{cases}$$

where $j_m(z)$ *is the mth order spherical Bessel function of the first kind.*

Proof. First note that Theorem 3 (iii) implies

$$b_m = \begin{cases} \dfrac{2m-1}{z} (\cos z/\sin z) b_{m-1} + b_{m-2}, & m \text{ is even} \\ dsize\frac{2m-1}{z} (-\sin z/\cos z) b_{m-1} + b_{m-2}, & m \text{ is odd} \end{cases}$$

with

$$b_{-1} = b_0 = \frac{\sin z}{2z} \cdot \frac{\cos z}{z}.$$

We shall prove the theorem based on induction. Note easily that the theorem is true for $m = -1, 0$. Now, assume that it is true for $m \leq 2n-1$. Then, for $m = 2n$, we have

$$\begin{aligned} b_m &= \frac{2m-1}{z} \left(\frac{\cos z}{\sin z} \right) \left\{ (-1)^n \frac{\sin z}{2z} j_{m-1}(z) \right\} \\ &\quad + \left\{ (-1)^{n-1} \frac{\cos z}{2z} j_{m-2}(z) \right\} \\ &= (-1)^n \frac{\cos z}{2z} \left\{ \frac{2m-1}{z} j_{m-1}(z) - j_{m-2}(z) \right\} \\ &= (-1)^n \frac{\cos z}{2z} j_m(z). \end{aligned}$$

Similarly, for $m = 2n + 1$, we obtain

$$b_m = (-1)^{n+1} \frac{\sin z}{2z} j_m(z).$$

This completes the proof of the theorem. □

Applying Theorem 6 into (18) and noting (19), one obtains

$$j_{p-1}(z) j_p(z) = 0. \tag{20}$$

Hence, the sequence of eigenvalues are obtained as solutions of either $j_{p-1}(z) = 0$ or $j_p(z) = 0$ where $\lambda_p = 1/(4z^2)$.

This completes the proof of solution in (6).

3. Applications. Let $\{\varepsilon_j\}_{j=1}^n$ be a sequence of unobservable independent random variables defined on the same probability space each with mean 0 and variance σ^2. Let $\{Y_{nj}, j = 1, \ldots, n, n \geq 1\}$ be a triangular array of observable random variables satisfying the linear regression model

$$Y_{nj} = \sum_{i=0}^{p} \beta_i f_i(j/n) + \varepsilon_j, \qquad j = 1, \ldots, n, \tag{21}$$

where $\beta_0, \ldots, \beta_p$ are regression parameters and $f_i(\cdot)$ are regressor functions defined on $[0, 1]$. In the above model, the regression parameters $\beta_0, \ldots, \beta_p$ remain unchanged for all the observations $Y_{n1}, \ldots, Y_{nn}$. In the change-point set-up, the model (21) is called a 'no-change' model. In a change-point model, the first m observations $Y_{n1}, \cdots, Y_{nm}$ consist of regression parameters $\beta_0, \ldots, \beta_p$ while the parameters change to $\gamma_0, \ldots, \gamma_p$ for the last $n - m$ observations $Y_{nm+1}, \ldots, Y_{nn}$. Thus, a change-point model may be formulated as

$$Y_{nj} = \begin{cases} \sum_{i=0}^{p} \beta_i f_i(j/n) + \varepsilon_j, & j = 1, \ldots, m \\ \sum_{i=0}^{p} \gamma_i f_i(j/n) + \varepsilon_j, & j = m+1, \ldots, n. \end{cases} \tag{22}$$

In the above m is called the change-point and is usually assumed unknown. The problem of testing for the no change model (21) against the change-point model (22) is called the 'change-point problem' in statistical literature. The problem has many applications including in areas such as quality control, environmental monitoring and econometrics. A special case of the general change-point problem considered frequently in the literature consists of testing for change in a single parameter, say, $\beta_i, i = 0, \dots, p$ occurring at the unknown point $m, m = 1, \dots, n-1$. Jandhyala and MacNeill [**3**] derived a Bayes-type statistic for this problem. Their test statistic is

$$Q_{pn}^{(i)} = \sum_{m=1}^{n-1} p_m \left\{ \frac{1}{\sigma\sqrt{n}} \sum_{k=m+1}^{n} f_i(k/n)(Y_{nk} - \hat{Y}_{nk}) \right\}^2 \tag{23}$$

where $\{Y_{nk} - \hat{Y}_{nk}\}_{k=1}^{n}$ are the least squares residuals and $p_m, m = 1, \dots, n-1$ denotes a prior probability function for the unknown change-point m. When the prior on the change-point is discrete uniform, Jandhyala and MacNeill [**2**] have shown that

$$Q_{pn}^{(i)} \xrightarrow{d} \int_0^1 \{B_p^{(f_i)}(t)\}^2 \, dt, \qquad i = 0, \dots, p \tag{24}$$

where $\xrightarrow{d}$ denotes convergence in distribution and $\{B_p^{(f_i)}(t), t\varepsilon[0,1]\}$ is a Gaussian process defined by

$$B_p^{(f_i)}(t) = \int_0^t f_i(x)dB(x) - \int_0^t f_i(x)\left\{ \int_0^t g_p(x,y)dB(y) \right\} dx \tag{25}$$

and $g_p(x,y) = \mathbf{f}'(s)F^{-1}\mathbf{f}(t)$ with $\mathbf{f}'(t) = (f_0(t), \dots, f_p(t))$ and $F = ((\int_0^1 f_i(t)f_j(t)dt))$, $i,j = 0,1,\dots,p$.

Thus, from (24) we note that the asymptotic distribution function of $Q_{pn}^{(i)}$ is equivalent to the distribution function of the stochastic integral $\int_0^1 \{B_p^{(f_i)}(t)\}^2 dt$, $i = 0,1,\dots,p$.

For the case $p = 0$ in the model (21), Anderson and Darling [**1**] have shown that the characteristic function of the stochastic integral $\int_0^1 \{B_0^{(f_0)}(t)\}^2 \, dt$ may be obtained as

$$\Phi_0(u) = \prod_{n=1}^{\infty} \{1 - 2iu\lambda_{o,n}\}^{-1/2}$$

where $\{\lambda_{0,n}\}$ are the eigenvalues satisfying the Fredholm integral equation

$$\int_0^1 K_0^{(f_0)}(s,t)\phi_0(s)\,ds = \lambda_0\phi_0(t).$$

The kernel $K_0^{(f_0)}(s,t)$ is the covariance kernel of the Gaussian process $\{B_0^{(f_0)}(t), t \in [0,1]\}$ and is given by

$$K_0^{(f_0)}(s,t) = \min(s,t) - st.$$

More generally, Jandhyala and MacNeill [**2**] have derived the covariance kernel $K_p^{(f_i)}(s,t)$ for the process $\{B_p^{(f_i)}(t), t \in [0,1]\}$ to be:

$$K_p^{(f_i)}(s,t) = \int_0^{\min(s,t)} f_i^2(x)dx - \int_0^s \int_0^t f_i(x)f_i(y)g_p(x,y)\,dx\,dy. \tag{26}$$

One may apply the method of Anderson and Darling [**1**] to find the characteristic function for the stochastic integral $\int_0^1 \{B_p^{(f_i)}(t)\}^2\,dt$ and hence for the asymptotic form of the change detection statistic $Q_{p_n}^{(i)}$. We shall now discuss some special cases.

One obtains a pth order polynomial regression model by letting $f_i(t) = t^i$, $t \in [0,1]$, $i = 0, 1, \ldots, p$ in the model given by (21). For this case, the statistic that tests for change in the intercept parameter β_0 is $Q_{p_n}^{(0)}$ and this statistic is asymptotically equivalent to $\int_0^1 \{B_p^{(f_0)}(t)\}^2\,dt$. For the case of polynomial regression, $\{B_p^{(f_0)}(t), t \in [0,1]\}$ is precisely the process given by $B_p(t)$ in expression (5) and its covariance kernel $K_p^{(f_0)}(s,t)$ is the kernel $K_p(s,t)$ given in (2). Thus, the eigenvalues in (6) obtained as solutions of (1) enable one to find the characteristic function of the statistic that tests for change in the intercept parameter β_0 of a general polynomial regression model.

Alternatively, through a random walk formulation, Nabeya and Tanaka [**7**] have found the explicit forms for the Fredholm determinants and hence for the characteristic functions associated with the following statistics.

(i) The model is: $Y_{nj} = \beta_i(j/n)^i + \varepsilon_j$, $i > -1/2$. The statistic of interest tests for change in the parameter β_i.

(ii) The model is $Y_{nj} = \beta_0 + \beta_i (j/n)^i + \varepsilon_j$, $i = 1, 2, 3, 4$. For each i, $i = 1, 2, 3, 4$, the statistic of interest tests for change in the parameter β_0.

The relationship between the random walk formulation and the change-point formulation has been recently discussed by Jandhyala and MacNeill [**4**].

Finally, a pth order harmonic regression model is stated by

$$Y_{nj} = \beta_0 + \sum_{i=0}^{p} \{\beta_i \cos 2\pi i(j/n) + \beta_{p+i} 2\pi i(j/n)\} + \varepsilon_j, \qquad j = 1, \dots, n. \tag{27}$$

For this case $Q_{pn}^{(0)}$ tests for change in the parameter β_0 and its asymptotic equivalence is given by $\int_0^1 \{B_p^{(f_0)}(t)\}^2 \, dt$.

The covariance kernel associated with the process $\{B_p^{(f_0)}(t), t \in [0,1]\}$ is given by:

$$K_p^{(f_0)}(s,t) = \min(s,t) - st - \sum_{j=1}^{p} (2\pi^2 j^2)^{-1} \times \{(1 - \cos 2\pi jt)(1 - \cos 2\pi js) + \sin 2\pi jt \cdot \sin 2\pi js\}. \tag{28}$$

For this case, the eigenvalues satisfying the associated Fredholm integral equation have been derived by Jandhyala and MacNeill [**2**]. The eigenvalues $\{\lambda_{p,n}\}_{n=1}^{\infty}$ are:

$$\lambda_{p,n} = 1/(4\pi^2 n^2), \qquad n = p+1, p+2, \dots \tag{29}$$

and those satisfying the equation

$$\tan\left(\frac{1}{2\sqrt{\lambda_p}}\right) = \frac{1}{4\sqrt{\lambda_p}} \left\{ \left(\sum_{j=1}^{p} \frac{1}{1 - 4\pi^2 j^2 \lambda_p} \right)^{-1} \right\}, \qquad n = 1, 2, \dots . \tag{30}$$

There are several other cases in this area for which the associated eigenvalues are unknown. For both polynomial and harmonic regressions, the eigenvalues of

$$\int_0^1 K_p^{(f_i)}(s,t)\phi_p(s)\, ds = \lambda_p \phi(t), \qquad i = 1, 2, \dots, p$$

where $K_p^{(f_i)}(s,t)$ is given by (26) are unknown. These are interesting open problems.

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Department of Pure and Applied Mathematics, Washington State University, Pullman, Washington 99164-3113

Department of Pure and Applied Mathematics, Washington State University, Pullman, Washington 99164-3113