JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 7, Number 4, Fall 1995

A QUADRATURE METHOD FOR THE CAUCHY SINGULAR INTEGRAL EQUATIONS

YOUNGMOK JEON

ABSTRACT. We consider a quadrature method for the Cauchy singular integral equation (CSIE):

$$a\phi(t) + \frac{b}{\pi}\text{p.v.}\int_0^1 \frac{1}{s-t}\phi(s)\,ds = f(t)$$

with the index 0 and -1. The quadrature method is based on a mesh grading transformation followed by the rectangular rule. Our method is universal in the sense that the method is barely dependent on the coefficients (a, b). A complete analysis is given by the use of localization and the Mellin transform technique. This method is also applicable to the Cauchy singular integral equation with nonconstant coefficients. For this case, only numerical results are presented.

1. Introduction. We consider the Cauchy singular integral equations (CSIE)

$$\mathcal{C}\phi(t) := (aI + b\mathcal{S})\phi(t) = a\phi(t) + \frac{b}{\pi}\text{p.v.}\int_0^1 \frac{\phi(s)}{s-t}\,ds = f(t) \tag{1.1}$$

with $a > 0$, $b \neq 0$ real constants. We also assume that $a^2 + b^2 = 1$ (a normalized CSIE) for convenience. If $a = 0$, it is of the first kind, and if $a \neq 0$, it is of the second kind. In this paper we concentrate on the analysis of the second kind Cauchy integral equation.

It is known that the solution of (1.1) with a smooth f has the endpoint singularities:

$$\phi(t) = t^{\alpha}(1-t)^{\beta}h(t), \quad -1 < \alpha, \beta < 1, \tag{1.2}$$

where

$$\begin{gathered} \alpha = \gamma_0 + M_1, \qquad \beta = -\gamma_0 + M_2, \\ \gamma_0 = -\frac{1}{2\pi i}\log\frac{a-bi}{a+bi} = -\frac{\tan^{-1}(b/a)}{\pi} \end{gathered} \tag{1.3}$$

Received by the editors on May 16, 1994, and in revised form on March 20, 1995.
This research is supported by a research fund of Ajou University.

for some integers M_1, M_2 so that $\nu = -(\alpha+\beta) = -(M_1+M_2) = -1, 0,$ or 1 [**8, 13**]. The integer ν is the *index*. For the index 0 solution, there can be two classes: that is, $M_1 = M_2 = 0$ (the natural airfoil solution) or $-M_1 = M_2 = \gamma_0/|\gamma_0|$.

We are interested in the natural airfoil solution. We also consider the index -1 problem. The index -1 solution satisfies $M_1 = 0$, $M_2 = 1$ when $\gamma_0 > 0$ or $M_1 = 1$, $M_2 = 0$ when $\gamma_0 < 0$. The index -1 solution is obtained when f satisfies the *consistency condition*,

$$\int_0^1 \frac{f(t)}{t^\alpha(1-t)^\beta} = 0. \tag{1.4}$$

So the index -1 solution can be considered as an index 0 solution with a special f. We also introduce an important result from [**13**]:

$$\begin{aligned}&\text{For } a > 0, \mathcal{C} : L_2[0,1] \longrightarrow L_2[0,1]\\ &\text{is a Fredholm integral operator of index 0.}\end{aligned} \tag{1.5}$$

This will be used in proving Theorem 5.2 and then Theorem 3.1 that appear in later sections.

Let us consider a transformation:

$$w(x) = \frac{v^q(x)}{v(x)^q + v^q(1-x)}, \qquad q \geq 2 \tag{1.6}$$

with $v(x) = (1/2 - 1/q)(2x-1)^3 + 1/q(2x-1) + 1/2$. This parameterization is considered in [**10**]. We call q the *order of the mesh grading or the mesh grading parameter*. The transformation, $w : [0,1] \to [0,1]$ is bijective and has the following properties:

1. $w(x) = x^q + O(x^{q+1})$ as $x \to 0$ and $w(x) = 1 - (1-x)^q + O((1-x)^{q+1})$ as $x \to 1$,
2. $w'(1/2) = 2$ so that $w(x) \approx 2x$ away from 0 and 1.

With a change of variables, $t = w(x)$ and $s = w(y)$ in (1.1), and multiplying (1.1) by the derivative $w'(x)$, we have

$$\mathcal{C}^q\tilde{\phi}(x) = a\tilde{\phi}(x) + \frac{b}{\pi}\text{p.v.}\int_0^1 \frac{w'(x)}{w(y)-w(x)}\tilde{\phi}(y)\,dy = \tilde{f}(x) \tag{1.7}$$

with $\tilde{\phi}(x) := w'(x)\phi(w(x))$ and $\tilde{f}(x) := w'(x)f(w(x))$.

But, unfortunately, (1.7) is not uniquely solvable in the L_2-space, but it is uniquely solvable in a weighted L_2-space. We introduce the weight function

$$\kappa(x) = [x(1-x)]^{\lambda}.$$

The number λ is the *weight parameter* in this article. Let

$$\psi(x) = \frac{\tilde{\phi}(x)}{\kappa(x)}. \tag{1.8}$$

Dividing equation (1.7) by $\kappa(x)$, we have

$$\mathcal{C}_{\lambda}^{q}\psi(x) := a\psi(x) + \frac{b}{\pi}\text{p.v.}\int_0^1 \frac{\kappa(y)}{\kappa(x)}\frac{w'(x)}{w(y)-w(x)}\psi(y)\,dy = g(x) \tag{1.9}$$

where $g(x) = \tilde{f}(x)/\kappa(x)$. Now (1.9) is uniquely solvable for ψ if the mesh grading parameter q and the weight parameter λ satisfies

$$\lambda \le q < q^* = \frac{(2\lambda+1)\pi}{2\tan^{-1}(|b|/|a|)} \tag{1.10}$$

(see Theorem 5.2), and then the solution ψ satisfies the following regularity:

$$\psi(x) = x^{\alpha q+(q-1)-\lambda}(1-x)^{\beta q+(q-1)-\lambda}h(x), \tag{1.11}$$

where h is a smooth function on $[0,1]$. Therefore, ψ is a smooth 1-periodic function if mesh grading parameter q is sufficiently large, and the trigonometric approximation for ψ may give a well-convergent numerical scheme.

Let us consider a quadrature method for equation (1.9). Using the subtraction of a singularity, followed by the rectangular rule, we consider a fully discrete approximation for $\mathcal{C}_{\lambda}^{q}$:

$$\begin{aligned}\mathcal{C}_{\lambda,h}^{q}\psi_h(x) := a\psi_h(x) &+ \frac{b}{\pi}h\sum_{k=0}^{N-1}\frac{\kappa(kh)}{\kappa(x)}\frac{w'(x)\psi_h(kh)}{w(kh)-w(x)} \\ &+ \frac{b}{\pi}\psi_h(x)\Bigg[\log\left(\frac{1-w(x)}{w(x)}\right) \\ &\qquad - h\sum_{k=0}^{N-1}\frac{w'(kh)}{w(kh)-w(x)}\Bigg] \approx g(x),\end{aligned} \tag{1.12}$$

where $h = 1/N$ and ψ_h is a trigonometric function of degree N.

Collocating (1.12) at $x = kh + \xi h$ with $0 < \xi < 1$, we have an $N \times N$ matrix system. If it is nonsingular, we can solve for $\{\psi_h(kh)\}_{k=0}^{N-1}$. In collocating (1.12), we need to evaluate ψ_h at points other than node points, and it will be evaluated by the use of the Dirichlet kernel.

$$\psi_h(x) = h \sum_{k=0}^{N-1} D(x, kh)\psi_h(kh), \tag{1.13}$$

where

$$D(x,y) = \begin{cases} \dfrac{\sin(N\pi(x-y))}{\sin(\pi(x-y))} & N \text{ odd}, \\ \cos(\pi(x-y))\dfrac{\sin(N\pi(x-y))}{\sin(\pi(x-y))} & N \text{ even}. \end{cases}$$

Once ψ_h is solved, $\phi_h(x) = \lambda(x)\psi_h(x)$ will be an approximate solution of (1.7).

For the second kind of CSIE, the Chebyshev or the Jacobi polynomials are generally used for the approximation of density functions. When a fully discrete method is considered, the Gauss quadrature or Lobatto quadrature methods are used for the approximation of integrals in most cases [**8, 16**]. But the singularity in (1.2) varies according to the coefficients (a, b). Therefore, the quadrature method and the trial function space need to cope with these changes to have a well convergence numerical scheme. But generating the Jacobi polynomial and the Gaussian quadrature method for the weight with arbitrary (α, β) may be costly. So, sometimes the Chebyshev polynomial with the Lobatto quadrature is used for the general second kind equation without regard to (α, β) [**16**].

Our method is a universal method in the sense that it is little dependent on (a, b). Moreover, it solves the problems of the indices 0 and -1 at the same time. The mesh grading transformation methods have been popular recently for the boundary integral equations of potential problems [**10,9,7**]. The solution of potential problems (the Symm's integral equation) has a similar regularity to that in (1.2) with $\nu = 0$, and it is our motivation. A quadrature method with the mesh grading transformation for the Cauchy singular integral equation has been used before in [**14**], in which the explicit singular representation of the solution is used for a weight, and the Simpson rule is used for the approximation of integral operators. Our method has different flavors with them in that we use a general weight (reduce the dependence on (a,b)), and we use the rectangular rule for the integral operator approximation, which results in a good convergence for a certain class of integral operators. Our quadrature method is rather similar to that of [**15**], in which they analyze the ε-collocation for the CSIE on closed curves.

2. Preliminary. We introduce Sobolev spaces and a collocation projection P_h. Let H^s be the Sobolev space of periodic functions on $[0,1]$ of order s with the norm $||\cdot||_s$. Then $f \in H^s$ if and only if

$$\sum_{m\in\mathbf{Z}} |m|^{2s}|\hat{f}(m)|^2 < \infty, \qquad f(x) = \sum_{m\in\mathbf{Z}} \hat{f}(m)e^{2\pi imx}.$$

Note that $H^0 = L_2[0,1]$. We denote $\mathbf{Z}^* \equiv \mathbf{Z}/\{0\}$. Let T^h be the space of trigonometric functions of dimension N. Let

$$\Lambda_h = \{m : -N/2 \le m \le N/2 - 1 \text{ for even } N\},$$

or

$$\{m : -[N/2] \le m \le [N/2] \text{ for odd } N\},$$

where $[x]$ is the greatest integer not greater than x, and let $\Lambda_h^* = \Lambda_h/\{0\}$. Then

$$T^h := \text{span}\,\{e^{2\pi imx} : m \in \Lambda_h\}.$$

For simplicity of notation, we denote $\phi_m(x) := e^{2\pi imx}$.

Define a collocation projection $P_{h,\xi} : H^s \to T^h$ as

$$(2.1)\quad (P_{h,\xi}f)(kh+\xi h) = f(kh+\xi h), \quad 0 \le \xi < 1,\ k = 0,\dots,N-1$$

for $s > 1/2$ because H^s, $s > 1/2$, is compactly embedded in the continuous periodic function space $C_p[0,1]$. From here on, we denote $P_h := P_{h,\xi}$ unless stated otherwise. Then we have the following standard convergence result for P_h.

Lemma 2.1 [**15**]. *Assume $f \in H^t$ with $t > 1/2$ and $0 \le s < t$. Then*

$$||f - P_hf||_s \le Ch^{t-s}||f||_t.$$

For an integral operator,

$$\mathcal{A}\phi(x) := \int_0^1 A(x,y)\phi(y)\,dy,$$

define a rectangular approximation $\mathcal{A}_N$ as

$$\mathcal{A}_N\phi(x) := h\sum_{k=0}^{N-1} A(x,kh)\phi(kh). \tag{2.2}$$

We introduce a useful result for the Hilbert transform $\mathcal{H}$:

$$\mathcal{H}\phi(x) = \text{p.v.}\int_0^1 \cot(\pi(y-x))\phi(y)\,dy.$$

Lemma 2.2. *For $\phi \in T^h$, $\mathcal{H}_N\phi - \phi\mathcal{H}_N(1) = \mathcal{H}\phi$.*

Proof. By a simple calculation,

$$\cot(\pi(y-x)) = i\frac{\phi_1(y-x)+1}{\phi_1(y-x)-1} = i\frac{1+\phi_{-1}(y-x)}{1-\phi_{-1}(y-x)}$$

where $\phi_m(x) = e^{2\pi i m x}$. Then for $m > 0$,

$$\begin{aligned}
\mathcal{H}_N\phi_m - \phi_m\mathcal{H}_N(1) &= h\sum_{k=0}^{N-1}\cot(\pi(kh-x))(\phi_m(kh)-\phi_m(x)) \\
&= ih\sum_{k=0}^{N-1}\frac{\phi_1(kh-x)+1}{\phi_1(kh-x)-1}\phi_m(x)(\phi_m(kh-x)-1) \\
&= i\phi_m(x)h \\
&\quad\cdot\sum_{k=0}^{N-1}[\phi_1(kh-x)+1][\phi_{m-1}(kh-x)+\cdots+1] \\
&= i\phi_m(x),
\end{aligned}$$

since $h\sum_{k=0}^{N-1}\phi_m(kh)=1$ if $m\equiv 0 \pmod N$ and 0 otherwise. By the same way for $m<0$,

$$\begin{aligned}\mathcal{H}_N\phi_m-\phi_m\mathcal{H}_N(1)&=h\sum_{k=0}^{N-1}\cot(\pi(kh-x))(\phi_m(kh)-\phi_m(x))\\&=ih\sum_{k=0}^{N-1}\frac{1+\phi_{-1}(kh-x)}{1-\phi_{-1}(kh-x)}\phi_m(x)(\phi_m(kh-x)-1)\\&=-i\phi_m(x)h\\&\quad\cdot\sum_{k=0}^{N-1}[\phi_{-1}(kh-x)+1][\phi_{m+1}(kh-x)+\cdots+1]\\&=-i\phi_m(x).\end{aligned}$$

Then the lemma follows. □

We introduce the singular integral operators:

$$\begin{aligned}\mathcal{L}\psi(x)&=\int_0^1\left(\frac{1}{\pi}\frac{\kappa(y)}{\kappa(x)}\frac{w'(x)}{w(y)-w(x)}-\cot(y-x)\right)\psi(y)\,dy,\\\mathcal{W}\psi(x)&=\int_0^1\left(\frac{1}{\pi}\frac{w'(y)}{w(y)-w(x)}-\cot(y-x)\right)\psi(y)\,dy.\end{aligned}\tag{2.3}$$

Using Lemma 2.2, equations (1.9) and (1.12) can be written in symbolic form:

$$\mathcal{C}_\lambda^q\psi=a\psi+b\mathcal{H}\psi+b\mathcal{L}\psi=g,\tag{2.4}$$

$$\begin{aligned}\mathcal{C}_{\lambda,h}^q\psi_h=a\psi_h+b\mathcal{H}\psi_h+b\mathcal{L}_N\psi_h+b\psi_h\\\cdot(\mathcal{W}(1)-\mathcal{W}_N(1))\approx g,\qquad\psi_h\in T^h.\end{aligned}\tag{2.5}$$

Then the collocation method for (2.5) is: find $\psi_h\in T^h$ such that

$$\begin{aligned}P_h\mathcal{C}_{\lambda,h}^q\psi_h&=a\psi_h+b\mathcal{H}\psi_h+bP_h\mathcal{L}_N\psi_h\\&\quad+bP_h\{\psi_h\cdot(\mathcal{W}(1)-\mathcal{W}_N(1))\}=P_hg.\end{aligned}\tag{2.6}$$

Unfortunately, the collocation method (2.6) is unstable, and we need to modify (2.6) around endpoints 0 and 1. Let us introduce a truncation operator T_r such that

$$(T_r v)(x) := \begin{cases} v(x) & x \in [r, 1-r] \\ 0 & \text{otherwise} \end{cases}, \tag{2.7}$$

and a smooth truncation function S_r such that $0 \le S_r(x) \le 1$ and

$$S_r(x) := \begin{cases} 1 & x \in [r, 1-r], \\ 0 & x \in [r/2, 1-r/2], \end{cases} \tag{2.8}$$

where its derivatives satisfy

$$|S_r^{(p)}| \le C \frac{1}{r^p}$$

for any integer p and a constant C independent of p.

Our modified method is: find $\psi_h \in T^h$ such that

$$\begin{aligned} P_h \mathcal{C}^q_{\lambda,h,i^*h} \psi_h := a\psi_h + b\mathcal{H}\psi_h + bP_h \mathcal{L}_N T_{i^*h} \psi_h \\ + bP_h\{(S_{i^*h} \cdot \psi_h \cdot (\mathcal{W}1 - \mathcal{W}_N 1)\} = P_h g. \end{aligned} \tag{2.9}$$

Here i^* is an integer independent of h, and it denotes the number of subintervals cut off. In fact, i^* is needed only for theoretical purposes, and in most cases $i^* = 0$ is sufficient for stability in practice.

3. Stability. In this section we prove the stability of our numerical system equation (2.9). For stability analysis we introduce the following stability result for the operator:

$$\mathcal{C}^q_{\lambda,r} \psi := (aI + b\mathcal{H} + b\mathcal{L}T_r)\psi. \tag{3.1}$$

Theorem 3.1. *There is an $r_0 > 0$ such that*

$$||(aI + b\mathcal{H} + b\mathcal{L}T_r)\psi||_0 \ge C||\psi||_0 \tag{3.2}$$

for all $0 \le r \le r_0$, $\psi \in H_0$ and $C > 0$ independent of r.

The proof of Theorem 3.1 will be given in Section 5.

In this section, assuming Theorem 3.1, we prove that for a properly chosen integer $i^* > 0$ independent of h,

$$(3.3)\quad ||(aI + b\mathcal{H} + bP_h\mathcal{L}_N T_{i^*h})\psi_h + bP_h[S_{i^*h}\cdot\psi_h\cdot(\mathcal{W}1 - \mathcal{W}_N 1)]||_0 \geq C||\psi_h||_0$$

for $\psi_h \in T^h$ and $C > 0$ independent of h.

For the proof of (3.3), we introduce a class of integral operators where the operators $\mathcal{L}$ and $\mathcal{W}$ belong. The integral operator space Ξ is a set of singular integral operators that satisfy the following.

(A1)

$$(3.4)\qquad (\mathcal{M}\phi)(x) = \int_0^1 M(x,y)\phi(y)\,dy,$$

where the kernel $M(x,y)$ is a 1-periodic function in $[0,1]\times[0,1]$ except at four corners of $[0,1]\times[0,1]$.

(A2) Let

$$(3.5)\quad \mathcal{M}_{i,j,k,l}\phi(x) = \int_0^1 [x(1-x)]^k[y(1-y)]^l|D_x^i D_y^j M(x,y)|\phi(y)\,dy$$

with $k+l = i+j$ where $k,l \in \mathbf{R}^+$ and $i,j \in \mathbf{Z}^+$. Then $\mathcal{M}_{i,j,k,l}$ is a bounded operator on H^0.

Here $D_x^i := \alpha^i\cdot/\partial x^i$ and $D_y^i := \partial^i\cdot/\partial y^i$.

It is not difficult to see that the operators $\mathcal{L}$ and $\mathcal{W}$ in (2.3) satisfy (A1) and (A2), and we refer to the Appendix.

Theorem 3.2. *Assume that the integral operator $\mathcal{M}$ satisfies the assumptions* (A1) *and* (A2). *For an arbitrary $\delta > 0$, there is an integer $i^* > 0$ independent of h such that*

(1)

$$(3.6)\qquad ||(I-P_h)\mathcal{M}T_{i^*h}\phi||_0 \leq \delta||\phi||_0, \qquad \phi \in H^0,$$

(2)

$$(3.7)\qquad ||(I-P_h)\mathcal{M}_N T_{i^*h}\phi||_0 \leq \delta||\phi||_0, \qquad \phi \in T^h,$$

(3)

$$||P_h[S_{i^*h} \cdot \phi \cdot (\mathcal{W}1 - \mathcal{W}_N 1)]||_0 \le \delta ||\phi||_0, \qquad \phi \in T^h. \tag{3.8}$$

Proof of (1).

$$||(\mathcal{M}T_{i^*h} - P_h \mathcal{M}T_{i^*h})\phi||_0 \le Ch^p ||D^p \mathcal{M}T_{i^*h}\phi||_0$$

for some integer $p \ge 1$. Now

$$\begin{aligned} ||D^p(\mathcal{M}T_{i^*h}\phi)||_0 &= \left\| \int_{i^*h}^{1-i^*h} D_x^p M(x,y)\phi(y)\,dy \right\|_0 \\ &\le C(i^*h)^{-p} \left\| \int_{i^*h}^{1-i^*h} [y(1-y)]^p |D_x^p M(x,y)\phi(y)|\,dy \right\|_0 \\ &\le \frac{C}{(i^*h)^p} ||\phi||_0. \end{aligned}$$

Then

$$\begin{aligned} Ch^p ||D^p(\mathcal{M}T_{i^*h}\phi)||_0 &\le \frac{C}{(i^*)^p} ||\phi||_0 \\ &\le \delta ||\phi||_0 \end{aligned}$$

for sufficiently large i^*. □

Proof of (2). For an integer $p \ge 1$,

$$||(I - P_h)\mathcal{M}_N T_{i^*h}\phi||_0 \le Ch^p ||D^p \mathcal{M}_N T_{i^*h}\phi||_0.$$

By the error formula for the rectangular rule, and using the inverse estimate on T^h, that is, $h^s ||\phi||_s \le ||\phi||_0$ for $\phi \in T^h$,

$$\begin{aligned} &||D^p \mathcal{M}T_{i^*h}\phi - D^p \mathcal{M}_N T_{i^*h}\phi||_0 \\ &\qquad \le Ch \left\| \int_{i^*h}^{1-i^*h} \left| \frac{\partial^{p+1}}{\partial x^p \partial y}[M(x,y)\phi(y)] \right| dy \right\|_0 \\ &\qquad \le Ch \sum_{l=0}^{1} \left\| \int_{i^*h}^{1-i^*h} \left| \frac{\partial^{p+l} M(x,y)}{\partial x^p \partial y^l} \phi(y)^{(1-l)} \right| dy \right\|_0 \\ &\qquad \le Ch \sum_{l=0}^{1} \frac{1}{(i^*h)^{p+l}} ||\phi^{(1-l)}||_0 \\ &\qquad \le C \sum_{l=0}^{1} \frac{h^l}{(i^*h)^{p+l}} ||\phi||_0. \end{aligned}$$

The proof of (1) and the above inequality give us (2) immediately for all sufficiently large i^*. □

Proof of (3). For an integer $s \geq 1$,

$$\begin{aligned}
&||S_{i^*h} \cdot \phi \cdot (\mathcal{W}1 - \mathcal{W}_N 1) - P_h[S_{i^*h} \cdot \phi \cdot (\mathcal{W}1 - \mathcal{W}_N 1)]||_0 \\
&\qquad \leq Ch^s \sum_{j=0}^{s} ||(S_{i^*h}\phi)^{(j)} \omega_{s-j+1}^{-1} \omega_{s-j+1} (\mathcal{W}1 - \mathcal{W}_N 1)^{(s-j)}||_0 \\
&\qquad \leq Ch^s \sum_{j=0}^{s} ||(S_{i^*h}\phi)^{(j)} \omega_{s-j+1}^{-1}||_0 ||\omega_{s-j+1} (\mathcal{W}1 - \mathcal{W}_N 1)^{(s-j)}||_\infty,
\end{aligned}$$

where $\omega_j(x) = [x(1-x)]^j$. Now, by the rectangular rule error formula,

$$\begin{aligned}
|\omega_{j+1}(\mathcal{W}1 - \mathcal{W}_N 1)^{(j)}| &\leq Ch \int_0^1 \left| [x(1-x)]^{j+1} \frac{\partial^{j+1} W(x,y)}{\partial x^j \partial y} \right| dy \\
&\leq Ch.
\end{aligned}$$

Using the inverse estimate on T^h and the property of S_{i^*h},

$$||(S_{i^*h}\phi)^{(j)} \omega_{s-j+1}^{-1}||_0 \leq \frac{C}{(i^*)^{s-j+1} h^{s+1}} ||\phi||_0.$$

Then

$$\begin{aligned}
||S_{i^*h} \cdot \phi \cdot (\mathcal{W}1 - \mathcal{W}_N 1) &- P_h[S_{i^*h} \cdot \phi \cdot (\mathcal{W}1 - \mathcal{W}_N 1)]||_0 \\
&\leq \sum_{j=0}^{s} C \frac{h^{s+1}}{(i^*)^{s-j+1} h^{s+1}} ||\phi||_0 \\
&\leq \frac{\delta}{2} ||\phi||_0
\end{aligned}$$

for all sufficiently large i^*.

To complete the proof, we show that

$$||S_{i^*h} \cdot \phi \cdot (\mathcal{W}1 - \mathcal{W}_N 1)||_0 \leq \frac{\delta}{2} ||\phi||_0.$$

For some integer $s \geq 1$,

$$\begin{aligned}||S_{i^*h} \cdot \phi \cdot (\mathcal{W}1 - \mathcal{W}_N 1)||_0 &\leq ||\phi||_0 S_{i^*h} \cdot w_s^{-1}||_\infty ||w_s \cdot (\mathcal{W}1 - \mathcal{W}_N 1)||_\infty \\ &\leq C\frac{h^s}{(i^*h)^s}||\phi||_0.\end{aligned}$$

Now (3) is immediate. □

We introduce theorems that trace the collectively compact operator theory [**1, 3, 11**].

Theorem 3.3. *Under the same assumption as in Theorem* 3.2, *for an arbitrary* $\delta > 0$ *there is an integer* $i^* > 0$ *independent of* h *such that*

(1)

$$||(\mathcal{M}T_{i^*h} - \mathcal{M}_N T_{i^*h})\mathcal{M}T_{i^*h}\phi||_0 \leq \delta||\phi||_0, \qquad \phi \in H^0 \tag{3.9}$$

(2)

$$||(\mathcal{M}T_{i^*h} - \mathcal{M}_N T_{i^*h})\mathcal{M}_N T_{i^*h}\phi||_0 \leq \delta||\phi||_0, \qquad \phi \in T^h. \tag{3.10}$$

Proof of (1). By the error formula for the rectangular rule,

$$\begin{aligned}&(\mathcal{M}T_{i^*h} - \mathcal{M}_N T_{i^*h})\mathcal{M}T_{i^*h}\phi \\ &= \int_{i^*h}^{1-i^*h} M(x,y)\psi(y)\,dy - h\sum_{k=i^*}^{N-i^*-1} M(x,ky)\psi(kh), \qquad \psi = \mathcal{M}T_{i^*h}\phi \\ &= Ch\int_{i^*h}^{1-i^*h} B\left(\frac{y}{h}\right)\frac{\partial}{\partial y}[M(x,y)\mathcal{M}T_{i^*h}\phi(y)]\,dy \\ &= Ch\sum_{j=0}^{1}\int_{i^*h}^{1-i^*h} B\left(\frac{y}{h}\right)\frac{\partial^j M(x,y)}{\partial y^j}\int_{i^*h}^{1-i^*h}\frac{\partial^{1-j}M(y,z)}{\partial y^{1-j}}\phi(z)\,dz\,dy,\end{aligned}$$

where B is a locally defined polynomial (Peano kernel). Then

$$\begin{aligned}
&||(\mathcal{M}T_{i^*h} - \mathcal{M}_N T_{i^*h})\mathcal{M}T_{i^*h}\phi||_0 \\
&\qquad \le \frac{Ch}{i^*h}\sum_{j=0}^{1}\bigg\| \int_0^1 [y(1-y)]^j \left|\frac{\partial^j M(x,y)}{\partial y^j}\right| \\
&\qquad\quad \cdot \left| \int_0^1 [y(1-y)]^{1-j} \frac{\partial^{1-j} M(y,z)}{\partial y^{1-j}}\phi(z)\,dz\right| dy \bigg\|_0 \\
&\qquad \le \frac{C}{i^*}||\phi||_0.
\end{aligned}$$

Proof of (2). By the same way as the above,

$$\begin{aligned}
&(\mathcal{M}T_{i^*h} - \mathcal{M}_N T_{i^*h})\mathcal{M}_N T_{i^*h}\phi \\
&\qquad = Ch\sum_{j=0}^{1}\int_{i^*h}^{1-i^*h} B\left(\frac{y}{h}\right)\frac{\partial^j M(x,y)}{\partial y^j} \\
&\qquad\quad \cdot \left[h\sum_{k=i^*}^{N-i^*-1}\frac{\partial^{1-j} M(y,kh)}{\partial y^{1-j}}\phi(kh)\,dy\right]
\end{aligned}$$

Now

$$\begin{aligned}
&h\sum_{k=i^*}^{N-i^*-1}\frac{\partial^{1-j} M(y,kh)}{\partial y^{1-j}}\phi(kh) \\
&\qquad = \int_{i^*h}^{1-i^*h}\frac{\partial^{1-j} M(y,z)}{\partial y^{1-j}}\phi(z)\,dz \\
&\qquad\quad - h\int_{i^*h}^{1-i^*h} B\left(\frac{y}{h}\right)\frac{\partial^{2-j}[M(y,z)\phi(z)]}{\partial y^{1-j}\partial z}\,dz \\
&\qquad = \int_{i^*h}^{1-i^*h}\frac{\partial^{1-j} M(y,z)}{\partial y^{1-j}}\phi(z)\,dz \\
&\qquad\quad - h\int_{i^*h}^{1-i^*h} B\left(\frac{y}{h}\right)\frac{\partial^{2-j} M(y,z)}{\partial y^{1-j}\partial z}\phi(z)\,dz \\
&\qquad\quad - h\int_{i^*h}^{1-i^*h} B\left(\frac{y}{h}\right)\frac{\partial^{1-j} M(y,z)}{\partial y^{1-j}}\phi'(z)\,dz.
\end{aligned}$$

By a similar way to the above proof with the inverse estimate on T^h, we have the proof of (2). $\square$

As a result of Theorem 3.2 with the stability estimate (3.2),

$$aI + b\mathcal{H} + bP_h\mathcal{L}T_{i^*h} : H^0 \longrightarrow H^0 \tag{3.11}$$

is invertible for all sufficiently small $h > 0$ and sufficiently large i^* fixed, and the inverse is uniformly bounded on h.

In the next corollary, we will show one of our main results, the stability of the operator $\{aI + b\mathcal{H} + bP_h\mathcal{L}T_{i^*h} + bP_h(S_{i^*h}(\cdot) \cdot (\mathcal{W}1 - \mathcal{W}_N1))\}$ on T^h. In the proof, we need Theorems 3.2 and 3.3 with the stability estimate (3.11) and the perturbation theorem in [**3, 10**].

Corollary 3.4. *For a sufficiently large i^*,*

$$aI + b\mathcal{H} + bP_h\mathcal{L}_NT_{i^*h} + bP_h[S_{i^*h}(\cdot) \cdot (\mathcal{W}1 - \mathcal{W}_N1)] : T^h \to T^h \tag{3.12}$$

is invertible, and the inverse is uniformly bounded on h.

Proof. First we will prove that the operator

$$aI + b\mathcal{H} + bP_h\mathcal{L}_NT_{i^*h} : T^h \longrightarrow T^h$$

is invertible, and the inverse is uniformly bounded on h. Then by the property (3) of Theorem 3.2,

$$||P_h(S_{i^*h} \cdot \phi \cdot (\mathcal{W}1 - \mathcal{W}_N1))||_0 \le \delta||\phi||_0$$

for arbitrary $\delta > 0$, and we have the theorem immediately.

Let

$$G := aI + b\mathcal{H}.$$

Then G is invertible (see Section 5), and the inverse is bounded. Then

$$aI + b\mathcal{H} + bP_h\mathcal{L}_NT_{i^*h} = G(I + bG^{-1}P_h\mathcal{L}_NT_{i^*h}),.$$

Now we show that $I + bG^{-1}P_h\mathcal{L}_NT_{i^*h}$ is invertible, and the inverse is uniformly bounded for all sufficiently small h.

From (3.11), we see that $(I + bG^{-1}P_h\mathcal{L}T_{i^*h})$ is invertible, and the inverse is uniformly bounded. Define

$$B_h := I - b(I + bG^{-1}P_h\mathcal{L}T_{i^*h})^{-1}G^{-1}P_h\mathcal{L}_N T_{i^*h}.$$

Then simple calculation shows that

$$B_h(I + bG^{-1}P_h\mathcal{L}_N T_{i^*h}) = I - S_h, \tag{3.13}$$

where

$$S_h := b^2(I + G^{-1}P_h\mathcal{L}T_{i^*h})^{-1}(G^{-1}P_h\mathcal{L}_N T_{i^*h} - G^{-1}P_h\mathcal{L}T_{i^*h})G^{-1}P_h\mathcal{L}_N T_{i^*h}.$$

Note that the operator $G^{-1}\mathcal{L}$ also satisfies (A1) and (A2). Using Theorem 3.3,

$$\begin{aligned}||S_h\phi||_0 &\le C||(G^{-1}\mathcal{L}_h T_{i^*h} - G^{-1}LT_{i^*h})P_h G^{-1}\mathcal{L}_N T_{i^*h}\phi||_0 \\ &\le C(||(G^{-1}\mathcal{L}_h T_{i^*h} - G^{-1}\mathcal{L}T_{i^*h})G^{-1}\mathcal{L}_N T_{i^*h}\phi||_0 \\ &\quad + ||(G^{-1}\mathcal{L}_N T_{i^*h} - G^{-1}\mathcal{L}T_{i^*h})(I - P_h)G^{-1}\mathcal{L}_N T_{i^*h}\phi||_0) \\ &\le \delta||\phi||_0\end{aligned}$$

for arbitrary $\delta > 0$ as $h \to 0$. Since the righthand side of (3.13) is invertible for $\delta < 1$ on T^h, therefore $(I + P_h\mathcal{L}_N T_{i^*h})$ is invertible on T^h and the inverse is bounded by

$$||(I + P_h G^{-1}\mathcal{L}_N T_{i^*h})^{-1}||_0 \le C\frac{||B_h||_0}{1 - ||S_h||_0}. \qquad \square$$

4. Convergence analysis. The next theorem is the main result of this paper. Let us introduce a space of functions. For any $p \in \mathbf{Z}^+$,

$$S^p := \{\psi : \psi^{(j)}(x) = [x(1-x)]^{p-j}\psi_j(x),\ \psi_j \in H^0\} \tag{4.1}$$

and the norm of S^p is defined as

$$||\psi||_{S^p} := \sum_{j=0}^{p} ||\psi_j||_0. \tag{4.2}$$

It is trivially identified that $S^p \subset H^p$, and for $f \in S^p$, $||f||_p \leq ||f||_{S^p}$.

Theorem 4.1. *Suppose* ψ *is the solution of equation* (2.4) *and* $\psi \in S^p$ *for* $p \geq 1$. *Then* (2.9) *is uniquely solvable, and we have an error estimate*

$$||\psi - \psi_h||_0 \leq Ch^p ||\psi||_{S^p}.$$

Proof. The unique solvability is obtained from Corollary 3.4. To look at the error estimate, write (2.9) using (2.4) for substitution of g.

$$[aI + b\mathcal{H} + bP_h\mathcal{L}_N T_{i^*h}]\psi_h + bP_h[S_{i^*h} \cdot \psi_h \cdot (\mathcal{W}1 - \mathcal{W}_N 1)] \\ = [aP_h + bP_h\mathcal{H} + bP_h\mathcal{L}]\psi.$$

Let ψ_N be a certain element in T^h. The above equation can be written as

$$P_h\mathcal{C}^q_{\lambda,h,i^*h}(\psi_h - \psi_N) = aP_h(\psi - \psi_N) + bP_h\mathcal{H}(\psi - \psi_N) \\ + bP_h[\mathcal{L}\psi - \mathcal{L}_N T_{i^*h}\psi_N] \\ - bP_h[S_{i^*h} \cdot \psi_N \cdot (\mathcal{W}1 - \mathcal{W}_N 1)].$$

Then, even though P_h is not bounded, roughly speaking,

$$||(\psi_h - \psi_N)||_0 \leq C(||(\psi - \psi_N)||_0 + ||\mathcal{L}\psi - \mathcal{L}_N T_{i^*h}\psi_N||_0 \\ + ||S_{i^*h} \cdot \psi_N \cdot (\mathcal{W}1 - \mathcal{W}_N 1)||_0) \\ = Q_1 + Q_2 + Q_3.$$

Let $\psi_N = P_{h,0}\psi$ (see Section 2 for the definition of $P_{h,0}$).

For Q_1, by Lemma 2.1,

$$||\psi - \psi_N||_0 \leq Ch^p ||\psi||_p, \qquad \psi \in H^p.$$

For Q_2, since $\psi_N(x) = \psi(x)$ at node points, by the nature of the quadrature method we have

$$\mathcal{L}\psi - \mathcal{L}_N T_{i^*h}\psi_N = \mathcal{L}\psi - \mathcal{L}_N T_{i^*h}\psi.$$

Then

$$||\mathcal{L}\psi - \mathcal{L}_N T_{i^*h}\psi||_0 \leq ||(\mathcal{L} - \mathcal{L}_N)\psi||_0 + ||(\mathcal{L}_N - \mathcal{L})(\psi - T_{i^*h}\psi)||_0 \\ + ||\mathcal{L}(\psi - T_{i^*h}\psi)||_0 \\ \leq P_1 + P_2 + C||\psi - T_{i^*h}\psi||_0 \\ \leq P_1 + P_2 + Ch^p ||\psi||_{S^p}.$$

Using the Euler-Maclaurin formula [**12**] and $\psi \in S^p$,

$$\begin{aligned}
P_1 = ||(\mathcal{L}-\mathcal{L}_N)\psi||_0 &\le Ch^p \bigg\| \int_0^1 |D_y^p[L(x,y)\psi(y)]|\,dy \bigg\|_0 \\
&\le Ch^p \sum_{l=0}^{p} \bigg\| \int_0^1 |[D_y^l L(x,y)]\psi^{(p-l)}(y)|\,dy \bigg\|_0 \\
&\le Ch^p \sum_{l=0}^{p} \bigg\| \int_0^1 |[D_y^l L(x,y)][y(1-y)]^l \psi_{p-l}(y)|\,dy \bigg\|_0 \\
&\le Ch^p ||\psi||_{S^p}.
\end{aligned}$$

$$\begin{aligned}
P_2 &= ||(\mathcal{L}-\mathcal{L}_N)(\psi - T_{i^*h}\psi)||_0 \\
&\le \bigg\| \int_0^{i^*h} L(x,y)\psi(y)\,dy - \sum_{k=0}^{i^*-1} L(x,kh)\psi(kh) \bigg\|_0 \\
&\quad + \bigg\| \int_{1-i^*h}^1 L(x,y)\psi(y)\,dy - \sum_{k=N-i^*}^{N-1} L(x,kh)\psi(kh) \bigg\|_0 .
\end{aligned}$$

Let us look only at

$$\bigg\| \int_0^{i^*h} L(x,y)\psi(y)\,dy - \sum_{k=0}^{i^*-1} L(x,kh)\psi(kh) \bigg\|_0 .$$

$$\begin{aligned}
&\bigg\| \int_0^{i^*h} L(x,y)\psi(y)\,dy - \sum_{k=0}^{i^*-1} L(x,kh)\psi(kh) \bigg\|_0 \\
&\qquad \le Ch \bigg\| \int_0^{i^*h} |D_y[L(x,y)\psi(y)]|\,dy \bigg\|_0 \\
&\qquad \le Ch \sum_{l=0}^{1} \bigg\| \int_0^{i^*h} |D_y^l L(x,y)\psi^{(1-l)}(y)|\,dy \bigg\|_0 \\
&\qquad \le Ch \sum_{l=0}^{1} \bigg\| \int_0^{i^*h} |D_y^l L(x,y) y^{p-1+l}\psi_{1-l}(y)|\,dy \bigg\|_0 \\
&\qquad \le C \sum_{l=0}^{1} h(i^*h)^{p-1} ||\psi_{1-l}||_0 \\
&\qquad \le Ch^p ||\psi||_{S^p}.
\end{aligned}$$

Then

$$Q_2 \le Ch^p||\psi||_{S^p}.$$

For Q_3, with $w_p(x) = [x(1-x)]^p$,

$$\begin{aligned} ||S_{i^*h} \cdot \psi_N \cdot (\mathcal{W}(1) - \mathcal{W}_N(1))]||_0 & \\ &\le ||S_{i^*h}\psi_N w_p^{-1} \cdot w_p(\mathcal{W}(1) - \mathcal{W}_N(1))||_0 \\ &\le C||w_p(\mathcal{W}(1) - \mathcal{W}_N(1))||_\infty ||S_{i^*h} w_p^{-1}\psi_N||_0 \\ &\le Ch^p||\psi||_{S^p}, \end{aligned}$$

since

$$||w_p(\mathcal{W}1 - \mathcal{W}_N 1)||_\infty \le Ch^p$$

and

$$\begin{aligned} ||S_{i^*h} w_p^{-1}\psi_N||_0 &\le ||S_{i^*h} w_p^{-1}(\psi_N - \psi)||_0 \\ &\quad + ||S_{i^*h} w_p^{-1}\psi||_0 \\ &\le C\frac{h^p}{(i^*h)^p}||\psi||_p + C||\psi||_{S^p} \\ &\le C||\psi||_{S^p}. \end{aligned}$$

Adding Q_1, Q_2 and Q_3,

$$||\psi_h - \psi_N||_0 \le Ch^p||\psi||_{S^p}.$$

By the triangle inequality,

$$\begin{aligned} ||\psi_h - \psi||_0 &\le ||\psi_h - \psi_N||_0 + ||\psi_N - \psi||_0 \\ &\le Ch^p||\psi||_{S^p}. \qquad \square \end{aligned}$$

Remark 1. For $p \in \mathbf{Z}^+$ and $0 \le \alpha < 1$, define

$$S^{p,\alpha} := \{\psi : \psi^{(j)}(x) = [x(1-x)]^{p-j}\psi_j(x),\ \psi_j \in H^\alpha\}. \tag{4.3}$$

Then using the generalized Euler-Maclaurin formula [**12**], we may be able to extend the convergence result in Theorem 4.1 as follows:

$$||\psi - \psi_h||_0 \le Ch^{p+\alpha}. \tag{4.4}$$

5. Proof of Theorem 3.1. Let us start localization of the operator $\mathcal{C}_\lambda^q$. Define smooth cutoff functions, χ and χ' such that supp $(\chi) \subset [0, \varepsilon]$ and $\chi = 1$ on $[0, \varepsilon_1]$ where $0 < \varepsilon_1 < \varepsilon < 1/2$ and $\chi'(x) = \chi(1-x)$. It is easy to see that

$$\begin{aligned} \frac{1}{\pi}\frac{1}{y-x} - \cot \pi(y-x) &= \frac{1}{\pi} K_1(x,y) + E_1(x,y) \\ \frac{1}{\pi}\frac{\kappa(y)}{\kappa(x)}\frac{w'(x)}{w(y)-w(x)} - \frac{1}{\pi}\frac{1}{y-x} &= \frac{1}{\pi}K_2(x,y) + E_2(x,y) \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} K_1(x,y) &= \chi(x)\chi'(y)\frac{1}{(1-y)+x} - \chi'(x)\chi(y)\frac{1}{y+(1-x)} \\ K_2(x,y) &= \chi(x)\chi(y)\left[\frac{qx^{q-\lambda-1}y^\lambda}{y^q - x^q} - \frac{1}{y-x}\right] \\ &\quad + \chi'(x)\chi'(y)\left[- \frac{q(1-x)^{q-\lambda-1}(1-y)^\lambda}{(1-y)^q - (1-x)^q} \right. \\ &\qquad\qquad \left. + \frac{1}{(1-y)-(1-x)}\right], \end{aligned} \tag{5.2}$$

and E_1 and E_2 are smooth. Then

$$\begin{aligned} \frac{1}{\pi}\frac{\kappa(y)}{\kappa(x)}\frac{w'(x)}{w(y)-w(x)} &= \cot(\pi(y-x)) + \frac{1}{\pi}K_1(x,y) \\ &\quad + \frac{1}{\pi}K_2(x,y) + E_1(x,y) + E_2(x,y), \end{aligned}$$

and (2.4) can be written as follows:

$$\mathcal{C}_\lambda^q \phi = (aI + b\mathcal{H} + b\mathcal{K} + \mathcal{E})\phi = g, \tag{5.3}$$

where

$$\begin{aligned} \mathcal{H}\phi(x) &= \text{p.v.} \int_0^1 \cot(\pi(y-x))\phi(y)\,dy \\ \mathcal{K}\phi(x) &= \frac{1}{\pi}\int_0^1 (K_1(x,y) + K_2(x,y))\phi(y)\,dy \\ \mathcal{E}\phi(x) &= \int_0^1 (E_1(x,y) + E_2(x,y))\phi(y)\,dy. \end{aligned}$$

Here $\mathcal{H}$ is the Hilbert transform and the operator $\mathcal{E}$ is compact. From here on, $\mathcal{E}$ represents a generic compact operator, and it can be a different compact operator for each occasion.

Note that

$$(aI + b\mathcal{H})(aI - b\mathcal{H}) = I - b^2\mathcal{J}$$

since $a^2 + b^2 = 1$ and $\mathcal{H}\mathcal{H} = -I + \mathcal{J}$, where

$$\mathcal{J}\phi = \int_0^1 \phi(y)\, dy.$$

Then $(aI + b\mathcal{H})^{-1}$ exists, and

$$\begin{aligned}(aI + b\mathcal{H})^{-1} &= (aI - b\mathcal{H})(I - b^2\mathcal{J})^{-1}\\ &= (aI - b\mathcal{H}) + \hat{\mathcal{E}},\\ \hat{\mathcal{E}} &= (aI - b\mathcal{H})(b^2\mathcal{J})(I - b^2\mathcal{J})^{-1}.\end{aligned}$$

We multiply (5.3) by $(aI + b\mathcal{H})^{-1}$ to obtain

$$(I + \mathcal{B} + \tilde{\mathcal{E}})\phi = \tilde{g}, \tag{5.4}$$

where

$$\begin{aligned}\mathcal{B} &:= b(aI - b\mathcal{H})\mathcal{K}\\ \tilde{\mathcal{E}} &:= b\hat{\mathcal{E}}\mathcal{K} + (aI - b\mathcal{H})\mathcal{E} + \hat{\mathcal{E}}\mathcal{E}\\ \tilde{g} &:= (aI + b\mathcal{H})^{-1}g.\end{aligned} \tag{5.5}$$

Now we seek the local representation of the operator $\mathcal{B}$ in terms of Mellin operators. Firstly, we introduce a set of Mellin convolution operators that resemble the local behavior of the operator $\mathcal{K}$. Define

$$\begin{aligned}\mathcal{G}\phi(x) &= \frac{1}{\pi}\text{p.v.}\int_0^\infty G\left(\frac{x}{y}\right)\frac{\phi(y)}{y}\, dy\\ \mathcal{G}_\lambda^q\phi(x) &= \frac{1}{\pi}\text{p.v.}\int_0^\infty G_\lambda^q\left(\frac{x}{y}\right)\frac{\phi(y)}{y}\, dy\\ \mathcal{R}\phi(x) &= \frac{1}{\pi}\text{p.v.}\int_0^\infty R\left(\frac{x}{y}\right)\frac{\phi(y)}{y}\, dy\end{aligned} \tag{5.6}$$

where

$$G(t) = \frac{1}{1-t}, \qquad G_\lambda^q(t) = \frac{qt^{q-\lambda-1}}{1-t^q}, \qquad R(t) = \frac{1}{1+t}.$$

The operator $\mathcal{G}$ is the well-known Cauchy singular integral operator on $[0,\infty]$. Then, in view of (5.1), (5.2) and (5.3), $\mathcal{K}$ can be expressed in terms of Mellin's convolution operators defined above.

$$\begin{aligned} \mathcal{K} &= \chi[\mathcal{G}_\lambda^q - \mathcal{G}]\chi + \chi'[-(\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}})]\chi' + \chi\tilde{\mathcal{R}}\chi' - \chi'\tilde{\tilde{\mathcal{R}}}\chi \\ &= \Pi \begin{bmatrix} [\mathcal{G}_\lambda^q - \mathcal{G}] & \tilde{\mathcal{R}} \\ -\tilde{\tilde{\mathcal{R}}} & -[\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}}] \end{bmatrix} \Pi^T \\ &= \Pi(\overline{\mathcal{G}}_\lambda^q - \overline{\mathcal{H}})\Pi^T, \end{aligned} \tag{5.7}$$

where the matrices of Mellin operators, $\overline{\mathcal{H}}$ and $\overline{\mathcal{G}}_\lambda^q$ are defined as

$$\overline{\mathcal{H}} := \begin{bmatrix} \mathcal{G} & -\tilde{\mathcal{R}} \\ \tilde{\tilde{\mathcal{R}}} & -\tilde{\mathcal{G}} \end{bmatrix}, \qquad \overline{\mathcal{G}}_\lambda^q = \begin{bmatrix} \mathcal{G}_\lambda^q & 0 \\ 0 & -\tilde{\mathcal{G}}_\lambda^q \end{bmatrix} \tag{5.8}$$

and

$$\Pi = (\chi, \chi').$$

Here $\tilde{\mathcal{G}}_\lambda^q$ and $\tilde{\mathcal{G}}$ are obtained from $\mathcal{G}_\lambda^q$ and $\mathcal{G}$ by the change of variables $x \to 1-x$ and $y \to 1-y$, $\tilde{\mathcal{R}}$ from $\mathcal{R}$ by the change of variables $x \to x$, $y \to 1-y$ and $\tilde{\tilde{\mathcal{R}}}$ from $\mathcal{R}$ by the change of variables $x \to 1-x$ and $y \to y$.

In a similar way (see Appendix B),

$$\begin{aligned} \mathcal{B} &= b(aI - b\mathcal{H})\mathcal{K} \\ &= \Pi b \begin{bmatrix} aI - b\mathcal{G} & b\tilde{\mathcal{R}} \\ -b\tilde{\tilde{R}} & aI + b\tilde{\mathcal{G}} \end{bmatrix} \begin{bmatrix} [\mathcal{G}_\lambda^q - \mathcal{G}] & \tilde{\mathcal{R}} \\ -\tilde{\tilde{\mathcal{R}}} & -[\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}}] \end{bmatrix} \Pi + \mathcal{E} \\ &= \Pi b(aI - b\overline{\mathcal{H}})(\overline{\mathcal{G}}_\lambda^q - \overline{\mathcal{H}})\Pi \\ &= \Pi\overline{\mathcal{B}}\Pi + \mathcal{E}, \end{aligned} \tag{5.9}$$

where

$$\overline{\mathcal{B}} = b(aI - b\overline{\mathcal{H}})(\overline{\mathcal{G}}_\lambda^q - \overline{\mathcal{H}}). \tag{5.10}$$

Using (5.9), write (5.4) as

$$(I+\mathcal{B}+\tilde{\mathcal{E}})\phi=(I+\Pi\overline{\mathcal{B}}\Pi+\tilde{\tilde{\mathcal{E}}})\phi=\tilde{g}. \tag{5.11}$$

Lemma 5.1. *Assume* $|a|-|b|\cot((2\lambda+1)\pi/(2q))\neq 0$ *that implies*

$$q\neq\frac{(2\lambda+1)\pi}{2\tan^{-1}(|b|/|a|)}.$$

Then

$$(I+\overline{\mathcal{B}}):L_2[0,\infty)\oplus L_2[0,\infty)\to L_2[0,\infty)\oplus L_2[0,\infty)$$

is an invertible Fredholm operator.

Proof. We prove the lemma by showing that the symbol of $(I+\overline{\mathcal{B}})$ is nonsingular. First, note that the symbols $\overline{\mathcal{H}},\overline{\mathcal{G}}^q_\lambda$ are (see Appendix A)

$$\sigma(\overline{\mathcal{H}})=\begin{bmatrix}\cot(\pi z) & -1/\sin(\pi z)\\ 1/\sin(\pi z) & -\cot(\pi z)\end{bmatrix},$$

and

$$\sigma(\overline{\mathcal{G}^q_\lambda})=\begin{bmatrix}\cot(\pi(z-\lambda-1)/q) & 0\\ 0 & -\cot(\pi(z-\lambda-1)/q)\end{bmatrix}$$

with $z=1/2+iy$, $y\in\mathbf{R}$. Using the fact that $a^2+b^2=1$ and $\sigma(\overline{\mathcal{H}})^2=\mathrm{Diag}\,(-1,-1)$, we have

$$\begin{aligned}\sigma(I+\overline{\mathcal{B}})&=1+b(a-b\sigma(\overline{\mathcal{H}}))(\sigma(\overline{\mathcal{G}}^q_\lambda)-\sigma(\overline{\mathcal{H}}))\\&=(aI-b\sigma(\overline{\mathcal{H}}))(aI+b\sigma(\overline{\mathcal{G}}^q_\lambda)).\end{aligned}$$

It is clear that

$$\det[(aI-b\sigma(\overline{\mathcal{H}}))]\neq 0$$

since

$$(aI-b\sigma(\overline{\mathcal{H}}))(aI+b\sigma(\overline{\mathcal{H}}))=I.$$

Now let us look at $(aI + b\sigma(\overline{\mathcal{G}}_\lambda^q))$. By a simple calculation,

$$(aI + b\sigma(\overline{\mathcal{G}}_\lambda^q)) = \mathrm{Diag}\,(a + b\cot(\pi(z-\lambda-1)/q), \\ a - b\cot(\pi(z-\lambda-1)/q)).$$

Because

$$\begin{aligned}&\cot(-(2\lambda+1)\pi/(2q) + i\pi y/q)\\ &\quad= \frac{-\sin((2\lambda+1)\pi/(2q))\cos((2\lambda+1)\pi/(2q)) - i\sinh(\pi y/q)\cosh(\pi y/q)}{\cosh^2(\pi y/q) - \cos^2((2\lambda+1)\pi/(2q))}\\ &\quad= \omega(y) + i\mu(y), \qquad y \in \mathbf{R},\end{aligned}$$

we will show that $a \pm b\omega(y)$ and $\mu(y)$ are not zero simultaneously. Since $\mu(y) = 0$ only for $y = 0$, we must have $a \pm b\omega(0) \neq 0$ to have $\det[(aI + b\sigma(\overline{\mathcal{G}}_\lambda^q))] \neq 0$. Since

$$a \pm b\omega(0) = a \pm b\cot\left(\frac{(2\lambda+1)\pi}{2q}\right),$$

we have $a \pm b\omega(0) \neq 0$ from the assumption. □

Theorem 5.2. *Suppose* λ *and* q *satisfy* (1.10). *The operator,* $\mathcal{C}_\lambda^q : L_2[0,1] \to L_2[0,1]$ *is an invertible Fredholm operator of index* 0.

Proof. We prove the theorem in the following way.

Step 1. $\mathcal{C}_\lambda^q$ is a Fredholm operator from $L_2[0,1]$ to itself.

Step 2. $\mathcal{C}_\lambda^q$ is homotopic to $\mathcal{C}$ of (1.1); then $\mathrm{ind}\,(\mathcal{C}_\lambda^q) = \mathrm{ind}\,(\mathcal{C}) = 0$.

Step 3. $\mathrm{null}\,(\mathcal{C}_\lambda^q) = \{0\}$.

Step 1. If λ and q satisfy (1.10), we have

$$\lambda \le q < \frac{(2\lambda+1)\pi}{2\tan^{-1}(|b|/|a|)}.$$

By Lemma 5.1, $(I+\overline{\mathcal{B}})$ is invertible, and let $(I+\overline{\mathcal{F}}) = (I+\overline{\mathcal{B}})^{-1}$. Define

$$I + \mathcal{F} := I + \Pi\overline{\mathcal{F}}\Pi^T.$$

Simple calculation shows that $(I+\mathcal{F})$ is a left and right regularizer of $I+\mathcal{B}+\tilde{\mathcal{E}}$. Then $I+\mathcal{B}+\tilde{\mathcal{E}}$ is a Fredholm operator. For details, see [**13**]. Since

$$\mathcal{C}_\lambda^q = (aI+b\mathcal{H})(I+\mathcal{B}+\tilde{\mathcal{E}}),$$

and $(aI+b\mathcal{H})$ is invertible on $L_2[0,1]$, $\mathcal{C}_\lambda^q$ is a Fredholm operator on $L_2[0,1]$.

Step 2. To prove $\operatorname{ind}(\mathcal{C}_\lambda^q)=0$, we will show that $\mathcal{C}_\lambda^q$ is homotopic to $\mathcal{C}_0^1=\mathcal{C}$. Define a homotopy map $F: L_2[0,1]\times[0,1]\to L_2[0,1]$ as

$$F(\phi,t)=\mathcal{C}_{\lambda(t)}^{q(t)}\phi$$

where $\lambda(t)=\lambda t$, and $q(t)=1+(q-1)t$. F is continuous for each variable, and $F(\cdot,0)=\mathcal{C}$ and $F(\cdot,1)=\mathcal{C}_\lambda^q$. Moreover, $\mathcal{C}_{\lambda(t)}^{q(t)}$ is a Fredholm operator because

$$\begin{aligned}\lambda(t)\le q(t)=(1-t)+qt &< (1-t)+\frac{(2\lambda+1)\pi}{2\tan^{-1}(|b|/|a|)}t\\ &< \frac{(2\lambda t+1)\pi}{2\tan^{-1}(|b|/|a|)}+(1-t)\bigg(1-\frac{\pi}{2\tan^{-1}(|b|/|a|)}\bigg)\\ &\le \frac{(2\lambda(t)+1)\pi}{2\tan^{-1}(|b|/|a|)},\end{aligned}$$

and by following Step 1. By [**13**, Chapter 1] and (1.5) in Section 1, $\operatorname{ind}(\mathcal{C}_\lambda^q)=\operatorname{ind}(\mathcal{C})=0$.

Step 3. Let us prove $\operatorname{null}(\mathcal{C}_\lambda^q)=0$. Suppose $\psi\in L_2[0,1]$ is a solution of $C_\lambda^q\psi=0$. then, with w in (1.6)

$$\Psi(t)=[w^{-1}(t)(1-w^{-1}(t))]^\lambda\psi(w^{-1}(t))(w^{-1})'(t)$$

is a solution of $\mathcal{C}\Psi=0$. But, using the Jacobi polynomial expansion formula for the Cauchy singular integral equation [**8**, page 260], we can see that $\Psi(t)=Ct^\alpha(1-t)^\beta$ for some constant C where

$$(\alpha,\beta)=(-1+\gamma_0,-\gamma_0)\quad\text{or}\quad(\alpha,\beta)=(-\gamma_0,-1+\gamma_0),$$
$$\gamma_0=\tan^{-1}(|b|/|a|)/\pi.$$

Using that

$$w^{-1}(t) = [t(1-t)]^{1/q} v(t), \qquad (w^{-1})' = [t(1-t)]^{(1/q)-1} u(t)$$

with continuous functions v and u, we have

$$\Psi(t) = t^{\alpha}(1-t)^{\beta} \cdot t^{\lambda//q-1/2\cdot(q-1)/q-\alpha}(1-t)^{\lambda/q-1/2\cdot(q-1)/q-\beta} h(t),$$

where

$$h(t) = \psi(w^{-1}(t))\sqrt{(w^{-1})'(t)}\gamma(t) \in L_2[0,1]$$

and some continuous function γ. Then we must have

$$t^{\lambda/q-1/2\cdot(q-1)/q-\alpha}(1-t)^{\lambda/q-1/2\cdot(q-1)/q-\beta} h(t) = C,$$

or, equivalently,

$$h(t)/C = t^{-\lambda/q+1/2\cdot(q-1)/q+\alpha}(1-t)^{-\lambda/q+1/2\cdot(q-1)/q+\beta} \in L_2[0,1].$$

Without loss of generality, assume $(\alpha, \beta) = (-1+\gamma_0, -\gamma_0)$. Then, since $\gamma_0 < (2\lambda+1)/(2q)$ from (1.10),

$$-\frac{\lambda}{q} + \frac{1}{2}\frac{q-1}{q} + \alpha = -\frac{1}{2} - \frac{2\lambda+1}{2q} + \gamma_0 < -\frac{1}{2},$$

which contradict $h(t)/C \in L_2[0,1]$. Then $C = 0$ and so $\Psi = 0$. □

Conjecture 1. *Under the same assumption as in Theorem* 5.2, *the operator* $(I + \overline{\mathcal{B}})$ *is strongly elliptic, that is,*

$$Re\,\langle[\mathbf{M}(I+\overline{\mathcal{B}})]\Phi, \Phi)\rangle \geq C\langle\Phi, \Phi\rangle \tag{5.12}$$

for some constant 2×2 *matrix operator* $\mathbf{M}$, *depending on* a, b, q, λ *and some constant* $C > 0$, *where* $\Phi \in L_2[0,\infty) \times L_2[0,\infty)$ *and* $\langle\cdot,\cdot\rangle$ *represents the usual inner product such that*

$$\langle(f_1, f_2), (g_1, g_2)\rangle = \int_0^{\infty} (f_1\bar{g}_1 + f_2\bar{g}_2)\,dx.$$

The conjecture is partially justified in Remark 2. As a result of Conjecture 1, we have the Gårding inequality for the operator $I+\mathcal{B}+\tilde{\mathcal{E}}$, that is, with the operator $\mathcal{N}$ induced by M,

$$(\mathcal{N}(I+\mathcal{B}+\tilde{\mathcal{E}})\phi,\phi) \geq C(\phi,\phi) - (\mathcal{E}'\phi,\phi) \tag{5.13}$$

for some compact operator $\mathcal{E}'$ and constant $C>0$. Here $(\cdot,\cdot)$ is the usual inner product in $L_2[0,1]$.

The Gårding inequality with Theorem 5.2 implies that the Galerkin method, i.e., find $\phi \in \operatorname{Ker}(I-T_r) \subset L_2[0,1]$ for any $0 \leq r < 1/2$, such that

$$(I+\mathcal{B}+\tilde{\mathcal{E}})(\phi) = T_r f \tag{5.14}$$

where $f \in L_2[0,1]$ is uniquely solvable, and the stability of the Galerkin method is independent of r. Then $T_r(I+\mathcal{B}+\tilde{\mathcal{E}})T_r$ is invertible, and the inverse is uniformly bounded on r. Since

$$(I+(\mathcal{B}+\mathcal{E})T_r))_{L_2[0,1]} \equiv \begin{bmatrix} T_r(I+\mathcal{B}+\mathcal{E})T_r & 0 \\ (I-T_r)(\mathcal{B}+\mathcal{E})T_r & I \end{bmatrix}_{\operatorname{Ker}(1-T_r)\times \operatorname{Ker} T_r}, \tag{5.15}$$

and the inverse satisfies

$$(I+(\mathcal{B}+\mathcal{E})T_r))^{-1} \equiv \begin{bmatrix} [T_r(I+\mathcal{B}+\mathcal{E})T_r]^{-1} & 0 \\ -(I-T_r)(\mathcal{B}+\mathcal{E})T_r[T_r(I+\mathcal{B}+\mathcal{E})T_r)]^{-1} & I \end{bmatrix}, \tag{5.16}$$

we have

$$||I+(\mathcal{B}+\tilde{\mathcal{E}})T_r||_0 \geq C. \tag{5.17}$$

Because $(aI+b\mathcal{H}+b\mathcal{L}T_r) = (aI+b\mathcal{H})(I+(\mathcal{B}+\tilde{\mathcal{E}})T_r)$, the proof of Theorem 3.1 is complete.

Remark 2. Conjecture 1 is tested by numerical experiments. Here our numerical procedure is explained. By Parseval equality, and using an elementary matrix algebra,

$$\begin{aligned}
&\operatorname{Re}\langle \mathbf{M}[I+b(aIb\overline{\mathcal{H}})(\overline{\mathcal{G}}^q_\lambda-\overline{\mathcal{H}})]\Phi,\Phi\rangle \\
&\quad=\frac{1}{2\pi}\int_{\operatorname{Re} z=1/2}\operatorname{Re}\left(\{\mathbf{M}\sigma(I+b(aI-b\overline{\mathcal{H}})(\overline{\mathcal{G}}^q_\lambda-\overline{\mathcal{H}}))\}\sigma(\Phi),\overline{\sigma(\Phi)}\right)d|z| \\
&\quad=\frac{1}{2\pi}\int_{\operatorname{Re} z=1/2}\left(\frac{\mathbf{L}(z)+\mathbf{L}^*(z)}{2}\sigma(\Phi),\sigma(\Phi)\right)d|z|,
\end{aligned}$$

where

$$\mathbf{L}:=\mathbf{M}\sigma(I+b(aI-b\overline{\mathcal{H}})(\overline{\mathcal{G}}^q_\lambda-\overline{\mathcal{H}})),$$

$\mathbf{L}^*$ is a complex conjugate transpose of the matrix $\mathbf{L}$ and $(\cdot,\cdot)$ is a usual inner product in $\mathbf{C}^2$. We define an inequality between matrices.

$$\mathbf{A}>0 \text{ if and only if } (\mathbf{A}X,X)>0.$$

To prove the conjecture, we need to show that $\mathbf{L}>0$ for some constant matrix $\mathbf{M}$. Let

$$\begin{aligned}
\cot(\pi(z-\lambda-1)/q)&=w(y)+i\mu(y),\\
\cot(\pi z)=iu(y),&\quad 1/\sin(\pi z)=v(y).
\end{aligned}$$

Then $w(y)$ and $\mu(y)$ are as in the proof of Lemma 5.1, and

$$u(y)=-\frac{\sinh(\pi y)}{\cosh(\pi y)},\qquad v(y)=\frac{1}{\cosh(\pi y)}.$$

Simple calculation shows

$$\begin{aligned}
&[\sigma(I+b(aI-b\overline{\mathcal{H}})(\overline{\mathcal{G}}^q_\lambda-\overline{\mathcal{H}}))]\\
&=[(aI-b\sigma(\overline{\mathcal{H}}))(aI+b\sigma(\overline{\mathcal{G}}^q_\lambda))]\\
&=\begin{bmatrix} a(a+bw(y))+b^2u(y)\mu(y) & -bv(y)(a-bw(y))\\ bv(y)(a+bw(y)) & a(a-bw(y))+b^2u(y)\mu(y)\end{bmatrix}\\
&\quad+i\begin{bmatrix} ab\mu(y)-bu(y)(a+bw(y)) & b^2v(y)\mu(y)\\ b^2v(y)\mu(y) & -ab\mu(y)+bu(y)(a-bw(y))\end{bmatrix}.
\end{aligned}$$

Choose

$$\mathbf{M}=\begin{bmatrix} a & b\\ -b & a\end{bmatrix}.$$

Then

$$\mathbf{P}(y) := \frac{\mathbf{L}+\mathbf{L}^*}{2}$$
$$= \begin{bmatrix} (a^2+b^2v(y))(a+bw(y))+ab^2u(y)\mu(y) & -ab^2(1-v(y))w(y)-ib^3u(y)w(y) \\ -ab^2(1-v(y))w(y)+ib^3u(y)w(u) & (a^2+b^2v(y))(a-bw(y))+ab^2u(y)\mu(y) \end{bmatrix}.$$

The matrix $\mathbf{P}(y)$ is Hermitian. Therefore, it is real-diagonalizable, and eigenvalues are positive if $\det(\mathbf{P}(y)) > 0$ for all $y \in R$ because $\mathbf{P}(\pm\infty) = aI$. Numerically, we observed that $\det(\mathbf{P}(y)) > 0$ for the choices of (a, b) and (q, λ) at which the numerical results, Examples 1 and 2 in Section 6, are obtained.

6. Numerical experiments. We do numerical experiments for equation (1.1) with various choices of (a, b) and f. For each (a, b) and f, we experiment with the effect of various mesh grading.

Example 1. Let $(a, b) = (1/2, \sqrt{3}/2)$. Then by the formulas (1.2) and (1.3), (α, β) can be $(-2/3, -1/3)$, $(-2/3, 2/3)$, $(1/3, -1/3)$ or $(1/3, 2/3)$ for the index 1,0,0 (the natural airfoil solution), -1 problems respectively. Consider a boundary function $f(t) = 2t - 3/4$. Since f is the Jacobi polynomial of degree 1 with regard to the weight $t^{-1/3}(1-t)^{-2/3}$, it satisfies the consistency condition (1.4) with $(\alpha, \beta) = (1/3, 2/3)$. Then (1.1) is a CSIE of the index -1, and the unique solution ϕ of (1.1) will have regularity:

$$\phi(t) = t^{1/3}(1-t)^{2/3}h(t), \qquad h \in C[0,1], \quad h \in C^\infty(0,1).$$

After a mesh grading transformation of order q, ψ is a unique solution of (1.9) if the mesh grading parameters q and the weight parameter λ satisfy the condition (1.10), that is,

$$\frac{2}{3}q < 2\lambda + 1.$$

Then ψ will have regularity:

$$\begin{aligned}\psi(x) &= x^{(1/3)q+(q-1)-\lambda}(1-x)^{(2/3)q+(q-1)-\lambda}h(x), \qquad h \in C[0,1] \\ &= x^{(1/3)q+(q-1)-\lambda+1/2-\varepsilon}(1-x)^{(2/3)q+(q-1)-\lambda+1/2-\varepsilon}\tilde{h}(x), \\ &\qquad \tilde{h} \in L_2[0,1]\end{aligned}$$

for an arbitrary $\varepsilon > 0$, and

$$\tilde{\phi}(x) = [x(1-x)]^{\lambda}\psi(x) = x^{(1/3)q+(q-1)+1/2-\varepsilon}(1-x)^{(2/3)q+(q-1)+1/2-\varepsilon}\tilde{h}(x)$$

is a solution of (1.7).

Theorem 4.1 and Remark 1 assert that the convergence of the approximation solution ψ_h to ψ will be

$$||\psi - \psi_h||_0 \approx Ch^p$$

with $p = (1/3)q + (q-1) - \lambda + 1/2 - \varepsilon$ for an arbitrary $\varepsilon > 0$. Actually, we are interested in the convergence of $\phi_h(x) = [x(1-x)]^{\lambda}\psi_h(x)$ to $\tilde{\phi}$. So far, we have no convergence analysis for ϕ_h, but in our numerical experiments ϕ_h converges consistently faster to $\tilde{\phi}$ than ψ_h does to ψ.

In this example we give results obtained by using the mesh grading parameters $q = 3$ and 5 with the weight parameters $\lambda = 1$ and $\lambda = 1.2$, respectively. According to Remark 1, the order of convergence may be $p \approx 2.5$ for $q = 3$ and $\lambda = 1$, and $p \approx 4.966\ldots$ for $q = 5$ and $\lambda = 1.2$. In Table 1, Columns 2 and 6 show the numerical results for ψ_h, and they concord well with Theorem 4.1 and Remark 1. Columns 4 and 8 represent the numerical results of ϕ_h.

TABLE 1. Numerical results for Example 1 with $q = 3, 5$.

h	$E_h(q=3)$	R_h	$\overline{E_h}$	$\overline{R_h}$	$E_h(q=5)$	R_h	$\overline{E_h}$	$\overline{R_h}$
1/4	.185E+00		0.459E-01		0.318E+00		0.570E-01	
1/8	0.538E-02	5.10	0.990E-03	5.54	0.434E-01	2.88	0.646E-02	3.14
1/16	0.261E-03	4.37	0.133E-04	6.22	0.135E-03	8.33	0.261E-04	7.95
1/32	0.441E-04	2.56	0.140E-05	3.25	0.306E-06	8.78	0.178E-07	10.52
1/64	0.777E-05	2.50	0.166E-06	3.08	0.101E-07	4.92	0.419E-09	5.41
1/128	0.138E-05	2.50	0.201E-07	3.04	0.336E-09	4.90	0.122E-10	5.10

Example 2. Let $(a,b) = (\sqrt{3}/2, 1/2)$ and $f(t) = 1$. The natural airfoil solution satisfies $(\alpha, \beta) = (1/6, -1/6)$ in (1.2). After a mesh grading transformation of order q, the solutions of (1.9) will have regularity:

$$\psi(x) = x^{(1/6)q+(q-1)-\lambda+1/2-\varepsilon}(1-x)^{(-1/6)q+(q-1)-\lambda+1/2-\varepsilon}\tilde{h}(x),$$
$$\tilde{h} \in L_2[0,1].$$

Then

$$\tilde{\phi}(x) = x^{(1/6)q+(q-1)+1/2-\varepsilon}(1-x)^{(-1/6)q+(q-1)+1/2-\varepsilon}\tilde{h}(x),$$

is a solution of (1.7). In this example, any $q < 9$ with $\lambda = 1$ satisfies (1.10). The numerical results are given for $q = 3$ and $q = 6$ with $\lambda = 1$. Then the order of convergence of ψ_h in the L_2-norm will be $p \approx 1$, 3.5 for $q = 3, 6$, respectively.

TABLE 2. Numerical results for Example 2 with $q = 3, 6$.

h	$E_h(q=3)$	R_h	$\overline{E_h}$	$\overline{R_h}$	$E_h(q=6)$	R_h	$\overline{E_h}$	$\overline{R_h}$
1/4	.175E+00		0.397E-01		0.228E+00		0.780E-01	
1/8	0.246E-01	2.83	0.278E-02	3.84	0.422E-01	2.44	0.108E-01	2.85
1/16	0.115E-01	1.09	0.688E-03	2.01	0.229E-03	7.53	0.463E-04	7.87
1/32	0.560E-02	1.04	0.219E-03	1.65	0.803E-05	4.83	0.401E-06	6.85
1/64	0.277E-02	1.01	0.738E-04	1.57	0.701E-06	3.52	0.253E-07	3.99
1/128	0.138E-02	1.01	0.255E-04	1.53	0.622E-07	3.49	0.161E-08	3.97

Example 3. Let (a, b) be nonzero real functions in (1.1) with $f = 1$. This will be an index 0 problem. Using the regularity result [**13**, Section 3.7, Chapter 17], $\alpha = -g(0)$, $\beta = g(1)$, where

$$g(x) = \frac{1}{2\pi i}\log\frac{a(x) - ib(x)}{a(x) + ib(x)}.$$

For $a(x) = x + 1$ and $b(x) = -\exp(x)$, we have $\alpha = -1/4$ and $\beta = \tan^{-1}(e/2)/\pi \approx .2980882980$. After a mesh grading transformation of order q, the solution of (1.9) will have regularity:

$$\psi(x) = x^{(-1/4)q+(q-1)+1/2-\varepsilon-\lambda}(1-x)^{\beta q+(q-1)+1/2-\varepsilon-\lambda}\tilde{h}(x),$$

and then

$$\tilde{\phi}(x) = x^{(-1/4)q+(q-1)+1/2-\varepsilon}(1-x)^{\beta q+(q-1)+1/2-\varepsilon}\tilde{h}(x).$$

Equation (1.10) leads to

$$q < (2\lambda + 1)/(2\tan^{-1}(e/2))$$

for this problem to have a unique solution. All $q \leq 5$ with $\lambda = 1$ satisfy the above inequality. For $q = 6$, we use $\lambda = 1.3$ so that the above inequality is satisfied. The order of convergence of ψ_h will be $p \approx 3/4$ for $q = 3$ and $\lambda = 1$, and $p \approx 2.7$ for $p = 6$ with $\lambda = 1.3$.

In the tables,

$$E_h = ||\psi_h - \psi_{h/2}||_0, \qquad R_h = \log(E_{2h}/E_h)/\log 2$$

and

$$\overline{E_h} = ||P_h(\phi_h) - P_{h/2}(\phi_{h/2})||_0, \qquad \overline{R_h} = \log(\overline{E_{2h}}/\overline{E_h})/\log 2$$

with the collocation projection P_h in Section 2. Our method especially has a superb convergence for the index -1 solution and the natural airfoil solution with $a \gg b$ in (1.1). When q is large and h is small, mesh tends to highly concentrate toward 0 and 1. We observe that some instability occurs in our experiments when $q \geq 7$ and $h \leq 1/256$. It seems that the instability is mainly caused by round off error rather than by not having cutoff around endpoints. In our numerical experiments, we didn't use the cutoff of intervals around endpoints.

TABLE 3. Numerical results for Example 3 with $q = 3, 6$.

h	$E_h(q=3)$	R_h	$\overline{E_h}$	$\overline{R_h}$	$E_h(q=6)$	R_h	$\overline{E_h}$	$\overline{R_h}$
1/4	0.229E+00		0.414E-01		0.539E+00		0.909E-01	
1/8	0.791E-01	1.53	0.697E-02	2.57	0.821E-01	2.71	0.124E-01	2.88
1/16	0.466E-01	0.76	0.246E-02	1.50	0.117E-02	6.14	0.103E-03	6.92
1/32	0.274E-01	0.76	0.961E-03	1.36	0.191E-03	2.61	0.126E-04	3.03
1/64	0.162E-01	0.76	0.382E-03	1.33	0.307E-04	2.64	0.194E-05	2.69
1/128	0.961E-02	0.76	0.153E-03	1.32	0.489E-05	2.65	0.298E-06	2.71

Acknowledgment. This work was performed when the author was a research associate at the University of New South Wales, Sydney, Australia.

Appendix

A. Here are some basic facts about the Mellin convolution operator on $[0, \infty)$, which are needed in Section 5. These are based on [**7**] and

[**17**]. The Mellin transform $\sigma(\phi)$ of $\phi \in L_2[0,1]$ is defined as

$$\sigma(\phi)(z) = \int_0^\infty s^{z-1}\phi(s)\,ds, \qquad z = 1/2 + iy, \quad y \in \mathbf{R}.$$

Then it has the following properties.

1. Mellin transform σ is an isomorphism from $L_2[0,\infty)$ to $L_2[\{\mathrm{Re}\,(z) = 1/2\}]$.

2. Parseval's equality.

$$\int_0^\infty \phi\bar{\psi}\,ds = \int_{\mathrm{Re}\,(z)=1/2} \sigma(\phi)\overline{\sigma(\psi)}|dz|$$

3. (Convolution Theorem.) If $\mathcal{K}$ is a Mellin convolution operator, i.e.,

$$\mathcal{K}\phi(t) = \int_0^\infty K\left(\frac{t}{s}\right)\frac{\phi(s)}{s}\,ds$$

with kernel $s^{-1/2}K(s) \in L_1[0,\infty)$, then $\sigma(\mathcal{K}\phi) = \sigma(K)\cdot\sigma(\phi)$. Then it is easy to see that $||\mathcal{K}||_0 \le \sup_{\mathrm{Re}\,(z)=1/2}|\sigma(K)(z)|$. Here we define $\sigma(\mathcal{K}) := \sigma(K)$, and $\sigma(K)$ is the *symbol* of the operator $\mathcal{K}$.

4. If $\mathcal{K}$ and $\mathcal{L}$ are convolution operators, then $\sigma(\mathcal{KL}) = \sigma(\mathcal{K})\cdot\sigma(\mathcal{L})$. Therefore, if $|\sigma(\mathcal{K})z)| \ge C > 0$, then $\mathcal{K}^{-1}$ exists and $\sigma(\mathcal{K}^{-1}) = 1/\sigma(\mathcal{K})$.

5. The properties (1)–(4) extend to the case of matrix operators if we replace symbols and kernels with matrix functions. In particular, if $\overline{\mathcal{K}}$ is a matrix of Mellin operators with a matrix symbol $\sigma(\overline{\mathcal{K}})$ with $|\det(\sigma(\overline{\mathcal{K}})(z))| > C > 0$, then $\overline{\mathcal{K}}$ is an invertible operator.

Let us consider singular integral operators considered in Section 5. The Cauchy singular operator on $\mathbf{R}^+$,

$$\mathcal{G}\phi(t) = \frac{1}{\pi}\text{p.v.}\int_0^\infty \frac{\phi(s)}{s-t}\,ds = \frac{1}{\pi}\text{p.v.}\int_0^\infty \frac{1}{1-(t/s)}\frac{\phi(s)}{s}\,ds,$$

is a Mellin operator with the symbol $\sigma(\mathcal{G})(z) = \cot(\pi z)$, and

$$\mathcal{R}\phi(t) = \frac{1}{\pi}\int_0^\infty \frac{\phi(s)}{s+t}\,ds = \frac{1}{\pi}\int_0^\infty \frac{1}{1+(t/s)}\frac{\phi(s)}{s}\,ds,$$

is a Mellin operator with the symbol $\sigma(\mathcal{R})(z) = 1/\sin(\pi z)$. For

$$\begin{aligned}
\mathcal{G}_\lambda^q \phi(t) &= \frac{1}{\pi}\text{p.v.}\int_0^\infty \left(\frac{s}{t}\right)^\lambda \frac{qt^{q-1}\phi(s)}{sq - t^q}\, ds \\
&= \frac{1}{\pi}\text{p.v.}\int_0^\infty \frac{q(t/s)^{q-\lambda-1}}{1-(t/s)^q}\frac{\phi(s)}{s}\, ds, \\
\sigma(\mathcal{G}_\lambda^q)(z) &= \frac{1}{\pi}\text{p.v.}\int_0^\infty s^{z-1}\frac{qs^{q-\lambda-1}}{1-s^q}\, ds, \\
&\quad \text{(by a change of variable, } \mu = s^q\text{)} \\
&= \frac{1}{\pi}\text{p.v.}\int_0^\infty \mu^{(z-\lambda-1)/q+1-1}\frac{1}{1-\mu}\, d\mu \\
&= \cot((z-\lambda-1)\pi/q + \pi) \\
&= \cot((z-\lambda-1)\pi/q).
\end{aligned}$$

B. Here we will prove that

$$\text{(B.1)}\qquad \begin{aligned}
\mathcal{B} &= (aI - b\mathcal{H})\mathcal{K} \\
&= \Pi \begin{bmatrix} aI - b\mathcal{G} & b\tilde{R} \\ -b\tilde{\tilde{R}} & aI + b\tilde{\mathcal{G}} \end{bmatrix} \begin{bmatrix} [\mathcal{G}_\lambda^q - \mathcal{G}] & \tilde{\mathcal{R}} \\ -\tilde{\tilde{R}} & -[\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}}] \end{bmatrix} \Pi^T + \mathcal{E}
\end{aligned}$$

where $\mathcal{E}$ is compact and $\Pi = (\chi, \chi')$, where χ and χ' in Section 5.

Remember that

$$\begin{aligned}
\mathcal{K} &= \chi[\mathcal{G}_\lambda^q - \mathcal{G}]\chi + \chi'[-(\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}})]\chi' + \chi\tilde{\mathcal{R}}\chi' - \chi'\tilde{\tilde{R}}\chi + \mathcal{E} \\
&= \Pi \begin{bmatrix} [\mathcal{G}_\lambda^q - \mathcal{G}] & \tilde{\mathcal{R}} \\ -\tilde{\tilde{\mathcal{R}}} & -[\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}}] \end{bmatrix} \Pi^T + \mathcal{E}.
\end{aligned}$$

Let us introduce other cut off functions ω and ω' such that $\omega = 1$ on $\text{supp}\,(\chi)$, $\text{supp}\,(\omega) \subset [0, 1/2)$, and $\omega'(x) = \omega(1-x)$. Then, using ω and ω', split $\mathcal{H}$ as

$$\mathcal{H} = -\omega\tilde{\mathcal{R}}\omega' + \omega'\tilde{\tilde{\mathcal{R}}}\omega + \mathcal{S} + \mathcal{E},$$

where

$$\mathcal{S}\phi = \frac{1}{\pi}\int_0^1 \frac{1}{y-x}\phi(y)\, dy.$$

The operator $\mathcal{S}$ is the Cauchy singular integral operator on $[0,1]$. Split $\mathcal{S}$ also as follows.

$$\begin{aligned}\mathcal{S} &= \omega\mathcal{S}\omega + \omega'\mathcal{S}\omega' + \omega\mathcal{S}(1-\omega-\omega') + \omega'\mathcal{S}(1-\omega-\omega')\\ &\quad + (1-\omega-\omega')\mathcal{S}\omega + (1-\omega-\omega')\mathcal{S}\omega'\\ &\quad + (1-\omega-\omega')\mathcal{S}(1-\omega-\omega') + \mathcal{E}\end{aligned}$$

since $\omega\mathcal{S}\omega'$ and $\omega'\mathcal{S}\omega$ are compact.

Now let us look at $\mathcal{SK}$. Since

$$(1-\omega-\omega')\chi = (1-\omega-\omega')\chi' = \omega\chi' = \omega'\chi = 0,$$

and

$$\omega\chi = \chi, \qquad \omega'\chi' = \chi',$$

$$\begin{aligned}\mathcal{SK} &= \omega\mathcal{S}\chi[\mathcal{G}_\lambda^q - \mathcal{G}]\chi + (1-\omega-\omega')\mathcal{S}\chi[\mathcal{G}_\lambda^q - \mathcal{G}]\chi + \omega'\mathcal{S}\chi'[-(\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}})]\chi'\\ &\quad + (1-\omega-\omega')\mathcal{S}\chi'[-(\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}})]\chi' + \omega\mathcal{S}\chi\tilde{\mathcal{R}}\chi' + (1-\omega-\omega')\mathcal{S}\chi\tilde{\mathcal{R}}\chi'\\ &\quad - \omega'\mathcal{S}\chi'\tilde{\tilde{\mathcal{R}}}\chi - (1-\omega-\omega')\mathcal{S}\chi'\tilde{\tilde{\mathcal{R}}}\chi + \mathcal{E}\\ &= \omega\mathcal{S}\chi[\mathcal{G}_\lambda^q - \mathcal{G}]\chi + \omega'\mathcal{S}\chi'[-(\tilde{\mathcal{G}}_\lambda^q - \tilde{\mathcal{G}})]\chi'\\ &\quad + \omega\mathcal{S}\chi\tilde{\mathcal{R}}\chi' - \omega'\mathcal{S}\chi'\tilde{\tilde{\mathcal{R}}}\chi + \mathcal{E}.\end{aligned}$$

The last line follows from the fact that $(1-\omega-\omega')\mathcal{S}\chi$, $(1-\omega-\omega')\mathcal{S}\chi'$ are compact. Moreover, since $\chi\mathcal{S} - \mathcal{S}\chi$ and $\chi'\mathcal{S} - \mathcal{S}\chi'$ are compact, and $\mathcal{S} = \mathcal{G}$ around $(0,0)$ and $\mathcal{S} = -\tilde{\mathcal{G}}$ around (1,1), $\mathcal{SK}$ can be further simplified as

$$\begin{aligned}\mathcal{SK} &= \chi\mathcal{G}[\mathcal{G}_\lambda^q - \mathcal{G}]\chi + \chi'(-\tilde{\mathcal{G}})[-\tilde{\mathcal{G}}_\lambda^q + \tilde{\mathcal{G}}]\chi'\\ &\quad + \chi^{\mathcal{G}}\tilde{\mathcal{R}}\chi' - \chi'(-\tilde{\mathcal{G}})\tilde{\tilde{\mathcal{R}}}\chi + \mathcal{E}\\ &= \Pi\begin{bmatrix}\mathcal{G} & 0\\ 0 & -\tilde{\mathcal{G}}\end{bmatrix}\cdot\begin{bmatrix}\mathcal{G}_\lambda^q - \mathcal{G} & \tilde{\mathcal{R}}\\ -\tilde{\mathcal{R}} & -\tilde{\mathcal{G}}_\lambda^q + \tilde{\mathcal{G}}\end{bmatrix}\Pi^T + \mathcal{E}.\end{aligned}$$

Noting that $\tilde{\mathcal{R}}\chi - \chi'\tilde{\mathcal{R}}$ and $\tilde{\tilde{\mathcal{R}}}\chi' - \chi\tilde{\tilde{\mathcal{R}}}$ are compact, and by the same way as above,

$$\begin{aligned}(-\omega\tilde{\mathcal{R}}\omega' + \omega'\tilde{\tilde{\mathcal{R}}}\omega)\mathcal{K} &= \Pi\begin{bmatrix}0 & -\tilde{\mathcal{R}}\\ \tilde{\tilde{R}} & 0\end{bmatrix}\\ &\quad\cdot\begin{bmatrix}\mathcal{G}_\lambda^q - \mathcal{G} & \tilde{\mathcal{R}}\\ -\tilde{\tilde{R}} & -\tilde{\mathcal{G}}_\lambda^q + \tilde{\mathcal{G}}\end{bmatrix}\Pi^T + \mathcal{E}.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{H}\mathcal{K} &= (\mathcal{S} - \omega\mathcal{R}\omega' + \omega'\tilde{\tilde{\mathcal{R}}}\omega + \mathcal{E})\mathcal{K}\\ &= \Pi \begin{bmatrix} \mathcal{G} & -\tilde{\mathcal{R}} \\ \tilde{\tilde{\mathcal{R}}} & -\tilde{\mathcal{G}} \end{bmatrix} \begin{bmatrix} \mathcal{G}^q_\lambda - \mathcal{G} & \tilde{\mathcal{R}} \\ -\tilde{\tilde{\mathcal{R}}} & -\tilde{\mathcal{G}}^q_\lambda + \tilde{\mathcal{G}} \end{bmatrix} \Pi + \mathcal{E}'.\end{aligned}$$

Now we have (B.1).

C. Divide the integral operator $\mathcal{M}$ into local Mellin operators and a compact perturbation.

$$\mathcal{M} := \mathcal{M}_{00} + \mathcal{M}_{10} + \mathcal{M}_{01} + \mathcal{M}_{11} + \mathcal{E}. \tag{C.1}$$

Here $\{\mathcal{M}_{\alpha\beta}\}_{\alpha,\beta=0,1}$ are Mellin operators, which represent the local behavior of $\mathcal{M}$ at (α, β), and $\mathcal{E}$ is a compact operator.

For simplicity, let us look at $\mathcal{M}_{00}$ only.

$$M_{0,0}z(x) = \int_0^\infty M\left(\frac{x}{y}\right)\frac{1}{y}z(y)\,dy, \tag{C.2}$$

where

$$M(s) := \frac{P(s)}{Q(s)},$$

with P and Q polynomials, and $\deg(P) = \deg(Q) - 1$, $Q \neq 0$.

Now look at the operator $\mathcal{M}_{i,j,k,l}$, where

$$\mathcal{M}_{i,j,k,l}z(s) := \int_0^1 x^l y^k \left| D_x^i D_y^j \left[M\left(\frac{x}{y}\right)\frac{1}{y}\right]\right| z(y)\,dy. \tag{C.3}$$

Mathematical induction gives us, for $m := i + j$,

$$D_y^i D_x^j \left[M\left(\frac{x}{y}\right)\frac{1}{y}\right] = G\left(\frac{x}{y}\right) y^{-m}\frac{1}{y},$$

where

$$G(s) := \frac{R(s)}{T(s)}$$

and $\deg(R) = \deg(T) - m - 1$. Then, if $i + j = k + l$,

$$x^l y^k D_y^i D_x^j \left[M\left(\frac{x}{y}\right)\frac{1}{y}\right] = \left(\frac{x}{y}\right)^l G\left(\frac{x}{y}\right)\frac{1}{y}, \tag{C.4}$$

and

$$\sup_{s\in\mathbf{R}^+} s^l|G(s)| < \infty \tag{C.5}$$

for $0 \leq l \leq m$. Thus, $s^{-1/2}(s^l|G(s)|) \in L_1(\mathbf{R}^+)$, and $\mathcal{L}_{i,j,k,l}$ is a bounded operator on $L_2(\mathbf{R}^+)$. For more details about Mellin operators, see [**7**].

REFERENCES

1. P.M. Anselone, *Collectively compact operator approximation theory and applications to integral equations*, Prentice-Hall, Englewood Cliffs, 1971.

2. K.E. Atkinson, *An introduction to numerical analysis*, Second edition, John Wiley, New York, 1989.

3. ———, *A survey of numerical methods for the solution of Fredholm integral equations of the second kind*, Part II, SIAM, 1976.

4. K.E. Atkinson and I.H. Sloan, *The numerical solution of first-kind logarithmic-kernel integral equations on smooth open arcs*, Math. Comp. **56** (1991), 119–139.

5. G.A. Chandler and I.G. Graham, *Product integration collocation methods for non-compact integral operator equations*, Math. Comp. **50** (1988), 125–138.

6. M. Costabel and E. Stephan, *On the convergence of collocation methods for boundary integral equations on polygons*, Math. Comp. **54** (1987), 461–478.

7. J. Elschner and I. Graham, *An optimal order collocation method for first kind boundary integral equations on polygons*, Institut für Angewandte Analysis und Stochastik (1993), preprint #54.

8. M. Golberg, *Numerical solution of integral equations*, in *Mathematical concepts and methods in science and engineering*, Plenum Press, New York, 1990.

9. Y. Jeon, *A Nyström method for boundary integral equations on domains with a piecewise smooth boundary*, J. Integral Equations Appl. (1993) **5**, 221–242.

10. R. Kress, *A Nyström method for boundary integral equations in domains with corners*, Numer. Math. **58** (1990), 145–161.

11. ———, *Linear integral equations*, Springer-Verlag, Berlin-New York, 1989.

12. J. Lyness and B. Ninham, *Numerical quadrature and asymptotic expansions*, Math. Comput. **21** (1967), 162–178.

13. S. Mikhlin and S. Prössdorf, *Singular integral operators*, Springer-Verlag, Berlin-New York, 1986.

14. S. Prössdorf and A. Rathsfeld, *Quadrature methods for strongly elliptic Cauchy singular integral equations on an interval*, Oper. Theory: Adv. Appl. **41** (1989), 435–471.

15. S. Prössdorf and I.H. Sloan, *Quadrature method for singular integral equations on closed curves*, Numer. Math. **61** (1992), 543–559.

16. E. Venturino, *Unconventional solution of singular integral equations*, J. Integral Equations Appl. **4** (1992), 443–463.

17. Y. Yan, *First kind boundary integral methods for two-dimensional potential problems with singularities*, Ph.D. thesis, University of New South Wales,

18. Y. Yan and I. Sloan, *Mesh grading for integral equations of the first kind with logarithmic kernel*, SIAM J. Numer. Anal. **26** (1989), 574–587.

Department of Mathematics, Ajou University, Suwon 441-749, Korea