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NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF BARBASHIN TYPE: TOPOLOGICAL AND MONOTONICITY METHODS

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ABSTRACT. The purpose of this paper is to illustrate the applicability of topological and monotonicity methods to the solution of nonlinear integro-differential equations of Barbashin type. Such equations arise in the mathematical modelling of various transport phenomena. We show first how to solve initial value problems for nonlinear Barbashin equations by means of a classical fixed point theorem due to M.A. Krasnosel'skij. Afterwards, we apply a nonclassical fixed point principle for nonlinear operators in so-called K-normed spaces to a certain boundary value problem for Barbashin equations. The main step consists here in transforming the boundary value problem into an equivalent operator equation involving Uryson-type integral operators. Finally, we show how to use Minty's monotonicity principle to prove (unique) solvability of a Barbashin equation containing Hammerstein-type integral operators.

1. A fixed point theorem by Krasnosel'skij.

In 1955, M.A. Krasnosel'skij proved the following fixed point principle:

Theorem 1 [**9**]. *Let E be a Banach space, $G_1 : E \to E$ a contraction, and $G_2 : E \to E$ a continuous compact operator. Suppose that there exists a nonempty convex closed bounded set $M \subset E$ such that $G_1(M) + G_2(M) \subseteq M$. Then the operator $G_1 + G_2$ has a fixed point in M.*

Obviously, Theorem 1 bridges the "gap" between the classical fixed point principles of Banach-Caccioppoli and Schauder. Theorem 1 is

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nowadays considered as a simple example of the Darbo-Sadovskij fixed point principle [**7, 15**] for condensing operators.

A classical application of Theorem 1 is concerned with initial value problems in Banach spaces. We give a sample result on the local solvability of such an initial value problem:

Proposition 1. *Let X be a Banach space, $t_0 \in \mathbf{R}$, $x_0 \in X$ and $Q = \{(t,x) \in \mathbf{R} \times X : |t-t_0| \leq a, \|x-x_0\| \leq b\}$. Suppose that $g_1 : Q \to X$ is continuous and also Lipschitz-continuous with respect to the second variable, i.e.,*

$$\|g_1(t,x) - g_1(t,y)\| \leq L(t)\|x-y\|, \tag{1}$$

and $g_2 : Q \to X$ is compact and continuous. Choose $c \in (0,a]$ such that both

$$\int_{t_0-c}^{t_0+c} L(t)\,dt < 1, \qquad \sup_Q \|g_1(t,x)\| + \sup_Q \|g_2(t,x)\| \leq \frac{b}{c}.$$

Then the initial value problem

$$x'(t) = g_1(t,x(t)) + g_2(t,x(t)), \qquad x(t_0) = x_0 \tag{2}$$

has a continuously differentiable solution x on the compact interval $[t_0-c, t_0+c]$ with values in X.

Proof. The proof simply consists in verifying the hypotheses of Theorem 1 for $E = C([t_0-c, t_0+c], X)$, $M = \{x \in E : \|x - x_0\| \leq b\}$ and $G_i x(t) = \int_{t_0}^t g_i(\tau, x(\tau))\,d\tau$, $i = 1,2$. □

2. Barbashin equations with Uryson integral operators. Consider the nonlinear *integro-differential equation of Barbashin type*

$$\frac{\partial x(t,s)}{\partial t} = c(t,s)x(t,s) + \int_{-1}^{1} k(t,s,\sigma,x(t,\sigma))\,d\sigma + f(t,s) \tag{3}$$

with initial condition

$$x(0,s) = x_0(s), \tag{4}$$

where $x_0 : [-1,1] \to \mathbf{R}$, $c : [-1,1] \times [-1,1] \to \mathbf{R}$, $k : [-1,1] \times [-1,1] \times [-1,1] \times \mathbf{R} \to \mathbf{R}$ and $f : [-1,1] \times [-1,1] \to \mathbf{R}$ are given measurable functions. Equations of this type occur in the mathematical modelling of certain transport phenomena (see, e.g., [**6, 10, 16, 17**]). Given an ideal space [**18**] of measurable functions over $[-1,1]$, under natural conditions one may write equation (3) as differential equation

$$\frac{dx}{dt} = C(t)x + K(t)x + f(t) \tag{5}$$

in X, where

$$C(t)x(s) = c(t,s)x(s) \tag{6}$$

is the *multiplication operator* generated by the function c,

$$K(t)x(s) = \int_{-1}^{1} k(t,s,\sigma,x(\sigma))\, d\sigma \tag{7}$$

is the *Uryson integral operator* generated by the kernel k and $f(t)(s) = f(t,s)$. By means of Proposition 1, we then get a local existence result for the problem (3)/(4) in a suitable function space. To this end, let us denote by $C_t^1 = C_t^1(X)$ the set of all functions $x : [-1,1] \times [-1,1] \to \mathbf{R}$ such that the map $t \mapsto x(t,\cdot)$ is continuously differentiable from $[-1,1]$ into X, equipped with the natural norm

$$\|x\|_{C_t^1} = \max_{|t| \le 1} \left[\|x(t,\cdot)\| + \left\| \frac{d}{dt} x(t,\cdot) \right\| \right]. \tag{8}$$

A standard choice is $X = L_p$ for $1 \le p < \infty$; in this case we write C_{tp}^1 instead of $C_t^1(L_p)$.

Proposition 2. *Suppose that the map $t \mapsto C(t)$ is strongly continuous in $\mathcal{L}(X)$ and the map $(t,x) \mapsto K(t)x$ is continuous and compact. Then the initial value problem* (3)/(4) *has, for each $f \in C([-1,1],X)$, a local solution $x \in C_t^1(X)$.*

Proof. Let $g_1(t,x) = C(t)x + f(t)$ and $g_2(t,x) = K(t)x$, and let Q be as in Proposition 1 with $t_0 = 0$. By assumption, g_1 :

$Q \to X$ is continuous and satisfies a Lipschitz condition (1) with $L(t) = \|C(t)\|_{\mathcal{L}(X)}$, and $g_2 : Q \to X$ is continuous and compact. By Proposition 1, the initial value problem (2) has a continuously differentiable solution x on $[-c, c]$ which is a C_t^1-solution of (3)/(4). $\square$

Since the operator family $t \mapsto C(t)$ is supposed to be strongly continuous in Proposition 2, the smallness condition $\int_{-c}^{c} \|C(\tau)\| \, d\tau < 1$ may always be achieved for sufficiently small $c > 0$. The crucial hypothesis is therefore the compactness of the map of two variables $(t, x) \mapsto K(t)x$. The following lemma reduces the problem of verifying the compactness of this map to that of verifying the compactness of each operator $K(t)$ (t fixed), as well as the equicontinuity of the family of functions $t \mapsto K(t)x$ (x fixed):

Lemma 1. *Let $B \subset X$ be bounded, and suppose that the following conditions are satisfied*:

1. *Each operator $K(t) : B \to X$ is compact and continuous.*
2. *The function set $\{K(\cdot)x : x \in B\}$ is equicontinuous.*

Then the map g defined by $g(t, x) = K(t)x$ is continuous and compact from $[-1, 1] \times B$ into X.

Proof. Continuity follows by the inequality

$$\|g(t, x) - g(t_0, x_0)\| \le \|K(t)x - K(t_0)x\| + \|K(t_0)x - K(t_0)x_0\|$$

from the hypotheses. For compactness, it suffices to prove that $g([-1, 1] \times B)$ has a finite ε-net for any $\varepsilon > 0$. By the second hypothesis, there exists a $\delta(\varepsilon) > 0$ such that $\|K(t)x - K(t')x\| < \varepsilon/2$ whenever $|t - t'| < \delta(\varepsilon)$ and $x \in B$. Choose some $n \in \mathbf{N}$ with $n^{-1} < \delta(\varepsilon)$, and consider a partition $\{t_i : t_i = i/n, i = 0, \pm 1, \pm 2, \dots, \pm n\}$ of $[-1, 1]$. Since $K(t_i)B$ is precompact, it contains a finite $\varepsilon/2$-net $\mathcal{N}_i$. We claim that $\mathcal{N}_{-n} \cup \cdots \cup \mathcal{N}_{-1} \cup \mathcal{N}_0 \cup \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_n$ is an ε-net for $g([-1, 1] \times B)$. In fact, let $(t, x) \in [-1, 1] \times B$ be arbitrary. Then $i/n \le t < (i + 1)/n$ for some i. Choose $y \in \mathcal{N}_i$ such that $\|g(i/n, x) - y\| < \varepsilon/2$. This implies

$$\|g(t, x) - y\| \le \|g(t, x) - g(i/n, x)\| + \|g(i/n, x) - y\| < \varepsilon$$

as claimed. □

As a typical application of Lemma 1, let us consider the case $k(t,s,\sigma,u) = k_0(t,s,\sigma)h(\sigma,u)$, i.e., the special family of *Hammerstein integral operators*

$$K(t)x(s) = \int_{-1}^{1} k_0(t,s,\sigma)h(\sigma,x(\sigma))\,d\sigma. \tag{9}$$

The operator (9) may be written as a composition of the nonlinear *Nemytskij operator*

$$Hx(s) = h(s,x(s)) \tag{10}$$

and the family of *linear integral operators*

$$K_0(t)y(s) = \int_{-1}^{1} k_0(t,s,\sigma)y(\sigma)\,d\sigma. \tag{11}$$

Proposition 3. *Suppose that the map $t \mapsto C(t)$ is strongly continuous in $\mathcal{L}(X)$, there exists some normed linear space Y such that $H : X \to Y$ is bounded and continuous, and each operator $K_0(t) : Y \to X$ is compact. Assume that the map $t \mapsto K_0(t)$ is continuous in the operator norm. Then the initial value problem (3)/(4) has, for each $f \in C([-1,1],X)$, a local solution $x \in C_t^1(X)$.*

Proof. It suffices to prove that $g : Q \to X$ is compact and continuous. Just use Lemma 1. □

3. Fixed point theorems in K-normed spaces. In the preceding section we have seen that, under some natural hypotheses, the initial value problem for the integro-differential equation (3) always has a local solution; this is completely analogous to ordinary differential equations. As usual, the existence of solutions of boundary value problems for (3) is more difficult to prove (and, as a matter of fact, is not always true). This will be illustrated in the following section. Our main tool is a nonclassical fixed point principle which we shall recall now.

A *K-normed space* is a Banach space E equipped with a functional ("K-norm") $]|\cdot|[: E \to K$ which has the usual properties of a norm, but takes its values in the positive cone K of some ordered Banach space Z. Usual norms are, of course, obtained by the trivial choice $Z = \mathbf{R}$ and $K = \mathbf{R}_+$; the simplest nontrivial example is $Z = \mathbf{R}^2$ and $K = \mathbf{R}_+^2$.

Theorem 2 [**19–21**]. *Let E be a Banach space with a K-norm $]|\cdot|[: E \to K \subset Z$ and $F : E \to E$ a bounded continuous operator which satisfies a condition*

$$]|Fx - Fy|[\leq G(]|x - y|[), \qquad x, y \in E, \tag{12}$$

where $G : K \to K$ is some positive linear operator in Z. Assume that G has spectral radius < 1. Then the operator F has a unique fixed point in E.

Of course, the choice $Z = \mathbf{R}$ and $K = \mathbf{R}_+$ leads to $Gz = qz$ with $q < 1$ which is the standard contraction condition in the Banach-Caccioppoli fixed point principle. In the following section we shall show how to obtain existence results for boundary value problems for integro-differential equations by a sophisticated "infinite-dimensional" choice of Z and K.

4. Generalized Barbashin equations with Uryson integral operators. In this section we replace equation (3) by the more general integro-differential equation

$$\begin{aligned} \frac{\partial x(t,\tau,s)}{\partial \tau} &= c(t,\tau,s)x(t,\tau,s) \\ &\quad + \int_{-1}^{1} l(t,\tau,s,\sigma,x(t,\tau,\sigma))\,d\sigma \\ &\quad + \int_{-1}^{1} m(t,\tau,s,r,x(r,\tau,s))\,dr \\ &\quad + \int_{-1}^{1}\int_{-1}^{1} n(t,\tau,s,\sigma,r,x(r,\tau,\sigma))\,d\sigma\,dr \\ &\quad + f(t,\tau,s), \qquad (t,\tau,s) \in Q, \end{aligned} \tag{13}$$

subject to the boundary condition

$$
\begin{cases} x(t,a,s) = \varphi(t,s) & \text{if } (t,s) \in Q_+, \\ x(t,b,s) = \psi(t,s) & \text{if } (t,s) \in Q_-; \end{cases} \tag{14}
$$

here we have put $Q = [-1,1] \times [a,b] \times [-1,1]$, $Q_+ = (0,1] \times [-1,1]$ and $Q_- = [-1,0) \times [-1,1]$. Equations like (13) occur as Fourier transforms of a certain type of Schrödinger equation. As above, $c : Q \to \mathbf{R}$, $\varphi : Q_+ \to \mathbf{R}$, $\psi : Q_- \to \mathbf{R}$ and $f : Q \to \mathbf{R}$ are given measurable functions, and $l : Q \times [-1,1] \times \mathbf{R} \to \mathbf{R}$, $m : Q \times [-1,1] \times \mathbf{R} \to \mathbf{R}$ and $n : Q \times [-1,1] \times [-1,1] \times \mathbf{R} \to \mathbf{R}$ are supposed to satisfy a Carathéodory condition.

We are going to study the boundary value problem (13)/(14) in the space $W^1_{\tau p} = W^1_{\tau p}(Q)$, $1 \le p < \infty$, of all measurable functions $x : Q \to \mathbf{R}$ for which the norm

$$
\|x\|_{W^1_{\tau p}} = \left\{ \int_a^b \int_{-1}^1 \int_{-1}^1 \left[|x(t,\tau,s)|^p + \left| \frac{\partial x(t,\tau,s)}{\partial \tau} \right|^p \right] ds\, dt\, d\tau \right\}^{1/p} \tag{15}
$$

$1 \le p < \infty$ is finite; this norm was introduced in [**12**] in a different context. Consider the kernel function of four variables

$$
e(t,\tau,\tau_0,s) = \exp \int_{\tau_0}^{\tau} c(t,\xi,s)\, d\xi \tag{16}
$$

which generates a partial integral operator (see [**8, 13**])

$$
Pf(t,\tau,s) = \begin{cases} \displaystyle\int_a^{\tau} e(t,\tau,\theta,s) f(t,\theta,s)\, d\theta & \text{if } (t,s) \in Q_+, \\ \displaystyle\int_b^{\tau} e(t,\tau,\theta,s) f(t,\theta,s)\, d\theta & \text{if } (t,s) \in Q_-. \end{cases} \tag{17}
$$

To begin with, we show how to solve the boundary value problem (13)/(14) explicitly in case $l = m = n \equiv 0$. The proof of the following lemma consists in a straightforward calculation:

Lemma 2. *For $c \in L_\infty(Q)$, $f \in L_p(Q)$, $\varphi \in L_p(Q_+)$ and $\psi \in L_p(Q_-)$, the problem*

$$
\begin{cases} \dfrac{\partial z(t,\tau,s)}{\partial \tau} = c(t,\tau,s) z(t,\tau,s) \\ \qquad\qquad + f(t,\tau,s) & \textit{if } (t,\tau,s) \in Q, \\ z(t,a,s) = \varphi(t,s) & \textit{if } (t,s) \in Q_+, \\ z(t,b,s) = \psi(t,s) & \textit{if } (t,s) \in Q_- \end{cases} \tag{18}
$$

has a unique solution $z \in W^1_{\tau p}$. *This solution is given by*

$$z = \begin{cases} Pf + \tilde{\varphi} & a.e. \;\; on \; (0,1] \times [a,b] \times [-1,1], \\ Pf + \tilde{\psi} & a.e. \;\; on \; [-1,0) \times [a,b] \times [-1,1], \end{cases} \tag{19}$$

where

$$\tilde{\varphi}(t,\tau,s) = \varphi(t,s)e(t,\tau,a,s), \qquad (t,s) \in Q_+$$

and

$$\tilde{\psi}(t,\tau,s) = \psi(t,s)e(t,\tau,b,s), \qquad (t,s) \in Q_-.$$

Lemma 2 allows us to transform the boundary value problem (13)/(14) into an operator equation which may be treated by the fixed point principle in K-normed spaces given above. To this end, we introduce the operators

$$\begin{aligned} Cx(t,\tau,s) &= c(t,\tau,s)x(t,\tau,s), \\ Lx(t,\tau,s) &= \int_{-1}^{1} l(t,\tau,s,\sigma,x(t,\tau,\sigma))\,d\sigma, \\ Mx(t,\tau,s) &= \int_{-1}^{1} m(t,\tau,s,r,x(r,\tau,s))\,dr, \end{aligned}$$

and

$$Nx(t,\tau,s) = \int_{-1}^{1}\int_{-1}^{1} n(t,\tau,s,\sigma,r,x(r,\tau,s))\,d\sigma\,dr.$$

The equation (13) may then be written more concisely in the form

$$\frac{\partial x}{\partial \tau} = [C + L + M + N]x + f. \tag{20}$$

Proposition 4. *Let the conditions of Lemma* 2 *be satisfied. Suppose that the nonlinear operator* $L+M+N$ *be continuous and bounded in* L_p.

Then every solution $x \in W^1_{\tau p}$ of the boundary value problem (13)/(14) *solves the nonlinear operator equation*

(21)
$$x = \begin{cases} P(Lx + Mx + Nx + f) + \tilde{\varphi} & \text{a.e. on } (0,1] \times [a,b] \times [-1,1], \\ P(Lx + Mx + Nx + f) + \tilde{\psi} & \text{a.e. on } [-1,0) \times [a,b] \times [-1,1]. \end{cases}$$

Conversely, every solution $x \in L_p$ of (21) *belongs to $W^1_{\tau p}$ and solves the boundary value problem* (13)/(14).

Proof. The proof follows immediately from Lemma 2 with f replaced by $Lx + Mx + Nx + f$. □

We begin now to study the operator equation (21) from the viewpoint of fixed point theorems in K-normed spaces. For $a \le \tau \le b$, $0 < t \le 1$ and $-1 \le s \le 1$, we put $x(t,\tau,s) = u(t,\tau,s)$, $x(-t,\tau,s) = v(t,\tau,s)$, $z(t,\tau,s) = g(t,\tau,s)$, $z(-t,\tau,s) = h(t,\tau,s)$ (z from Lemma 2), $e(t,\tau,\tau_0,s) = i(t,\tau,\tau_0,s)$ and $e(-t,\tau,\tau_0,s) = j(t,\tau_0,s)$. Moreover, we define four operators A, B, C and D by

$$\begin{aligned} Au(t,\tau,s) = \int_a^\tau \bigg[&\int_{-1}^1 i(t,\tau,\theta,s) l(t,\theta,s,\sigma,u(t,\theta,\sigma))\, d\sigma \\ &+ \int_0^1 i(t,\tau,\theta,s) m(t,\theta,s,r,u(r,\theta,s))\, dr \\ &+ \int_0^1 \int_{-1}^1 i(t,\tau,\theta,s) n(t,\theta,s,\sigma,r,u(r,\theta,\sigma))\, d\sigma\, dr \bigg] d\theta, \end{aligned}$$

$$\begin{aligned} Bv(t,\tau,s) = \int_a^\tau \bigg[&\int_0^1 i(t,\tau,\theta,s) m(t,\theta,s,-r,v(r,\theta,s))\, dr \\ &+ \int_0^1 \int_{-1}^1 i(t,\tau,\theta,s) n(t,\theta,s,\sigma,-r,v(r,\theta,\sigma))\, d\sigma\, dr \bigg] d\theta, \end{aligned}$$

$$\begin{aligned} Cu(t,\tau,s) = \int_b^\tau \bigg[&\int_0^1 j(t,\tau,\theta,s) m(-t,\theta,s,r,u(r,\theta,s))\, dr \\ &+ \int_0^1 \int_{-1}^1 j(t,\tau,\theta,s) n(-t,\theta,s,\sigma,r,u(r,\theta,\sigma))\, d\sigma\, dr \bigg] d\theta, \end{aligned}$$

and

$$Dv(t,\tau,s) = \int_b^\tau \bigg[\int_{-1}^1 j(t,\tau,\theta,s) l(-t,\theta,s,\sigma,v(t,\theta,\sigma))\, d\sigma$$

$$+\int_0^1 j(t,\tau,\theta,s)m(-t,\theta,s,-r,v(r,\theta,s))\,dr$$
$$+\int_0^1\int_{-1}^1 j(t,\tau,\theta,s)n(-t,\theta,s,\sigma,-r,v(r,\theta,\sigma))\,d\sigma\,dr\Big]\,d\theta.$$

The operator equation (21) may then be written as a system

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g \\ h \end{pmatrix}. \tag{22}$$

Suppose now that the kernel functions l, m and n satisfy Lipschitz conditions

$$\begin{aligned}
|l(t,\tau,s,\sigma,u) - l(t,\tau,s,\sigma,v)| &\le a_1(t,\tau,s,\sigma)|u-v|,\\
|m(t,\tau,s,r,u) - m(t,\tau,s,r,v)| &\le a_2(t,\tau,s,r)|u-v|,\\
|n(t,\tau,s,\sigma,r,u) - n(t,\tau,s,\sigma,r,v)| &\le a_3(t,\tau,s,\sigma,r)|u-v|,\\
|m(t,\tau,s,-r,u) - m(t,\tau,s,-r,v)| &\le b_1(t,\tau,s,r)|u-v|,\\
|n(t,\tau,s,\sigma,-r,u) - n(t,\tau,s,\sigma,-r,v)| &\le b_2(t,\tau,s,\sigma,r)|u-v|,\\
|m(-t,\tau,s,r,u) - m(-t,\tau,s,r,v)| &\le c_1(t,\tau,s,r)|u-v|,\\
|n(-t,\tau,s,\sigma,r,u) - n(-t,\tau,s,\sigma,r,v)| &\le c_2(t,\tau,s,\sigma,r)|u-v|,\\
|l(-t,\tau,s,\sigma,u) - l(-t,\tau,s,\sigma,v)| &\le d_1(t,\tau,s,\sigma)|u-v|,\\
|m(-t,\tau,s,-r,u) - m(-t,\tau,s,-r,v)| &\le d_2(t,\tau,s,r)|u-v|,\\
|n(-t,\tau,s,\sigma,-r,u) - n(-t,\tau,s,\sigma,-r,v)| &\le d_3(t,\tau,s,\sigma,r)|u-v|.
\end{aligned}$$

Moreover, assume that

$$\begin{aligned}
\Big\| \int_{-1}^1 & i(\cdot,\tau,\theta,\cdot)a_1(\cdot,\theta,\cdot,\sigma)u(\cdot,\sigma)\,d\sigma\\
&+\int_0^1 i(\cdot,\tau,\theta,\cdot)a_2(\cdot,\theta,\cdot,r)u(r,\cdot)\,dr\\
&+\int_0^1\int_{-1}^1 i(\cdot,\tau,\theta,\cdot)a_3(\cdot,\theta,\cdot,\sigma,r)u(r,\sigma)\,d\sigma\,dr\Big\|\\
&\le \alpha\|u\|,
\end{aligned}$$

$$\begin{aligned}
\Big\| \int_0^1 & i(\cdot,\tau,\theta,\cdot)b_1(\cdot,\theta,\cdot,r)v(r,\cdot)\,dr\\
&+\int_0^1\int_{-1}^1 i(\cdot,\tau,\theta,\cdot)b_2(\cdot,\theta,\cdot,\sigma,r)v(r,\sigma)\,d\sigma\,dr\Big\|\\
&\le \beta\|v\|,
\end{aligned}$$

$$\left\| \int_0^1 j(\cdot,\tau,\theta,\cdot)c_1(\cdot,\theta,\cdot,r)u(r,\cdot)\,dr + \int_0^1\int_{-1}^1 j(\cdot,\tau,\theta,\cdot)c_2(\cdot,\theta,\cdot,\sigma,r)u(r,\sigma)\,d\sigma\,dr \right\| \le \gamma\|u\|,$$

and

$$\left\| \int_{-1}^1 j(\cdot,\tau,\theta,\cdot)d_1(\cdot,\theta,\cdot,\sigma)v(\cdot,\sigma)\,d\sigma + \int_0^1 j(\cdot,\tau,\theta,\cdot)d_2(\cdot,\theta,\cdot,r)v(r,\cdot)\,dr + \int_0^1\int_{-1}^1 j(\cdot,\tau,\theta,\cdot)d_3(\cdot,\theta,\cdot,\sigma,r)v(r,\sigma)\,d\sigma\,dr \right\| \le \delta\|v\|.$$

We define a linear operator G by

$$Gz(\tau) = G\begin{pmatrix} u(\tau) \\ v(\tau) \end{pmatrix} = \begin{pmatrix} \int_a^\tau [\alpha u(\theta) + \beta v(\theta)]\,d\theta \\ \int_\tau^b [\gamma u(\theta) + \delta v(\theta)]\,d\theta \end{pmatrix}. \tag{23}$$

Now we get the result:

Proposition 5. *Let the assumptions of Proposition 4 be satisfied. Assume, moreover, that the numbers α, β, γ and δ can be defined as above and satisfy one of the following four conditions*:

1. $\Delta = (\alpha+\delta)^2 - 4\beta\gamma > 0$ and $\sqrt{\Delta} < \log(\alpha+\delta+\sqrt{\Delta})/(\alpha+\delta-\sqrt{\Delta})$,
2. $(\alpha+\delta)^2 = 4\beta\gamma < 4$;
3. $\Delta < 0$, $\alpha+\delta \neq 0$, *and* $\sqrt{-\Delta} < 2\arctan\sqrt{-\Delta}/(\alpha+\delta)$;
4. $\Delta < 0$, $\alpha+\delta = 0$, *and* $\sqrt{-\Delta} < \pi$.

Then the operator equation (21), *and hence also the boundary value problem* (13)/(14), *has a unique solution $x \in W^1_{\tau p}$; this solution may be obtained by the usual method of successive approximations.*

Proof. We have to construct a suitable K-normed space E such that the operator

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g \\ h \end{pmatrix} \tag{24}$$

satisfies the contraction condition (12). We take $X = L_p(Q_+) \times L_p(Q_+)$, equipped with the norm $\|(u,v)\|_X = \|u\|_{L_p} + \|v\|_{L_p}$. Moreover, let $E = L_p([a,b], X)$ be the Bochner-Lebesgue space of all X-valued functions $\tau \mapsto x(\cdot,\tau,\cdot) = (u(\cdot,\tau,\cdot), v(\cdot,\tau,\cdot))$, equipped with the norm

$$\|x\|_E = \left\{ \int_a^b [\|u(\cdot,\tau,\cdot)\|_p + \|v(\cdot,\tau,\cdot)\|_p]^p \, d\tau \right\}^{1/p}$$

and the K-norm

$$]|x|[= (\|u(\cdot,\tau,\cdot)\|_p, \|v(\cdot,\tau,\cdot)\|_p).$$

Thus, the K-norm takes its values in the natural cone of the Banach space $Z = L_p([a,b], \mathbf{R}^2)$. Our assumptions ensure that the estimate (12) is true for the operators (23) and (24). As in Lemma 3 of [**1**] we see that the spectral radius of G is less than 1. Consequently, Theorem 2 applies. $\square$

5. Surjectivity results for monotone operators. In the preceding sections we have proved existence and uniqueness theorems for various types of integro-differential equations of Barbashin type in Lebesgue spaces by means of topological methods. Since most Lebesgue spaces are reflexive, and many nonlinearities arising in applications are monotonically increasing or decreasing, it is a useful device to apply also monotonicity methods, rather than topological methods. This will be illustrated in this and the following section by means of the equation

$$\begin{aligned} y(t,s) = & \int_{-1}^1 l(t,s,\sigma) f(t,\sigma,y(t,\sigma)) \, d\sigma \\ & + \int_{-1}^1 m(t,s,\tau) f(\tau,s,y(\tau,s)) \, d\tau \\ & + \int_{-1}^1 \int_{-1}^1 n(t,s,\tau,\sigma) f(\tau,\sigma,y(\tau,\sigma)) \, d\sigma \, d\tau \\ & + g(t,s) \end{aligned} \tag{25}$$

which is similar to that considered in the previous section. Here $l : [-1,1] \times [-1,1] \times [-1,1] \to \mathbf{R}$, $m : [-1,1] \times [-1,1] \times [-1,1] \to \mathbf{R}$, $n : [-1,1] \times [-1,1] \times [-1,1] \times [-1,1] \to \mathbf{R}$, and $g : [-1,1] \times [-1,1] \to \mathbf{R}$ are given measurable functions, while $f : [-1,1] \times [-1,1] \times \mathbf{R} \to \mathbf{R}$ is supposed to satisfy a Carathéodory condition.

The existence results given below essentially build on Browder's generalization [**4**] of Minty's celebrated monotonicity principle [**11**]. Recall that a subset S of the product space $X \times X^*$ is said to be *monotone* if

$$(26) \qquad \langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \qquad (x_i, y_i) \in S,\ i = 1, 2.$$

The monotone set S is *maximal monotone* if it is not properly contained in any other monotone set. A (multivalued) mapping $A : D(A) \subset X \to 2^{X^*}$ is called *monotone* (*maximal monontone*) if its graph $G(A) = \{(x,y) \in X \times X^* : x \in D(A), y \in Ax\}$ is a monotone (maximal monotone) set and *weakly coercive* if either $D(A)$ is bounded or $D(A)$ is unbounded and

$$(27) \qquad \inf_{y \in Ax} \|y\| \longrightarrow \infty \quad \text{as} \quad \|x\| \longrightarrow \infty, \qquad x \in D(A).$$

Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing between X^* and X. Browder's monotonicity principle may be stated as follows:

Theorem 3 [**4**]. *Let X be a reflexive Banach space, and let $A : D(A) \subset X \to 2^{X^*}$ be maximal monotone and weakly coercive. Then $R(A) = X^*$.*

Another tool which will be useful in the sequel is that of spaces of mixed norm. If U and V are two ideal spaces of measurable functions over $[-1,1]$, the *space with mixed norm* $[U \to V]$ consists, by definition, of all measurable functions $x : [-1,1] \times [-1,1] \to \mathbf{R}$ for which the norm

$$\|x\|_{[U \to V]} = \|t \longmapsto \|x(t,\cdot)\|_U\|_V$$

is finite [**8**]; similarly, the space $[U \leftarrow V]$ is defined by the norm

$$\|x\|_{[U \leftarrow V]} = \|s \longmapsto \|x(\cdot,s)\|_V\|_U.$$

For example, in case $U = L_p$ and $V = L_q$, $1 \le p,\ q < \infty$, we have

$$\|x\|_{[L_p \to L_q]} = \left\{ \int_{-1}^{1} \left[\int_{-1}^{1} |x(t,s)|^p \, ds \right]^{q/p} dt \right\}^{1/q},$$
$$\|x\|_{[L_p \leftarrow L_q]} = \left\{ \int_{-1}^{1} \left[\int_{-1}^{1} |x(t,s)|^q \, dt \right]^{p/q} ds \right\}^{1/p}.$$

A detailed discussion of Lebesgue spaces with mixed norm may be found in [**3**].

6. Barbashin equations with Hammerstein integral operators. We define operators L, M and N by

$$Lx(t,s) = \int_{-1}^{1} l(t,s,\sigma)x(t,\sigma)\,d\sigma, \tag{28}$$

$$Mx(t,s) = \int_{-1}^{1} m(t,s,\tau)x(\tau,s)\,d\tau, \tag{29}$$

and

$$Nx(t,s) = \int_{-1}^{1}\int_{-1}^{1} n(t,s,\tau,\sigma)x(\tau,\sigma)\,d\sigma\,d\tau. \tag{30}$$

Moreover, we denote by

$$Fy(t,s) = f(t,s,y(t,s)) \tag{31}$$

the Nemytskij operator generated by the function f. We may then rewrite equation (25) as an operator equation

$$y = (L + M + N)Fy + g. \tag{32}$$

Since all operators occurring here act on functions of two variables, it is reasonable to study equation (32) in spaces with mixed norm. To this end, we first reduce the operators (28) and (29) to families of operators acting on functions of one variable by putting

$$L(t)u(s) = \int_{-1}^{1} l(t,s,\sigma)u(\sigma)\,d\sigma, \qquad -1 \le t \le 1 \tag{33}$$

and

$$M(s)v(t) = \int_{-1}^{1} m(t,s,\tau)v(\tau)\,d\tau, \qquad -1 \le s \le 1. \tag{34}$$

Lemma 3. *Let $2 \le p,\ q < \infty$, $1/p + 1/p' = 1$, and $1/q + 1/q' = 1$. Suppose that the following three conditions are satisfied:*

1. *The linear integral operators* (33) *are bounded from L_p into L'_p for each $t \in [-1,1]$ and the map $t \mapsto \|L(t)\|_{\mathcal{L}(L_p,L_{p'})}$ belongs to $L_{qq'/(q-q')}$.*

2. *The linear integral operators* (34) *are bounded from L_q into $L_{q'}$ for each $s \in [-1,1]$, and the map $s \mapsto \|M(s)\|_{\mathcal{L}(L_q,L_{q'})}$ belongs to $L_{pp'/(p-p')}$.*

3. *The linear integral operator* (30) *is bounded from $[L_p \to L_q]$ into $[L_{p'} \to L_{q'}]$ if $p \ge q$, or from $[L_p \leftarrow L_q]$ into $[L_{p'} \leftarrow L_{q'}]$ if $p \le q$.*

Then the operator $K = L + M + N$ is bounded from $[L_p \to L_q]$ into $[L_{p'} \to L_{q'}]$ if $p \ge q$, or from $[L_p \leftarrow L_q]$ into $[L_{p'} \leftarrow {L_q}']$ if $p \le q$.

Proof. By Theorem 9 of [**8**], the operator L is bounded from $[L_p \to L_q]$ into $[L_{p'} \to L_{q'}]$ and the operator M is bounded from $[L_p \leftarrow L_q]$ into $[L_{p'} \leftarrow L_{q'}]$. Moreover, by the Minkowski inequality, we have the continuous embeddings

$$\begin{aligned} [L_p \longrightarrow L_q] &\subseteq [L_p \longleftarrow L_q] \quad \text{if } p \ge q, \\ [L_p \longrightarrow L_q] &\supseteq [L_p \longleftarrow L_q] \quad \text{if } p \le q. \end{aligned}$$

Thus, we get the conclusion. □

With p, q, p' and q' as above, let us put for brevity $X = [L_p \to L_q]$ if $p \ge q$ and $X = [L_p \leftarrow L_q]$ if $p \le q$. Since $[L_p \to L_q]^* = [L_{p'} \to L_{q'}]$ and $[L_p \leftarrow L_q]^* = [L_{p'} \leftarrow L_{q'}]$, Lemma 3 gives a set of sufficient conditions under which the operator $K = L + M + N$ maps X into its dual X^*. Observe, moreover, that the space X is reflexive, by our choice of p and q.

In order to apply Theorem 3, we make the following further assumptions:

Assumptions. (A1) *There exists* $c > 0$ *such that* $\langle -Kx, x\rangle \geq c\|Kx\|^2$ *for all* $x \in X$.

(A2) *There exist a function* $a \in X$ *and* $b \geq 0$ *such that*

$$|f(t,s,w)| \leq a(t,s) + b|w|^{r(p,q)}, \tag{35}$$

where $r(p,q) = p'/p$ *if* $p \geq q$ *and* $r(p,q) = q'/q$ *if* $p \leq q$.

(A3) *The function* $f(t,s,\cdot)$ *is monotonically increasing for almost all* $t, s \in [-1,1]$.

(A4) *There exist a function* $h \in L_1$ *and* $d > 0$ *such that*

$$f(t,s,w)w \geq d|w|^{s(p,q)} + h(t,s), \tag{36}$$

where $s(p,q) = q'$ *if* $p \geq q$ *and* $s(p,q) = p'$ *if* $p \leq q$.

Proposition 6. *Suppose that the hypotheses of Lemma* 3 *as well as the assumptions* (A1)–(A3) *are satisfied. Then the operator equation* (32) *has, for each* $g \in X^*$, *a unique solution* $y \in X^*$.

Proof. We first show the existence of a solution. The operator equation (32), i.e., $y = KFy + g$, is equivalent to the relation

$$0 \in (-K)^{-1}(y-g) + Fy \tag{37}$$

(cf. [**5**]). Let $z = y - g$, and let F_g be defined by $F_g z = F(z+g)$. Then equation (37) holds if and only if

$$0 \in [(-K)^{-1} + F_g]z. \tag{38}$$

Under the assumptions (A2) and (A3), it is easy to show that the Nemytskij operator F_g acts from X^* into X and is monotone. By Theorem 2.6 of [**2**], F_g is also continuous. Since $(-K)^{-1}$ and F_g are both maximal monotone, by a result of Rockafellar [**14**], the mapping $(-K)^{-1} + F_g$ is also maximal monotone. Moreover, the estimate

$$\begin{aligned}\inf_{x\in(-K)^{-1}z}\|x + F_g z\| &\geq \inf_{x\in(-K)^{-1}z}\frac{\langle x + F_g z, z\rangle}{\|z\|}\\ &\geq c\|z\| - \|F_g 0\|\end{aligned}$$

shows that $(-K)^{-1}+F_g$ is weakly coercive. The solvability of equation (38) thus follows from Theorem 3.

For the proof of uniqueness, let $y_1, y_2 \in X^*$ such that $y_1 = KFy_1+g$, $y_2 = KFy_2 + g$. Then

$$\langle y_1 - y_2, Fy_1 - Fy_2\rangle + \langle(-K)Fy_1 - (-K)Fy_2, Fy_1 - Fy_2\rangle = 0,$$

and, by the monotonicity of F and the assumption (A1),

$$0 \geq \langle(-K)Fy_1 - (-K)Fy_2, Fy_1 - Fy_2\rangle \geq c\|K(Fy_1 - Fy_2)\|^2.$$

This implies that $KFy_1 = KFy_2$ and, hence, $y_1 = y_2$. □

In fact, under the assumptions (A2) and (A3), we only need that $(-K)$ is monotone, i.e., $\langle -Kx, x\rangle \geq 0$ for all $x \in X$, such that $(-K)^{-1} + F_g$ is maximal monotone. Furthermore, if the assumption (A4) is satisfied, by the Hölder inequality we have

$$\langle Fy, y\rangle \geq d'\|y\|^{s(p,q)} + e$$

for some $d' > 0$ and $e \in \mathbf{R}$. Now, let $g = 0$; since

$$\begin{aligned}\inf_{x\in(-K)^{-1}y} \|x + Fy\| &\geq \inf_{x\in(-K)^{-1}y} \frac{\langle x + Fy, y\rangle}{\|y\|}\\ &\geq \frac{d'\|y\|^{s(p,q)} + e}{\|y\|},\end{aligned}$$

$(-K)^{-1} + F$ is weakly coercive. Therefore, we have:

Proposition 7. *Suppose that the hypotheses of Lemma* 3 *as well as the assumptions* (A2)–(A4) *are satisfied. If* $(-K)$ *is monotone and* $g = 0$*, then the operator equation* (32) *has a solution* $y \in X^*$.

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