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APPROXIMATE METHODS FOR SINGULAR INTEGRAL EQUATIONS WITH A NON-CARLEMAN SHIFT

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ABSTRACT. It is known that some problems of synthesis with continuous time and stationary parameters can be reduced to the solution of Wiener-Hopf equations on the semi-axis $\mathbf{R}_+ = [0, \infty)$. If the problem of synthesis is not stationary, then the Wiener-Hopf method is not applicable. In this case the problem of synthesis is reduced to a singular integral equation $T\varphi = f$ on the unit circle $\mathbf{T}$ with a non-Carleman shift of $\mathbf{T}$ onto itself, which has a finite set of fixed points. An estimate for $\dim \ker T$ is obtained and an approximation algorithm of this estimate is given. For the case $\dim \ker T = 0$ we construct an approximate solution of the equation $T\varphi = f$.

1. Introduction. Let $\alpha(t)$ be a shift (diffeomorphism) of a closed Lyapunov contour Γ onto itself preserving the orientation on Γ. Suppose that $\alpha(t)$ is a non-Carleman shift which has on Γ a finite number of fixed points, say $\{\tau_1, \dots, \tau_l\}$, $1 \le l < \infty$, i.e., $\alpha(\tau_j) = \tau_j$, $j = 1, 2, \dots, l$. We consider on $L_2(\Gamma)$ the following operators: the isometric shift operator

$$(U\varphi)(t) = \sqrt{|\alpha'(t)|}\, \varphi(\alpha(t)),$$

the operator of singular integration

$$(S\varphi)(t) = (\pi i)^{-1} \int_\Gamma \varphi(\tau)(\tau - t)^{-1} d\tau,$$

the mutually complementary projection operators

$$P_\pm = \frac{1}{2}(I \pm S),$$

the functional operators

$$A = aI + bU, \qquad B = cI + dU$$

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and the singular integral operator with shift $\alpha(t)$

$$K = AP_+ + BP_-,$$

where I is the identity operator and $a, b, c, d \in C(\Gamma)$. Recall that a shift $\alpha(t)$ is called a Carleman shift if $\alpha_k(t) \equiv t$: $(\alpha_k(t) = \alpha(\alpha_{k-1}(t)), \alpha_0(t) \equiv t)$ for all $t \in \Gamma$ and some $k \geq 2$. The operator K is a singular integral operator with non-Carleman shift having a non-empty set $M(\alpha, \Gamma)$ of fixed points. For the operator K only its Fredholm theory is known [**4, 5**]. It can be formulated as follows: the operator K is Fredholm in $L_2(\Gamma)$ if and only if the functional operators A and B are continuously invertible in $L_2(\Gamma)$. It turns out that the operator A (analogously B) is invertible in $L_2(\Gamma)$ if and only if either the condition

$$\text{(i)} \quad a(t) \neq 0 \quad \text{on } \Gamma \quad \text{and} \quad |a(\tau_j)| > |b(\tau_j)|, \qquad j = \overline{1, l}$$

or the condition

$$\text{(ii)} \quad b(t) \neq 0 \quad \text{on } \Gamma \quad \text{and} \quad |a(\tau_j)| < |b(\tau_j)|, \qquad j = \overline{1, l}$$

holds.

A formula for index of K is known. For example, if for A and B conditions of type (i) are fulfilled, then

$$\operatorname{ind} K = \dim\ \ker K - \dim\ \operatorname{coker} K = \frac{1}{2\pi}\left\{\arg \frac{c(t)}{a(t)}\right\}_\Gamma .$$

The problem of computing or, at least, estimating the defect numbers $\dim\ \ker K$ and $\dim\ \operatorname{coker} K$ and, what is more, the problem of describing the defect subspaces $\ker K$ and $\dim\ \operatorname{coker} K$ (we have in mind explicit or approximate methods for constructing their bases), the problem of obtaining explicit or approximate solutions of the equation

$$K\varphi = f \tag{1.1}$$

and also various spectral problems connected with the operator K are extremely difficult, even for the case of a Carleman shift (see, for instance, [**9, 2, 6**]). This group of questions is usually referred as the solvability theory for the operator K. Whereas just the solvability theory is of most interest for numerous and very natural applications of the integral equation (1.1) to problems of mathematical physics. Of course, first of all, the problem of obtaining an approximate solution of the equation (1.1) is very important. It is known (see, for example, [**1**]) that Wiener-Hopf equations are a natural apparatus for the solution of problems of synthesis of signals for linear systems with continuous time and stationary parameters. However, if the parameters of linear system are not stationary then the Wiener-Hopf method is not applicable and we get more complicated equations, in particular, equation (1.1) with a non-Carleman shift. A problem of synthesis has, in some sense, a universal character. It arises in the theories of hydroacoustics, radiolocation, seismology and some others. Let us describe one of the typical situations. Let $f(t)$, $-\infty < t < \infty$, be an acoustic signal. We represent it in the form

$$f(t) = f_+(t) + f_-(t),$$

where $f_+(t) \equiv 0$ for $t \leq 0$ and $f_-(t) \equiv 0$ for $t > 0$. Let $h(t, \tau)$ be the impulse characteristic of an acoustic object and $g(t)$ be a reflected signal. It is known (see [**1**]) that this system is described by the integral equation

$$f(t) + \int_{-\infty}^{\infty} h(t, \tau) f_+(\tau)\, d\tau = g(t).$$

For the stationary case $h(t, \tau) = k(t - \tau)$. Now suppose that a linear system of location is not stationary and this system is characterized by an oscillating parameter a. For example, it means that the acoustic object has a non-homogeneous reflecting surface and it revolves. Then we obtain an integral equation with kernel $h(t, \tau) = k(t - \tau)e^{-ia\tau}$, i.e.,

$$f(t) + \int_{-\infty}^{\infty} k(t - \tau) e^{-ia\tau} f_+(\tau)\, d\tau = g(t). \tag{1.2}$$

In problems of synthesis we need to solve equation (1.2) for *a priori* given impulse characteristic and reflected signal, which is selected with the aid of some criterion of optimality. For example, as a criterion it is possible to take the maximum of the ratio of the power of signal by the dispersion of noise on the entrance of the analyzer. To find this maximum it is necessary to know in advance the solution of equation (1.1) for given $h(t,\tau)$ and $g(t)$. Applying to both sides of equation (1.1) the Fourier transformation we obtain the singular integral equation

$$((I - \hat{k}\,U_1)P_+ + P_-)\hat{f} = \hat{g}, \tag{1.3}$$

where $(U_1\varphi)(x) = \varphi(x-a)$ is the non-Carleman shift operator with fixed point $x=\infty$ and $\hat{k}, \hat{f}, \hat{g}$ are the Fourier transformations of k, f, g, respectively. With the aid of a conformal mapping equation (1.3) can be reduced to the singular integral equation

$$T\varphi = ((I - cU)P_+ + P_-)\varphi = \psi, \tag{1.4}$$

which can also be written in the form

$$T\varphi = (I - cUP_+)\varphi = \psi, \tag{1.5}$$

where U is the non-Carleman shift operator with a non-empty set of fixed points, $M(\alpha, \mathbf{T}) = \{\tau_1, \tau_2, \dots, \tau_l\}$, $l \geq 1$, on the unit circle $\mathbf{T}$.

The criterion of Fredholmness gives us that either $|c(\tau_j)| < 1$, for all the $\tau_j \in M(\alpha, \mathbf{T})$, or $c(t) \neq 0$ on $\mathbf{T}$ and $|c(\tau_j)| > 1$, $j = \overline{1,l}$. In the sequel we shall suppose that the first of these conditions is fulfilled, i.e., $|c(\tau_j)| < 1$, $j = \overline{1,l}$.

For this case in the present paper an estimate for the defect number $\dim\ \ker T$ is obtained and an algorithm of approximate computing of this estimate is given. For the case $\dim\ \ker\ T = 0$ an algorithm for obtaining an approximate solution of the non-homogeneous equation (1.4) is proposed.

2. Auxiliary results. The following two auxiliary propositions play the main role.

Lemma 2.1. *Let $\alpha(t)$ be the non-Carleman shift on the unit circle $\mathbf{T}$ which has a non-empty set of fixed points, $M(\alpha, \mathbf{T}) = \{\tau_1, \dots, \tau_l\}$, $l \geq 1$. For any function $c(t) \in C(\mathbf{T})$ such that*

$$|c(\tau_j)| < 1, \qquad j = \overline{1,l}$$

and for sufficiently large n there exists a polynomial

$$r(t) = \prod_{k=1}^{n} (t - \lambda_k), \qquad |\lambda_k| > 1, \; k = 1, 2, \dots, n \tag{2.1}$$

such that the condition

$$\left| \frac{c(t) r(t)}{r(\alpha(t))} \right| < 1 \tag{2.2}$$

holds for any $t \in \mathbf{T}$.

Proof. It is sufficient to consider only the case $\|c(t)\|_{C(\Gamma)} \geq 1$, because otherwise we have simply $n = 0$. Then we represent the function $c(t)$ in the form

$$c(t) = c_0(t) b(t)$$

where $c_0(t) \in C(\mathbf{T})$, $\|c_0\|_{C(\Gamma)} = \gamma < 1$, $b(t)$ is a continuous real valued function on $\mathbf{T}$ such that

$$\begin{aligned} &b(t) > 0, \qquad t \in \mathbf{T}, \\ &b(t) < 1, \qquad t \in \delta_i, \; i = 1, 2, \dots, l \end{aligned} \tag{2.3}$$

where δ_i is some neighborhood of the point τ_i. Now we construct a continuous real function $f(t)$ such that

$$f(\alpha(t)) \geq f(t)\, b(t). \tag{2.4}$$

Let

$$\begin{aligned} f(t) = b(\alpha_{-1}(t)) &+ b(\alpha_{-1}(t))\, b(\alpha_{-2}(t)) \\ &+ \dots + \prod_{k=1}^{n} b(\alpha_{-k}(t)), \end{aligned}$$

where $\alpha(\alpha_{-1}(t)) \equiv t$, $\alpha_{-k}(t) = \alpha_{-1}(\alpha_{-k+1}(t))$. It is easy to verify that condition (2.4) holds if

$$\prod_{k=1}^{n} b(\alpha_{-k}(t)) < 1. \tag{2.5}$$

Due to condition (2.3) and known properties of a shift function $\alpha(t)$ (see, for instance, [**7**, Lemma 2.2]) the inequality (2.5) is valid for sufficiently large n.

Now we introduce the function

$$\chi(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{\tau + z}{\tau - z} \ln f(\tau)\, |d\tau|.$$

It is known (see, for instance, [**3**]), that the function $\chi(z)$ is continuous on $\mathbf{T}$, analytic for $|z| < 1$ and satisfies the following properties

a) $|\chi(t)| = f(t)$, $t \in \mathbf{T}$,

b) $|\chi(z)| \neq 0$, $|z| \leq 1$,

c) $\chi(t)$ can be uniformly approximated on $\mathbf{T}$ by a polynomial of a finite degree with any prescribed exactness ε (see, for instance, [**8**]) and all the zeros of this polynomial lie outside the unit circle $\mathbf{T}$.

Let $r(t)$ be such a polynomial and

$$|\chi(t) - r(t)| < \varepsilon. \tag{2.7}$$

Then

$$\|c_0(t)\, b(t)\| \leq \|c_0(t)\|\, \|b(t)\| = \gamma\, \|b(t)\| \tag{2.8}$$

where as it is proposed above $\gamma < 1$.

Taking into account (2.8), (2.4), (2.7) and condition a) above, we can estimate the norm of the function

$$\frac{c(t)\, r(t)}{r(\alpha(t))}$$

in the space $C(\mathbf{T})$. We get

$$\begin{aligned} \left\| \frac{c(t)\, r(t)}{r(\alpha(t))} \right\| &\leq \gamma \left\| \frac{b(t)\, r(t)}{r(\alpha(t))} \right\| \\ &\leq \gamma \left\| \frac{b(t)\, f(t)}{r(\alpha(t))} \right\| \left\| \frac{r(t)}{f(t)} \right\| \\ &\leq \gamma \left\| \frac{f(t)}{r(t)} \right\| \left\| \frac{r(t)}{f(t)} \right\|. \end{aligned} \tag{2.9}$$

Further, we have

$$\left\|\frac{f(t)}{r(t)}\right\| = \left\|\frac{\chi(t)}{r(t)}\right\| = \left\|\frac{\chi(t)-r(t)}{r(t)}+1\right\| \le \frac{\varepsilon}{\|r(t)\|}+1 < \frac{\varepsilon}{\tilde f-\varepsilon}+1 = \frac{\tilde f}{\tilde f-\varepsilon}, \tag{2.10}$$

$$\left\|\frac{r(t)}{f(t)}\right\| = \left\|\frac{r(t)-\chi(t)}{\chi(t)}+1\right\| < \frac{\varepsilon+\tilde f}{\tilde f}, \tag{2.11}$$

where the notation $\tilde f = \inf(f)$ was used.

Taking into account (2.10) and (2.11), from (2.9) we get

$$\left\|\frac{c(t)\,r(t)}{r(\alpha(t))}\right\|_{C(\mathbf{T})} \le \gamma\,\frac{\tilde f+\varepsilon}{\tilde f-\varepsilon}.$$

Now, if we choose ε such that

$$\varepsilon < \tilde f\,\frac{1-\gamma}{1+\gamma}$$

we obtain inequality (2.2). □

As an example we consider the case of a linear fractional non-Carleman shift of the unit circle $\mathbf{T}$ onto itself. In this case the polynomial $r(t)$ can be found quite easily.

The linear fractional non-Carleman shift preserving the orientation on $\mathbf{T}$ has the form (see, for instance, [**7**]):

$$\alpha(t) = \frac{at+b}{\overline{b}t+\overline{a}},$$

where $|a|^2 - |b|^2 = 1$. The fixed points of this shift are given by the formula

$$\tau_{1,2} = \frac{a-\overline{a}\pm\sqrt{D}}{2\overline{b}},$$

where $D = (a + \bar{a})^2 - 4$. Obviously $\tau_1 \neq \tau_2$ if $|\mathrm{Re}\, a| \neq 1$. Without loss of generality we can suppose that the fixed points lie on the real line $\mathbf{R}$, i.e., $\tau_1 = 1, \tau_2 = -1$ or, what is the same, the coefficients a and b are real numbers. Indeed, it is not difficult to see that the set of linear fractional shifts of $\mathbf{T}$ onto itself form a non-commutative group. From this it follows, in particular, that for any two shifts $\alpha(t)$ and $\beta(t)$ there exists a shift $\gamma(t)$ such that

$$\gamma(\alpha(\gamma_{-1}(t))) = \beta(t), \qquad \gamma(\gamma_{-1}(t)) \equiv t,$$

and, therefore, the case of any linear fractional shift $\alpha(t)$ can be reduced to the case of a linear fractional shift $\beta(t)$ with real coefficients. It is known (see, for instance, [**7**]), that for the above considered shift $\alpha(t)$ one of its fixed points is attracting and another one is repelling. Suppose that $\tau_1 = 1$ is the attracting point. Then, for example, for $\lambda < -1$ the inequality

$$|t - \lambda| < |\alpha(t) - \lambda|$$

holds. Indeed, let $t = e^{i\theta}$, $\alpha(t) = e^{i\vartheta}$ and λ be a real number.

We get

$$|t - \lambda|^2 = (e^{i\theta} - \lambda)(e^{-i\theta} - \lambda) = \lambda^2 - 2\lambda\cos\theta + 1$$

and, analogously,

$$|\alpha(t) - \lambda|^2 = \lambda^2 - 2\lambda\cos\vartheta + 1.$$

It is clear that for the above mentioned supposition, concerning the points $\tau_1 = 1$ and $\tau_2 = -1$, we have $|\theta| > |\vartheta|$ and, consequently, $\cos\theta < \cos\vartheta$ if $\lambda < 0$.

Now let $c(t) \in C(\mathbf{T})$ be a function such that

$$|c(\tau_i)| < 1, \quad i = 1, 2, \quad \text{with} \quad \|c(t)\|_{C(\mathbf{T})} = M > 1 \quad \text{and} \quad \lambda < -1.$$

Then there exists a natural number n such that

$$\left| c(t) \left(\frac{t - \lambda}{\alpha(t) - \lambda} \right)^n \right| < 1. \tag{2.12}$$

Thus $r(t) = (t - \lambda)^n$. Put

$$L = \left\| \frac{t - \lambda}{\alpha(t) - \lambda} \right\|_{C(\mathbf{T})}.$$

Then $L < 1$ for $\lambda < -1$ and inequality (2.12) is fulfilled if $n > -\ln M / \ln L$.

Lemma 2.2. *Consider the operators*

$$N = I - a(t)\, U\, P_+, \qquad R = r(t)\, P_- + (I - a(t)\, U)\, P_+$$

where $a(t) \in C(\mathbf{T})$, $\|a\|_{C(\mathbf{T})} < 1$ *and* $r(t)$ *is a polynomial of the degree* n *with roots satisfying the conditions*

$$|\lambda_k| > 1, \qquad k = 1, 2, \dots, n.$$

Then

$$\dim\ \ker R > 0$$

if and only if there exits a polynomial $p(t)$ *with a degree which is not greater than the degree of the polynomial* $r(t)$ *and the condition*

$$(I - rP_+ r^{-1} P_- N^{-1})\, p(t) = 0 \tag{2.13}$$

is fulfilled.

Proof. Necessity. Suppose that $\ker R$ is not trivial, and let $\varphi \in \ker R$, i.e.,

$$R\varphi \equiv ((r-1)P_- + N)\, \varphi = 0. \tag{2.14}$$

Since $\|a(t)\|_{C(\mathbf{T})} < 1$, $\|U\| = 1$ and $\|P_+\| = 1$ it folllows that N is an invertible operator whose inverse can be represented as

$$N^{-1} = I + a\, UP_+ + (a\, UP_+)^2 + \cdots .$$

Obviously,

$$N^{-1} = P_- + N^{-1} P_+ \tag{2.15}$$

and

$$P_+ r P_- \varphi = p \tag{2.16}$$

where p is a polynomial such that $\deg p$ is not greater than $\deg r$. Using relations (2.15) and (2.16) we proceed by transforming relation (2.14). We have

$$N^{-1} R \varphi = (N^{-1}(r-1)P_- + I)\,\varphi = 0$$

and, consequently,

$$\begin{aligned} N^{-1} R \varphi &\equiv [(P_- + N^{-1}P_+)(r-1)P_- + I]\,\varphi \\ &= (P_-(r-1)P_- + I)\,\varphi + N^{-1}P_+ r P_- \varphi \\ &= P_-(r-1)P_- \varphi + N^{-1} p + \varphi. \end{aligned}$$

Further

$$P_- N^{-1} R \varphi \equiv P_-(r-1)P_- \varphi + P_- N^{-1} p + P_- \varphi = 0. \tag{2.17}$$

Transforming the expression $P_-(r-1)P_-\varphi$, we get

$$\begin{aligned} P_-(r-1)P_- \varphi &= (I - P_+)[(r-1)P_- \varphi] \\ &= (r-1)P_- \varphi - P_+ r P_- \varphi \\ &= r P_- \varphi - P_- \varphi - p. \end{aligned}$$

From (2.17) we obtain

$$\begin{aligned} P_- N^{-1} R \varphi &= r P_- \varphi - P_- \varphi + P_- N^{-1} p + P_- \varphi - p \\ &= r P_- \varphi + P_- N^{-1} p - p = 0. \end{aligned} \tag{2.18}$$

Application of $r^{-1} I$ to both sides of (2.18) on the left, yields

$$r^{-1} P_- N^{-1} R \varphi = P_- \varphi + r^{-1} P_- N^{-1} p - r^{-1} p = 0$$

and

$$r P_+ r^{-1} P_- N^{-1} R \varphi = r P_+ r^{-1} P_- N^{-1} p - r(I - P_-) r^{-1} p = 0.$$

Since all the roots of the polynomial $r(z)$ lie outside of unit disk, we have $P_- r^{-1} p = 0$ and so we obtain condition (2.13).

Sufficiency. Let us check that if condition (2.13) holds then the function

$$\hat{\varphi} = (P_+ + P_- r^{-1} P_-) N^{-1} p$$

belongs to the kernel of the operator R or, similarly, $\hat{\varphi} \in \ker N^{-1}R$. Computing $N^{-1}R\,\hat{\varphi}$ we have

$$\begin{aligned} N^{-1}R\,\hat{\varphi} &= (N^{-1}(r-1)P_- + I)(P_+ + P_- r^{-1} P_-)N^{-1}p \\ &= P_+ N^{-1}p + N^{-1} r P_- r^{-1} P_- N^{-1} p \\ &\quad - N^{-1} P_- r^{-1} P_- N^{-1} p + P_- r^{-1} P_- N^{-1} p. \end{aligned}$$

Taking into account that $N^{-1}P_- = P_-$, we get

$$\begin{aligned} N^{-1}R\,\hat{\varphi} &= P_+ N^{-1}p + N^{-1} r (I - P_+) r^{-1} P_- N^{-1} p \\ &\quad - P_- r^{-1} P_- N^{-1} p + P_+ r^{-1} P_- N^{-1} p \\ &= P_+ N^{-1} p + P_- N^{-1} p - N^{-1} r P_+ r^{-1} P_- N^{-1} p \\ &= N^{-1}(p - r P_+ r^{-1} P_- N^{-1} p). \end{aligned}$$

Due to condition (2.13) we finally obtain that $\hat{\varphi} \in \ker N^{-1}R$. □

3. Estimate for $\dim\ker T$. If we put together Lemmas 2.1 and 2.2 we are able to estimate the dimension of the kernel of the operator T, defined by (1.4).

Theorem 3.1. *If the conditions of Lemmas* (2.1) *and* (2.2) *are fulfilled and* n *is the degree of the polynomial*

$$r(t) = \prod_{k=1}^{n} (t - \lambda_k),$$
$$|\lambda_k| > 1, \quad k = 1, 2, \dots, n,$$

then the estimate

$$\dim\ker T \le n$$

holds.

Proof. Making use of $P_+ r = r$ and $U\,r(t)\,P_+ = r(\alpha(t))\,U\,P_+$, we can give the operator T the following form

$$T = r^{-1}(rP_- + (I - a\,U)\,P_+)\,(P_- + rP_+), \tag{3.1}$$

where

$$a(t) = \frac{c(t)r(t)}{r(\alpha(t))}.$$

Since $r(t) \neq 0$ on $\mathbf{T}$ and $\mathrm{Ind}_{\mathbf{T}}\, r(t) = (1/2\pi)\{\arg r(t)\}_{\mathbf{T}} = 0$, both operators $r^{-1}I$ and $P_- + rP_+$ are continuously invertible and, consequently, their kernels are trivial. Therefore

$$\dim\ \ker T = \dim\ker\left(rP_- + (I - aU)P_+\right) = \dim\ker R.$$

Now from Lemma 2.2 it follows that the number dim ker R coincides with the number of linear independent polynomials p. Since this number is not greater than n, we obtain

$$\dim\ \ker\ T \leq n. \qquad \square$$

Now it is not difficult to get an approximate estimate of the number $\dim\ker T$. To this end we rewrite condition (2.13) in the form of an homogenuous system of linear algebraic equations for the coefficients of the polynomial $p(t) = \sum_{k=0}^{n-1} P_k t^k$. Without loss of generality we suppose that all the roots of the polynomial $p(t)$ are distinct. Put

$$\varphi_- = P_- N^{-1} p, \qquad r(t) = (t - \lambda_k) r_k(t)$$

Applying successively the transformation

$$rP_+ r^{-1}\varphi_- = rP_+\left[\frac{\varphi_-(t) - \varphi_-(\lambda_k)}{(t-\lambda_k)r_k(t)} - \frac{\varphi_-(\lambda_k)}{r(t)}\right],$$
$$k = 1, 2, \dots, n-1,$$

we conclude that the function

$$\mu(t) = (rP_+ r^{-1}\psi_-)(t)$$

is a polynomial with a degree which is not greater than $n-1$, i.e.,

$$\mu(t) = \prod_{i=1}^{n-1}(t-\lambda_i)\overset{n-1}{\varphi_-}(\lambda_n)$$
$$+ \prod_{i=1}^{n-2}(t-\lambda_i)\overset{n-2}{\varphi_-}(\lambda_{n-1}) + \cdots$$
$$+ (t-\lambda_1)\overset{1}{\varphi_-}(\lambda_2) + \overset{0}{\varphi_-}(\lambda_1),$$

where

$$\overset{k}{\varphi}_-(t) = \frac{\overset{k-1}{\varphi}_-(t) - \overset{k-1}{\varphi}_-(\lambda_k)}{t - \lambda_k}, \qquad k = 1, 2, \dots, n-1,$$
$$\overset{0}{\varphi}_-(t) = \psi_-(t), \qquad \mu(\lambda_i) = \psi_-(\lambda_i), \qquad i = 1, 2, \dots, n.$$

Let us now introduce some notation. Let $\tilde{p} = (p_0, p_1, \dots, p_{n-1})$ be the set of the coefficients of the polynomial $p(t)$, Λ the matrix transposed to the Vandermond matrix constructed from the roots of the polynomial $r(t)$, $g_k = N^{-1}t^k$, E the identity matrix and

$$G_- = \begin{pmatrix} (P_-\, g_0)(\lambda_1) & \cdots & (P_-\, g_{n-1})(\lambda_1) \\ \vdots & \ddots & \vdots \\ (P_-\, g_0)(\lambda_n) & \cdots & (P_-\, g_{n-1})(\lambda_n) \end{pmatrix}.$$

Using these notations we rewrite condition (2.13) in the form

$$(E - \Lambda^{-1} G_-)\, \tilde{p} = 0$$

or, equivalently, in the form

$$\det(G_- - \Lambda) = 0$$

Due to Theorem 3.1 and Lemma 2.2 we obtain

$$\dim\ \ker T = \dim\ \ker R = n - \operatorname{rank}\ (G_- - \Lambda).$$

Thus approximating the function χ by the polynomial $r(t)$ and finding its roots λ_k, $k = \overline{1, n}$, we can construct the matrix $G_- - \Lambda$, calculate its rank and obtain an approximate estimate of $\dim\ \ker T$.

4. Approximate solution of the non-homogeneous equation $T\varphi = f$. Since (see the formula (3.1))

$$T = r^{-1}(rP_- + (I - a\,U)\,P_+)\,(P_- + rP_+),$$

the equation $T\varphi = f$ is equivalent to the equation

$$rP_-\varphi + NP_+ rP_+\varphi = h \tag{4.1}$$

where $h = rf$. Indeed,

$$R = rP_- + (I - aU)\,P_+ = rP_- + (I - aUP_+)\,P_+ = rP_- + NP_+,$$

and since $P_- rP_+ = 0$, it follows that $R\,(P_- + rP_+) = rP_- + NP_+ rP_+$. Obviously equation (4.1) is equivalent to the equation

$$N^{-1} rP_- \varphi + P_+ rP_+ \varphi = N^{-1} h, \tag{4.2}$$

Due to (2.15) and the obvious relation $P_+ rP_+ \varphi = rP_+ \varphi$, we can transform the left hand side of equation (4.2), obtaining

$$P_- rP_- \varphi + N^{-1} P_+ rP_- \varphi + rP_+ \varphi = N^{-1} h. \tag{4.3}$$

Since

$$P_- rP_- \varphi = rP_- \varphi - P_+ rP_- \varphi = rP_- \varphi - p \tag{4.4}$$

and

$$rP_+ \varphi + rP_- \varphi = r\varphi,$$

equation (4.3) can be given the form

$$r\varphi = N^{-1} h + p - N^{-1} p$$

and, consequently,

$$\varphi = r^{-1} (N^{-1} h + p - N^{-1} p).$$

It only remains to find p. To this end we return again to equation (4.3). We have

$$P_- N^{-1} P_+ rP_- \varphi + P_- rP_- \varphi = P_- N^{-1} h,$$

and if we take into account the first relation in (4.4) and (2.16), then

$$P_- N^{-1} p + rP_- \varphi - p = P_- N^{-1} h,$$

which after some obvious transformations can be written as

$$rP_+ r^{-1} P_- N^{-1} p - p = rP_+ r^{-1} N^{-1} h. \tag{4.5}$$

Equation (4.5) is just an equation for finding p. If it has a unique solution then that is just that p we need.

Writting (4.5) as a linear algebraic system of equations for the vector $\tilde{p}$ we get

$$(\Lambda^{-1}G_- - E)\,\tilde{p} = \Lambda^{-1}\,\tilde{h}$$

or

$$(G_- - \Lambda)\,\tilde{p} = \tilde{h}, \tag{4.6}$$

where the components of the vector $\tilde{h}$ are the values of the function $P_-N^{-1}h$ at the points λ_i, $i = 1, 2, \ldots, n$.

In order to construct an approximate solution of the equation $T\varphi = f$ we carry out the following operations. First of all, we estimate as far as $\det(G_- - \Lambda)$ is separated from zero. If it turns out that $|\det(G_- - \Lambda)|$ is greater than the absolute value of the error made in the calculations, then the kernel of the operator T is trivial and solving system (4.6) we find the solution of the initial equation.

It is clear that errors can occur in the calculation of the functions P_-g_k and of the vector $\tilde{h}$. Let $A = G_- - \Lambda$ and δA, $\delta\tilde{h}$ be the errors in the initial data of system (4.6) and $\delta\tilde{p}$ be the error of its solution. Then

$$A = A_0 - \delta A, \qquad \tilde{h} = h_0 - \delta\tilde{h}, \qquad \tilde{p} = p_0 - \delta\tilde{p}$$

where p_0 is a solution of system (4.6) with distorted initial data, i.e.,

$$A_0 p_0 = h_0. \tag{4.7}$$

We calculate the relative error in the computation of the vector $\tilde{p}$. Let us denote by $\kappa(A)$ the condition number of the matrix A, i.e., $\kappa(A) = \|A^{-1}\|\,\|A\|$. Then, according to (4.6), we have

$$(A_0 - \delta A)(p_0 - \delta\tilde{p}) = h_0 - \delta\tilde{h}$$

or

$$A_0 p_0 - A_0\delta\tilde{p} = h_0 - \delta\tilde{h} + \delta A\tilde{p}.$$

Due to equation (4.7) we get

$$A_0\delta\tilde{p} = \delta\tilde{h} - \delta A\tilde{p}.$$

Further

$$\|\delta\tilde{p}\| \le \|A_0^{-1}\| \left(\|\delta\tilde{h}\| + \|\delta A\| \, \|\tilde{p}\|\right),$$

$$\|\delta\tilde{p}\| \le \|A_0^{-1}\|\left(\frac{\|\delta\tilde{h}\| \, \|A_0\, p_0\|}{\|h_0\|} + \|\delta A\| \, \|\tilde{p}\|\right),$$

$$\|\delta\tilde{p}\| \le \|A_0^{-1}\| \, \|A_0\|\left(\frac{\|\delta\tilde{h}\| \, \|p_0\|}{\|h_0\|} + \frac{\|\delta A\| \, \|\tilde{p}\|}{\|A_0\|}\right),$$

$$\|\delta\tilde{p}\| \le \kappa(A_0)\left(\frac{\|\delta\tilde{h}\|}{\|h_0\|}(\|\tilde{p}\| + \|\delta\tilde{p}\|) + \frac{\|\delta A\|}{\|A_0\|}\|\tilde{p}\|\right),$$

$$\frac{\|\delta\tilde{p}\|}{\|\tilde{p}\|} \le \kappa(A_0)\left(\frac{\|\delta\tilde{h}\|}{\|h_0\|} + \frac{\|\delta A\|}{\|A_0\|} + \frac{\|\delta\tilde{h}\|}{\|h_0\|}\,\frac{\|\delta\tilde{p}\|}{\|\tilde{p}\|}\right),$$

$$\left(1 - \kappa(A_0)\frac{\|\delta\tilde{h}\|}{\|h_0\|}\right)\frac{\|\delta\tilde{p}\|}{\|\tilde{p}\|} \le \kappa(A_0)\left(\frac{\|\delta\tilde{h}\|}{\|h_0\|} + \frac{\|\delta A\|}{\|A_0\|}\right),$$

and, finally,

$$\frac{\|\delta\tilde{p}\|}{\|\tilde{p}\|} \le \kappa(A_0)\frac{\|\delta\tilde{h}\|/\|h_0\| + \|\delta A\|/\|A_0\|}{\left(1 - \kappa(A_0)\|\delta\tilde{h}\|/\|h_0\|\right)}.$$

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