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A QUADRATURE METHOD FOR THE HYPERSINGULAR INTEGRAL EQUATION ON AN INTERVAL

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ABSTRACT. This paper is concerned with a quadrature method for the approximate solution of the hypersingular integral equation

$$\text{p.f.} \int_0^1 \frac{u(\tau)}{\mid \tau - t \mid^2} \, d\tau = f(t), \qquad 0 \le t \le 1.$$

Stability and error estimates are proved. Numerical experiments are presented which confirm the theoretical estimates.

1. Introduction. In this paper we consider the hypersingular integral equation on the interval

$$(1) \qquad (Du)(t) := \text{p.f.} \int_0^1 \frac{u(\tau)}{\mid \tau - t \mid^2} \, d\tau = f(t), \qquad 0 \le t \le 1,$$

where f is a given function and u is to be found. The integral in (1) is to be interpreted as a Hadamard finite part integral. For the definition of such a finite part integral we refer, e.g., to [**11**, Section 3.2].

Notice that it is possible to consider the hypersingular integral equation of the form (1), disturbed by an integral operator with a smooth kernel function or the hypersingular integral equation on a smooth open curve by using the same methods as in the present paper.

The hypersingular integral equation (1) results from a certain boundary integral method which has attracted the attention of several mathematicians in recent years. In particular, we mention the paper [**3**] of Costabel and Stephan, where the Galerkin method for the hypersingular integral equation on polygons is studied, and the article [**2**] of Costabel, which gives a survey of several boundary integral operators on Lipschitz domains and investigates the Galerkin method for these. In the paper [**19**] of von Petersdorff and Stephan, a multigrid method on graded meshes is considered for the hypersingular integral equation.

In [**1**, Sections 1.6 and 5.1], a quadrature method for the hypersingular integral equation on an interval is derived and an error estimate is proved. The first rigorous analysis of a fully discretized method for the hypersingular integral equation has been given by Kieser, Kleemann and Rathsfeld in [**13**]. There a very easy discretization scheme is used to get a quadrature method for this equation on a smooth closed curve, and stability and error estimates for this method are obtained.

Another approach is given by Golberg in [**8**] and [**9**], where the hypersingular integral equation on the interval is discussed with the help of Chebyshev polynomials. The methods are based on expanding u into a finite series of Chebyshev polynomials of the second kind and then determining the unknown coefficients by either the Galerkin method or collocation. A similar approach is given by Erwin and Stephan in [**7**], where a collocation method using Chebyshev polynomials has been considered for the hypersingular integral equation on the interval. In [**7**], the operator D is considered as an operator acting between some spaces of Sobolev type, which are defined by means of Chebyshev polynomials.

In [**16**] a general formula for the solution to the hypersingular integral equation on the interval without compact perturbation is given by Martin. There the relation of this integral equation to a singular equation with a known solution is used.

In the present paper, we shall propose a fully discretized quadrature method for the hypersingular integral equation (1). Because the solution of this equation has an end-point behavior like $s^{1/2}(1-s)^{1/2}$ (see [**7**]), we carry out a refinement of the grid near the end points of the interval. To this end, we perform a change of the variables $\tau = \gamma(\sigma)$, $t = \gamma(s)$ in the integral (1), where γ has an end-point behavior like s^α. Transformations of this type have already been used for some integral equations, for example, in the case of the Cauchy singular integral equation (see [**18**] and [**20**]), in the case of boundary integral equations of the second kind for the harmonic Dirichlet problem in plane domains with corners (cf. [**14**]) and for Mellin convolution equations arising in crack problems (cf. [**5**]). In the present paper, the transformation $\gamma : [0,1] \to [0,1]$ is chosen as in [**14**],

$$\gamma(s) = \frac{[v(s)]^\alpha}{[v(s)]^\alpha + [v(1-s)]^\alpha}, \qquad 0 \le s \le 1, \tag{2}$$

with

$$(3) \qquad v(s) = \left(\frac{1}{\alpha} - \frac{1}{2}\right)(1-2s)^3 + \frac{1}{\alpha}(2s-1) + \frac{1}{2}, \qquad \alpha > \frac{3}{2}.$$

The function γ has an end-point behavior like s^α near 0 and like $1-(1-s)^\alpha$ near 1. Note that the cubic polynomial v is chosen such that $v(0)=0$, $v(1)=1$, and $\gamma'(1/2)=2$. The latter property ensures, roughly speaking, that one half of the grid points is equally distributed over the total interval, whereas the other half is accumulated towards the two end points.

Multiplying Equation (1) by $\gamma'(s)$, we obtain the transformed equation

$$(4) \qquad \text{p.f.} \int_0^1 \frac{\gamma'(s)\gamma'(\sigma)}{|\gamma(\sigma)-\gamma(s)|^2} w(\sigma)\, d\sigma = g(s), \qquad 0 \le s \le 1,$$

with

$$w(s) := u(\gamma(s)), \qquad g(s) := f(\gamma(s))\gamma'(s).$$

Using the quadrature rule

$$(5) \qquad \int_{-\infty}^{\infty} f(t)\, dt \approx \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} f(t_j)\frac{2}{n}, \qquad t_k = \frac{k}{n},$$

for n even and applying a kind of regularization to the finite part integral (see the next section for more details), we get the quadrature method

$$(6) \quad g(t_k) = \sum_{\substack{j=1 \\ j\equiv k+1 \bmod 2}}^{n-1} \frac{2}{n}\frac{\gamma'(t_j)\gamma'(t_k)}{|\gamma(t_j)-\gamma(t_k)|^2}\xi_j - \frac{n\pi^2}{2}\xi_k, \qquad k=1,\dots,n-1.$$

The term $-(n\pi^2/2)\xi_k$ results from the aforementioned regularization. A corresponding term occurs in the case of a closed curve (see [**13**]).

The paper is organized as follows. In Section 2 the quadrature method (6) is derived. In Section 3, the mapping properties of the approximate operators corresponding to (6) and the corresponding discretized spaces are investigated. In Section 4, the stability of the method is proved. Let us denote the matrix of the linear system of Equation (6) by A_n. The main point of the proof is that there holds the equivalence $\langle -A_n\xi, \xi\rangle_{l_2(n)} \sim \langle B_n\xi, \xi\rangle_{l_2(n)}$ for all finite sequences $\xi = \{\xi_k\}_{k=1}^{n-1}$ uniformly with respect to n. We write $a \sim b$, if positive constants C_1, $C_2 > 0$ exist with $C_1 a < b < C_2 a$. Here B_n is the norm isomorphism of the regarded discrete spaces and $\langle\cdot,\cdot\rangle_{l_2(n)}$ is the scalar-product in $l_2(n)$ (cf. Section 2).

The error estimate is derived in Section 5. Let f be sufficiently smooth. If $u_n = w_n \circ \gamma^{-1}$, i.e., $u_n \circ \gamma = w_n$, where w_n is a high order interpolation of the approximate values $w_n(t_j) = \xi_j$, $j = 1, \dots, n-1$, obtained by solving (6), then the Sobolev norm error $\|u - u_n\|_{1/2}$ can be estimated by $Cn^{-\alpha/2+1+\varepsilon}$ with ε arbitrarily small (cf. Theorem 5.1).

In Section 6, another transformation is used, namely, a cos-transformation like that used for the numerical solution of first-kind integral equations with logarithmic kernel (cf. [**22**, Section 3.8]). The quadrature method derived with the help of this transformation is shown to be stable, too. Here the proof is reduced to the case of the unit circle. The stability of this method is easier to prove than that of the method with γ defined by (2) and the order of convergence is arbitrarily high for smooth f (cf. Section 6). However, the techniques used in Sections 2–5 seem to be more convenient for applications to the two-dimensional case and to the case of more general integral equations on the interval or on the polygon, provided that the asymptotic behavior of the solution near to the end or corner points is known.

The numerical results are discussed in Section 7.

2. The discretization of the hypersingular integral equation. Consider the hypersingular integral equation on the interval $I = [0, 1]$,

$$(Du)(t) := \text{p.f.} \int_0^1 \frac{u(\tau)}{|\tau - t|^2}\, d\tau = f(t), \qquad t \in I.$$

By [**19, 2**] and [**3**], the mapping

$$D : \tilde{H}_{1/2}(I) \longrightarrow H_{-1/2}(I)$$

is bijective and continuous. Here the space $\tilde{H}_{1/2}(I)$ is defined by

$$\tilde{H}_{1/2}(I) := \{u|_I : u \in H_{1/2}(\mathbf{R}), u|_{\mathbf{R}\setminus I} \equiv 0\}$$

equipped with the norm of $H_{1/2}(\mathbf{R})$. By $H_s(\mathbf{R})$ we denote the usual Sobolev space, i.e., the completion of $C_0^\infty(\mathbf{R})$ with respect to the norm

$$\|u\|_s = \left(\sum_{j=0}^{m} \int_{-\infty}^{\infty} |u^{(j)}(x)|^2 \, dx + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|u^{(m)}(x) - u^{(m)}(y)|^2}{|x-y|^{1+2\theta}} \, dx \, dy \right)^{1/2}$$

where $s = m + \theta$, $m \in \mathbf{N}$, $\theta \in [0, 1)$, $u^{(j)} = (d/dx)^j u$ and the second term of the righthand side is omitted for $\theta = 0$. The space $H_{-1/2}(I)$ is defined as the dual space of $\tilde{H}_{1/2}(I)$ with respect to the L_2-scalar product (see [**3**]).

Remark 2.1. For $u \in H_{1/2}(\mathbf{R})$ with $u|_{\mathbf{R}\setminus I} \equiv 0$, there holds $\|u\|_{1/2}^2 = \|u\|_{L_2}^2 + |u|_{1/2}^2$ with

$$|u|_{1/2}^2 = \int_0^1 \int_0^1 \frac{|u(x) - u(y)|^2}{|x - y|^2} \, dx \, dy + 2 \int_0^1 \frac{|u(x)|^2}{x(1 - x)} \, dx,$$

and $||u||_{1/2} \sim |u|_{1/2}$.

The proof is well known and not hard.

Remark 2.2. Let $f \in C^\infty$ and u be the solution of Equation (1). Then there holds $u(t) = t^{1/2}(1 - t)^{1/2}\tilde{g}(t)$ with $\tilde{g}(t)$ smooth.

We do not give the proof here. We only remark that it is a consequence of the mapping properties of the hypersingular integral operator in the case of the unit circle. The interval has to be transformed to the unit circle as in Section 6. For the case $f \notin C^\infty$, a statement about the asymptotics can be found in [**7**].

In order to get a refinement of the grid near the end points of the interval I, we shall apply a transformation of coordinates. Consider the transformation function $\gamma : I \to I$ of R. Kress [**14**] given by (2) and (3). The condition $\alpha > 4/3$ is necessary to guarantee the monotonicity of v. The stronger condition $\alpha > 3/2$ will be needed in the proof of stability.

By [**12**] it is possible to apply the usual rules of transformation for the finite part integral in (1) if $t \in (0, 1)$. Thus (1) is equivalent to (4).

Now we set

$$h(s,\sigma) := \frac{|\sigma - s|^2 \gamma'(s)\gamma'(\sigma)}{|\gamma(\sigma)-\gamma(s)|^2}.$$

With this notation Equation (4) is equivalent to

$$Aw(s) := \text{p.f.} \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2} h(s,\sigma)\, d\sigma = g(s). \tag{7}$$

We shall derive a quadrature method for Equation (7). To this end, we use the well-known quadrature rule (5). Obviously, there holds

$$\begin{aligned} \text{p.f.} \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2} h(s,\sigma)\, d\sigma = \text{p.f.} &\int_0^1 \frac{w(\sigma)}{|\sigma - s|^2}\, d\sigma \\ &+ \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2}(h(s,\sigma)-1)\, d\sigma. \end{aligned} \tag{8}$$

Now we continue the function w to a function on $\mathbf{R}$ by setting $w(t) := 0$ for all $t \notin [0,1]$. Note that w remains to be continuous since $u(s)$ has the end-point asymptotics $s^{1/2}(1-s)^{1/2}$ and s is replaced by $s = \gamma(\sigma)$ with $\gamma(\sigma) \sim \sigma^\alpha(1-\sigma)^\alpha$, α being sufficiently large. The first integral on the righthand side of (8) is a finite part integral. At the point $s = t_k$, $1 \le k \le n-1$ we can compute it by using the regularization

$$\begin{aligned} \text{p.f.} \int_0^1 \frac{w(\sigma)}{|\sigma - t_k|^2}\, d\sigma &= \text{p.f.} \int_{-\infty}^{\infty} \frac{w(\sigma)}{|\sigma - t_k|^2}\, d\sigma \\ &= \int_{-\infty}^{\infty} \frac{w(\sigma) - w(t_k) - w'(t_k)(\sigma - t_k)}{|\sigma - t_k|^2}\, d\sigma \\ &\quad + w(t_k)\ \text{p.f.} \int_{-\infty}^{\infty} \frac{1}{|\sigma - t_k|^2}\, d\sigma \\ &\quad + w'(t_k)\ \text{p.f.} \int_{-\infty}^{\infty} \frac{\sigma - t_k}{|\sigma - t_k|^2}\, d\sigma. \end{aligned}$$

Now it follows from the definition of the finite part integral, that

$$\text{p.f.} \int_{-\infty}^{\infty} \frac{1}{|\sigma - t_k|^2}\, d\sigma = 0, \qquad \text{p.f.} \int_{-\infty}^{\infty} \frac{1}{\sigma - t_k}\, d\sigma = 0.$$

Thus with $\xi_j = w(t_j)$, $j = 1, \ldots, n-1$, and $\xi_j = 0$, $j \le 0$ or $j \ge n$, we obtain

$$\text{(9)} \qquad \text{p.f.} \int_0^1 \frac{w(\sigma)}{|\sigma - t_k|^2}\, d\sigma \approx \sum_{\substack{j=1 \\ j \equiv k+1 \bmod 2}}^{n-1} \frac{1}{|t_j - t_k|^2} \frac{2}{n} \xi_j - \frac{n\pi^2}{2} \xi_k,$$

since short calculations show that

$$\sum_{\substack{j=-\infty \\ j \equiv k+1 \bmod 2}}^{\infty} \frac{1}{|t_j - t_k|^2} \frac{2}{n} = \frac{n\pi^2}{2}, \qquad \sum_{\substack{j=-\infty \\ j \equiv k+1 \bmod 2}}^{\infty} \frac{t_j - t_k}{|t_j - t_k|^2} \frac{2}{n} = 0.$$

Now we consider the second part of the sum in (8). First define

$$\text{(10)} \qquad l(s, \sigma) := \frac{h(s, \sigma) - 1}{|\sigma - s|^2}.$$

The function $l(t_k, \sigma)$ is continuous, because

$$\lim_{\sigma \to t_k} l(t_k, \sigma) = \frac{1}{6} \frac{\gamma'''(t_k)}{\gamma'(t_k)} - \frac{1}{4} \left(\frac{\gamma''(t_k)}{\gamma'(t_k)} \right)^2, \qquad k = 1, \ldots, n-1.$$

Thus

$$\begin{aligned}
\int_0^1 & \frac{w(\sigma)}{|\sigma - t_k|^2} (h(\sigma, t_k) - 1)\, d\sigma \\
&= \int_0^1 w(\sigma)\; l(t_k, \sigma)\, d\sigma \\
&\approx \sum_{\substack{j=1 \\ j \equiv k+1 \bmod 2}}^{n-1} \frac{2}{n}\; l(t_k, t_j) \xi_j \\
&= \sum_{\substack{j=1 \\ j \equiv k+1 \bmod 2}}^{n-1} \frac{2}{n} \left(\frac{\gamma'(t_j)\gamma'(t_k)}{|\gamma(t_j) - \gamma(t_k)|^2} - \frac{1}{|t_j - t_k|^2} \right) \xi_j.
\end{aligned}$$

Combining this with (9), we get the quadrature method (6) for the approximate solution of (7).

Now we define an approximate solution w_n for w by

$$w_n = \sum_{j=1}^{n-1} \xi_j \phi_j^{(d)}(s), \tag{11}$$

where $\phi_j^{(d)}$ denotes the interpolation basis of piecewise polynomials of degree d, i.e., $\phi_j^{(d)}(t_k) = \delta_{k,j}$, $\phi_j^{(d)}$ is continuous and the restriction of $\phi_j^{(d)}$ to the interval $[dk/n, d(k+1)/n]$ is a polynomial of degree d for $k = 0, \dots, \frac{n}{d} - 1$. Here we choose n such that $n/d \in \mathbf{N}$. If $d = 1$, we write ϕ_j instead of $\phi_j^{(1)}$. There holds

$$\phi_j(s) = \begin{cases} n(s - (j-1)/n), & s \in [(j-1)/n, j/n], \\ n((j-1)/n - s), & s \in [j/n, (j+1)/n], \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.3. The linear spline functions are the simplest splines with $\phi_j \in \tilde{H}_{1/2}$ and $\phi_j \circ \gamma^{-1} \in \tilde{H}_{1/2}$.

This result is an easy consequence of the definitions of ϕ_j , γ and $\tilde{H}_{1/2}$.

Let us denote the matrix of the linear system of equations (6) by A_n, i.e.,

$$\begin{aligned} A_n &= n(h(t_k, t_j) a_{k-j})_{k,j=1}^{n-1}, \\ a_k &= \begin{cases} -\pi^2/2, & k = 0, \\ 2/|k|^2, & k \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{12}$$

We shall interpret this matrix A_n as an operator from the discrete space of finite sequences equipped with the norm induced by $\tilde{H}_{1/2,\alpha}$ (see Equation (13), below) into its dual space. The definition of these discrete spaces is given in the next section.

3. The discrete spaces E_n and F_n. Proving the stability for the operator sequence $\{A_n\}$ means that we have to show the invertibility of the operators A_n and the uniform boundedness of their inverses. The operators A_n have to be considered from the discrete space of finite sequences equipped with the norm induced by $\tilde{H}_{1/2,\alpha}$ into its dual space. It will be easier to prove the stability by using some equivalent norms of these spaces. Before introducing these norms we mention some properties of the transformation γ.

Remark 3.1. The following results are readily shown and follow from definitions.

1. $\gamma(0) = 0$, $\gamma(1) = 1$, $\gamma(1/2) = 1/2$, $\gamma(s) = 1 - \gamma(1-s)$, and γ is strictly monotonically increasing.

2. $\gamma'(s) \sim s^{\alpha-1}$ if $s \in [0, 1/2]$, $\gamma'(s) \sim (1-s)^{\alpha-1}$ if $s \in [1/2, 1]$ and

$$\gamma'(s) = \frac{\alpha v'(s) v(s)^{\alpha-1} v(1-s)^{\alpha-1}}{[v(s)^\alpha + v(1-s)^\alpha]^2} > 0, \qquad 0 < s < 1.$$

3. $(\gamma'(s)s(1-s))/(\gamma(s)(1-\gamma(1-s))) \sim 1$. Therefore, if $s \in [0, n/2]$, then $(1/n)(\gamma'(s/n)/\gamma(s/n)) \sim 1/s$ and, if $s \in [n/2, n]$, then $(1/n)(\gamma'(s/n))/(1-\gamma(s/n)) \sim 1/(n-s)$.

4. If $s \in [0, \delta(\alpha)]$, then $\gamma'' > 0$; if $s \in [1-\delta(\alpha), 1]$, then $\gamma'' < 0$, with a $\delta(\alpha) \in (0, 1/2]$. If $\alpha \leq 86/30$, then we can choose $\delta(\alpha) = 1/2$.

5. If $s \in [\delta(\alpha)/2, 1 - \delta(\alpha)/2]$, then $\gamma'(s) \sim 1$.

6. $h(s, \sigma) \leq C$ and $h(s, s) = 1$. If $\alpha \leq 4.54$, then $h(s, \sigma) \leq 1$.

Concerning property 6, we make only a comment on the proof. In the case when $\tilde{\gamma}(s) = s^\alpha$,

$$h(s, \sigma) = \frac{\alpha^2 s^{\alpha-1} \sigma^{\alpha-1} (s-\sigma)^2}{(s^\alpha - \sigma^\alpha)^2}.$$

Setting $x = s/\sigma$, we get

$$h(s, \sigma) = \tilde{h}(x) := \frac{\alpha^2 x^{\alpha-1}(1-x)^2}{(1-x^\alpha)^2} = \frac{x^{\alpha-1}}{\left(\int_0^1 (x + h(1-x))^{\alpha-1} dh\right)^2}.$$

An easy calculation shows that

$$\int_0^1 (x + h(1-x))^{\alpha-1}\, dh \geq 2 \int_{1/2}^1 x^{(\alpha-1)/2}\, dh = x^{(\alpha-1)/2},$$

and we arrive at $h(s,\sigma) = \tilde{h}(x) \leq 1$. For the transformation γ introduced in (2) the proof seems to be more complicated. We have pictured the function $h(s,\sigma)$ for several α with the help of Maple V.

Now we study the mapping properties of the operators and the spaces, in which the operators are acting, more precisely. The operator D is a bijective and continuous mapping between $\tilde{H}_{1/2}(I)$ and $H_{-1/2}(I)$ which are dual spaces with respect to the L_2-scalar product. Then the operator A transformed according to (7) is a bijective and continuous mapping in the transformed spaces,

$$A : \tilde{H}_{1/2,\alpha}(I) \longrightarrow H_{-1/2,\alpha}(I).$$

Here the spaces $\tilde{H}_{1/2,\alpha}(I)$ and $H_{-1/2,\alpha}(I)$ are defined in a natural way by

$$\tilde{H}_{1/2,\alpha}(I) := \{\phi : \phi(t) = \psi(\gamma(t)), \psi \in \tilde{H}_{1/2}(I)\}, \tag{13}$$

$$H_{-1/2,\alpha}(I) := \{\tilde{\phi} : \tilde{\phi}(t) = \tilde{\psi}(\gamma(t))\gamma'(t), \tilde{\psi} \in H_{-1/2}(I)\} \tag{14}$$

and the norms are given by $|\phi|_{1/2,\alpha} := |\psi|_{1/2}$ and $|\tilde{\phi}|_{-1/2,\alpha} := |\tilde{\psi}|_{-1/2}$, respectively.

Because

$$\int_0^1 \phi(s)\phi_1(s)\,ds = \int_0^1 \phi(\gamma(s))\phi_1(\gamma(s))\gamma'(s)\,ds = \int_0^1 \psi(s)\psi_1(s)\,ds,$$

the spaces $\tilde{H}_{1/2,\alpha}(I)$ and $H_{-1/2,\alpha}(I)$ are dual with respect to the L_2-scalar product. The operator $A : w \mapsto g$ is mapping from $\tilde{H}_{1/2,\alpha}(I)$ into the space dual with respect to the L_2-scalar product.

We shall consider the approximate operators A_n in discrete spaces, using a notation which is similar to that in the theory of Vainikko [**24**].

Let us consider a sequence of Banach spaces E and E_n. Let $P = (p_n)_{n\in\mathbf{N}}$ be a sequence of operators $p_n : E' \to E_n$ where p_n is linear and $E' \subset E$ is a dense subset. By definition, a sequence $\{x_n\}$ with $x_n \in E_n$ is discretely P-convergent to $x \in E$, if and only if $\|x_n - p_n x\|_{E_n} \to 0$. This convergence is denoted by $x_n \xrightarrow{P} x$. A sequence $\{A_n\}_{n\in\mathbf{N}}$, $A_n \in L(E_n, F_n)$ is called discretely stable if there is a number n_0 such that $A_n^{-1} \in L(F_n, E_n)$ exists for all $n > n_0$ and the inverses are uniformly bounded. Using $\|A_n^{-1}\| \le C$, $n > n_0$ we get the following discrete estimate:

$$\begin{aligned}\|\tilde{w}_n - p_n w\|_{E_n} &= \|A_n^{-1}(A_n\tilde{w}_n - A_n p_n w)\|_{E_n}\\ &\le C\|q_n Aw - A_n p_n w\|_{F_n},\end{aligned}$$

if $w \in E' = \tilde{H}_{1/2,\alpha} \cap C(I)$ and $Aw = g \in F' = H_{-1/2,\alpha} \cap C(I)$. This is important in order to get a convergence result.

Here we set $E = \tilde{H}_{1/2,\alpha}(I)$, $F = H_{-1/2,\alpha}(I)$ and define the system of discrete spaces $(E_n)_{n\in\mathbf{N}}$ by $E_n := \{\{\xi_j\}_{j=1}^{n-1}\}$ equipped with the norm

$$\|\{\xi_j\}_{j=1}^{n-1}\|_{E_n} = \left\| \sum_{j=1}^{n-1} \xi_j \phi_j \right\|_{\tilde{H}_{1/2,\alpha}(I)}.$$

Let $P = (p_n)_{n\in\mathbf{N}}$ be the sequence of operators

$$p_n : E' := \tilde{H}_{1/2,\alpha}(I) \cap C(I) \to E_n, \qquad p_n(\phi) := \{\phi(t_j)\}_{j=1}^{n-1}. \tag{15}$$

For each fixed n there holds

$$\frac{1}{\sqrt{n}} \left(\sum_{j=1}^{n-1} |\xi_j|^2 \right)^{1/2} \sim \left\| \sum_{j=1}^{n-1} \xi_j \phi_j \right\|_{L_2}.$$

Thus we denote the finite l_2-space by $l_2(n) := \{\{\xi_j\}_{j=1}^{n-1}\}$, and equip it with the norm

$$\|\{\xi_j\}_{j=1}^{n-1}\|_{l_2(n)} = \frac{1}{\sqrt{n}} \left(\sum_{j=1}^{n-1} |\xi_j|^2 \right)^{1/2}$$

and the scalar product

$$\langle \{\xi_j\}_{j=1}^{n-1}, \{\eta_j\}_{j=1}^{n-1} \rangle_{l_2(n)} = \frac{1}{n} \sum_{j=1}^{n-1} \xi_j \eta_j.$$

Now it makes sense to define the second system of discrete spaces $(F_n)_{n\in\mathbf{N}}$ by $F_n := \{\{\xi_j\}_{j=1}^{n-1}\}$ equipped with the norm, which is dual to the norm of E_n with respect to the $l_2(n)$-scalar product. Analogously, we set $Q = (q_n)_{n\in\mathbf{N}}$ with

(16) $$q_n : F' := H_{-1/2,\alpha}(I) \cap C(I) \to F_n, \qquad q_n(\tilde{\phi}) := \{\tilde{\phi}(t_j)\}_{j=1}^{n-1}.$$

The approximate operators A_n are mappings in these dual discrete spaces, $A_n : E_n \to F_n$.

Theorem 3.1. *There holds*

(17)
$$\begin{aligned}\|\{\xi_j\}\|_{E_n}^2 \sim \sum_{l=1}^{n/2-1} \frac{1}{l}\xi_l^2 &+ \sum_{l=n/2}^{n-1} \frac{1}{n-l}\xi_l^2 \\ &+ \sum_{\substack{l,i=1\\ i\not\equiv l \bmod 2}}^{n-1} h\left(\frac{l}{n},\frac{i}{n}\right)\frac{1}{|\,i-l\,|^2}(\xi_l-\xi_i)^2 \\ &=: \langle B_n\{\xi_j\},\{\xi_j\}\rangle_{l_2(n)} = \|\sqrt{B_n}\{\xi_j\}\|_{l_2(n)}^2\end{aligned}$$

where B_n is a positive self-adjoint matrix. Additionally,

$$\|\{\xi_j\}\|_{l_2(n)} \sim \|\sqrt{B_n}\{\xi_j\}\|_{F_n}.$$

Proof. First we use the definitions of the norms in E_n and $\tilde{H}_{1/2,\alpha}$ and the definition of the linear splines ϕ_j to evaluate $\|\{\xi_j\}\|_{E_n}$. Then we obtain

(18)
$$\begin{aligned}\|\{\xi_i\}\|_{E_n}^2 = &\sum_{i,l=0}^{n-1}\int_i^{i+1}\int_l^{l+1}\frac{s_{l,i}^2(x,y)}{(x-y)^2}h\left(\frac{x}{n},\frac{y}{n}\right)dx\,dy \\ &+ 2\sum_{l=0}^{n-1}\int_l^{l+1}\frac{1}{n}\frac{(\xi_l(l+1-x)+\xi_{l+1}(x-l))^2}{\gamma(x/n)(1-\gamma(x/n))}\gamma'\left(\frac{x}{n}\right)dx\end{aligned}$$

with

$$s_{l,i}(x,y) := \xi_l(l+1-x)+\xi_{l+1}(x-l)-\xi_i(i+1-y)-\xi_{i+1}(y-i).$$

We set

$$S_{l,i} := \int_i^{i+1} \int_l^{l+1} \frac{s_{l,i}^2(x,y)}{(x-y)^2} h\left(\frac{x}{n}, \frac{y}{n}\right) dx\, dy. \tag{19}$$

Next we consider the second term of the sum (18). Because $(\gamma'(s)s(1-s))/(\gamma(s)(1-\gamma(s))) \sim 1$ (cf. Proposition 3 of Remark 3.1) and $1-x/n \sim 1$ if $x/n < 1/2$, $x/n \sim 1$ if $x/n > 1/2$, there holds

$$\begin{aligned}\int_l^{l+1} \frac{\gamma'(x/n)}{n} \frac{(\xi_l(l+1-x) + \xi_{l+1}(x-l))^2}{\gamma(x/n)(1-\gamma(x/n))}\, dx \\ \sim \int_l^{l+1} \frac{(\xi_l(l+1-x) + \xi_{l+1}(x-l)^2}{n(x/n)(1-x/n)}\, dx \\ \sim \begin{cases} (1/l)(\xi_l^2 + \xi_{l+1}^2), & l \le n/2 - 1 \\ (1/(n-l))(\xi_l^2 + \xi_{l+1}^2), & l \ge n/2. \end{cases}\end{aligned}$$

We arrive at

$$\begin{aligned}\sum_{l=0}^{n-1} \int_l^{l+1} \frac{1}{n} \frac{(\xi_l(l+1-x) + \xi_{l+1}(x-l))^2}{\gamma(x/n)(1-\gamma(x/n))} \gamma'(x/n)\, dx \\ \sim \sum_{l=1}^{n/2-1} \frac{1}{l}\xi_l^2 + \sum_{l=n/2}^{n-1} \frac{1}{n-l}\xi_l^2.\end{aligned} \tag{20}$$

Now we investigate $S_{l,i}$. Without loss of generality let $i \le l$. First we give a lemma. Its proof follows immediately from the definitions of h and γ and is left to the reader.

Lemma 3.1. *If* $t, s \in [0,1]$, $1 \le i$, $l \le n-2$, $|l-i| \ge 2$, *then*

$$h\left(\frac{l}{n}, \frac{i}{n}\right) \sim h\left(\frac{l+t}{n}, \frac{i+s}{n}\right). \tag{21}$$

If $x, y \in [1/n, (n-1)/n]$, $|x-y| \le 2/n$, *then* $h(x,y) \sim 1$.

First let $1 \le l$, $i \le n-2$, $|l-i| \ge 2$. Using Lemma 3.1 and the definition of $S_{l,i}$, we see that

$$S_{l,i} \sim h\left(\frac{l}{n}, \frac{i}{n}\right) \frac{1}{(l-i)^2} \int_i^{i+1} \int_l^{l+1} s_{l,i}^2(x,y)\, dx\, dy,$$

since $1/(x-y)^2 \sim 1/(l-i)^2$. With the definition of $s_{l,i}$, we compute

$$\int_i^{i+1}\int_l^{l+1} s_{l,i}^2(x,y)\,dx\,dy$$
$$= \frac{1}{3}\left(\xi_l^2+\xi_{l+1}^2+\xi_i^2+\xi_{i+1}^2+\xi_l\xi_{l+1}+\xi_i\xi_{i+1}\right)$$
$$-\frac{1}{2}\left(\xi_i\xi_l+\xi_l\xi_{i+1}+\xi_{l+1}\xi_i+\xi_{i+1}\xi_{l+1}\right).$$

This is a quadratic form which is zero when $\xi_i=\xi_l=\xi_{l+1}=\xi_{i+1}$ and therefore equivalent to the following quadratic form which is zero when $\xi_i=\xi_l=\xi_{l+1}=\xi_{i+1}$:

$$\int_i^{i+1}\int_l^{l+1} s_{l,i}^2(x,y)\,dx\,dy \sim (\xi_l-\xi_i)^2+(\xi_{l+1}-\xi_i)^2$$
$$+(\xi_l-\xi_{i+1})^2+(\xi_{l+1}-\xi_{i+1})^2.$$

Thus we get for $1\le l,\ i\le n-2,\ |l-i|\ge 2$

$$S_{l,i}\sim h\left(\frac{l}{n},\frac{i}{n}\right)\frac{1}{(l-i)^2}((\xi_l-\xi_i)^2 \tag{22}$$
$$+(\xi_{l+1}-\xi_i)^2+(\xi_l-\xi_{i+1})^2+(\xi_{l+1}-\xi_{i+1})^2).$$

Now we consider the second case $1\le i,\ l\le n-2,\ |l-i|<2$. Using the definition of $S_{l,i}$ and Lemma 3.1, we obtain

$$S_{l,l}\sim(\xi_l-\xi_{l+1})^2,\qquad 1\le i=l\le n-2, \tag{23}$$

and

$$S_{l,l-1}\sim\int_{l-1}^{l}\int_l^{l+1}\frac{s_{l,i}^2(x,y)}{(x-y)^2}\,dx\,dy$$
$$=(\xi_l^2-\xi_l(\xi_{l+1}+\xi_{l-1}))(3-4\ln 2)$$
$$+(\xi_{l+1}^2+\xi_{l-1}^2)(1-\ln 2)+\xi_{l-1}\xi_{l+1}(1-2\ln 2)$$

if $2\le l=i+1\le n-2$. This is a quadratic form which is zero when $\xi_l=\xi_{l+1}=\xi_{l-1}$ and therefore equivalent to the following quadratic form which is zero when $\xi_l=\xi_{l+1}=\xi_{l-1}$:

$$S_{l,l-1}\sim(\xi_l-\xi_{l-1})^2+(\xi_l-\xi_{l+1})^2. \tag{24}$$

Assume $i = 0$. For the present consideration let $l \geq 2$. Then there holds $1/(x-y)^2 \leq 4/l^2$ for all $y \in [0,1], x \in [l, l+1]$. Using this property and $h(x,y) \leq C$, we have

$$\begin{aligned} 0 \leq S_{l,0} &\leq \frac{C}{l^2} \int_0^1 \int_0^1 (\xi_l(1-x) + \xi_{l+1}x - \xi_1 y)^2 \, dx \, dy \\ &\leq \frac{C}{l^2}(\xi_l^2 + \xi_{l+1}^2 + \xi_1^2). \end{aligned}$$

Now let $i = 0$ and $l = 1$. Using $h(x,y) \leq C$ again, we get

$$\begin{aligned} 0 \leq S_{1,0} &\leq \int_0^1 \int_1^2 \frac{(\xi_1(2-x) + \xi_2(x-1) - \xi_1 y)^2}{(x-y)^2} \, dx \, dy \\ &\leq C(\xi_1^2 + \xi_2^2). \end{aligned}$$

Thus we arrive at

$$\begin{aligned} 0 \leq \sum_{l=1}^{n-1} S_{l,0} &\leq C\left(\sum_{l=1}^{n-1} \frac{1}{l} \xi_l^2 \right) \\ &\leq C\left(\sum_{l=1}^{n/2-1} \frac{1}{l} \xi_l^2 + \sum_{l=n/2}^{n-1} \frac{1}{n-l} \xi_l^2 \right). \end{aligned} \tag{25}$$

Analogously we show that

$$\begin{aligned} 0 \leq \sum_{l=1}^{n-1} S_{l,n-1} &\leq C\left(\sum_{l=1}^{n-1} \frac{1}{n-l} \xi_l^2 \right) \\ &\leq C\left(\sum_{l=1}^{n/2-1} \frac{1}{l} \xi_l^2 + \sum_{l=n/2}^{n-1} \frac{1}{n-l} \xi_l^2 \right). \end{aligned} \tag{26}$$

From (18), (19), (20), (22), (23), (24), (25) and (26) it follows that

$$\begin{aligned} \|\{\xi_j\}\|_{E_n}^2 \sim \sum_{l=1}^{n/2-1} \frac{1}{l} \xi_l^2 + \sum_{l=n/2}^{n-1} \frac{1}{n-l} \xi_l^2 + \sum_{l=1}^{n-1} (\xi_l - \xi_{l+1})^2 \\ + \sum_{\substack{l,i=1 \\ |i-l| \geq 2}}^{n-1} h\left(\frac{l}{n}, \frac{i}{n} \right) \frac{1}{|i-l|^2} (\xi_l - \xi_i)^2, \end{aligned}$$

because, for $|l-i| \geq 2$, there holds $1/|l-i|^2 \sim 1/|l-i-1|^2 \sim 1/|l-i+1|^2$ and $h(l/n, i/n) \sim h((l+1)/n, i/n) \sim h(l/n, (i+1)/n)$. Using $h(l/n, (l+1)/n) \sim 1$, we get

$$\|\{\xi_j\}\|_{E_n}^2 \sim \sum_{l=1}^{n/2-1} \frac{1}{l}\xi_l^2 + \sum_{l=n/2}^{n-1} \frac{1}{n-l}\xi_l^2 + \sum_{\substack{l,i=1 \\ i\neq l}}^{n/2-1} h\left(\frac{2l}{n}, \frac{2i}{n}\right) \frac{q(\xi_{2l}, \xi_{2l+1}, \xi_{2i}, \xi_{2i+1})}{|2i-2l|^2}$$

with

$$q(\xi_{2l}, \xi_{2l+1}, \xi_{2i}, \xi_{2i+1}) = (\xi_{2l} - \xi_{2i})^2 + (\xi_{2l} - \xi_{2i+1})^2 + (\xi_{2l+1} - \xi_{2i})^2 + (\xi_{2l+1} - \xi_{2i+1})^2.$$

Obviously, $q(\xi_{2l}, \xi_{2l+1}, \xi_{2i}, \xi_{2i+1})$ is a quadratic form which is zero when $\xi_{2l} = \xi_{2l+1} = \xi_{2i} = \xi_{2i+1}$ and thus equivalent to the quadratic form $(\xi_{2l} - \xi_{2i+1})^2 + (\xi_{2l+1} - \xi_{2i})^2 + (\xi_{2l} - \xi_{2l+1})^2 + (\xi_{2i} - \xi_{2i+1})^2$. Using this equivalence, we see that

$$\|\{\xi_j\}\|_{E_n}^2 \sim \sum_{l=1}^{n/2-1} \frac{1}{l}\xi_l^2 + \sum_{l=n/2}^{n-1} \frac{1}{n-l}\xi_l^2 + \sum_{\substack{l,i=1 \\ i\not\equiv l \bmod 2}}^{n-1} h\left(\frac{l}{n}, \frac{i}{n}\right) \frac{1}{|i-l|^2}(\xi_l - \xi_i)^2 + \sum_{l=1}^{n-2} \left(\sum_{i=1}^{n-2} h\left(\frac{l}{n}, \frac{i}{n}\right) \frac{1}{|i-l|^2} \right) (\xi_l - \xi_{l+1})^2.$$

Furthermore, there holds

$$\sum_{i=1}^{n-2} h\left(\frac{l}{n}, \frac{i}{n}\right) \frac{1}{|i-l|^2} \leq \frac{\pi^2}{3},$$

and we arrive at (cf. Lemma 3.1)

$$\|\{\xi_j\}\|_{E_n}^2 \sim \sum_{l=1}^{n/2-1} \frac{1}{l}\xi_l^2 + \sum_{l=n/2}^{n-1} \frac{1}{n-l}\xi_l^2 + \sum_{\substack{l,i=1 \\ i\not\equiv l \bmod 2}}^{n-1} h\left(\frac{l}{n}, \frac{i}{n}\right) \frac{1}{|i-l|^2}(\xi_l - \xi_i)^2,$$

which completes the proof of the theorem, because the second assertion follows immediately by duality. □

4. The stability of the quadrature method. With the help of Theorem 3.1 we are able to show that the sequence $\{A_n\}$ is discretely stable.

The mapping properties of $\sqrt{B_n}$ are a direct consequence of Theorem 3.1:

$$\sqrt{B_n} : E_n \longrightarrow l_2(n), \qquad \sqrt{B_n} : l_2(n) \longrightarrow F_n$$

are isomorphic mappings and their norms are independent of n.

Now the mapping $A_n : E_n \to F_n$ is invertible if and only if $A'_n := -A_n : E_n \to F_n$ is invertible. Furthermore, B_n is positive and selfadjoint. Thus $A'_n : E_n \to F_n$ is invertible if and only if $\sqrt{B_n}^{\,-1} A'_n \sqrt{B_n}^{\,-1}$ is invertible in $l_2(n)$. The last assertion is equivalent to the relation

$$\left\langle \sqrt{B_n}^{\,-1} A'_n \sqrt{B_n}^{\,-1} \xi, \xi \right\rangle_{l_2(n)} \sim \langle \xi, \xi \rangle_{l_2(n)}$$

for all $\xi = \{\xi_j\}_{j=1}^{n-1} \in l_2(n)$ which is equivalent to

$$\langle A'_n \xi, \xi \rangle_{l_2(n)} \sim \langle B_n \xi, \xi \rangle_{l_2(n)}.$$

Thus, existence and the uniform boundedness of the inverses of A_n $(n > n_0)$ follow directly from the subsequent theorem and Theorem 3.1.

Theorem 4.1. *If* $1.5 < \alpha \leq 4.54$*, then there holds*

$$\langle A'_n \xi, \xi \rangle \sim \sum_{l=1}^{n/2-1} \frac{1}{l} \xi_l^2 + \sum_{l=n/2}^{n-1} \frac{1}{n-l} \xi_l^2 + \sum_{\substack{l,i=1 \\ i \not\equiv l \bmod 2}}^{n-1} h\left(\frac{l}{n}, \frac{i}{n}\right) \frac{1}{|l-i|^2} (\xi_l - \xi_i)^2.$$

Remark 4.1. The restriction $\alpha \leq 4.54$ is not essential. It is only used in the proof to get $\sum_{i=1,\, i \not\equiv l \bmod 2}^{n-1} (2/(l-i)^2)(1 - h(l/n, i/n)) \geq 0$, which is clear if $h(s,\sigma) \leq 1$. However, numerical tests show that this sum is positive for greater α, too (approximately up to $\alpha = 20$). Because numerical computations with greater α are not of practical interest, we do not consider this case. We only remark that $h(s,\sigma) \leq 1$ holds in a neighborhood of the critical values, i.e., $s = \sigma$, $s = 0$ or $\sigma = 0$. Thus, it seems to be possible to show stability for greater α using a local principle.

Proof of Theorem 4.1. Using $h(x,y) = h(y,x)$ and $a_{l-i} = a_{i-l}$, we obtain

$$\begin{aligned}
\langle A_n'\xi,\xi\rangle_{l_2(n)} &= -\sum_{l,i=1}^{n-1} h\left(\frac{l}{n},\frac{i}{n}\right) a_{l-i}\xi_l\xi_i \\
&= \sum_{\substack{l,i=1\\ i\neq l}}^{n-1} h\left(\frac{l}{n},\frac{i}{n}\right) a_{l-i}\frac{1}{2}(\xi_l-\xi_i)^2 \\
&\quad -\sum_{l=1}^{n-1}\left(a_0+\sum_{\substack{i=1\\ i\neq l}}^{n-1} h\left(\frac{l}{n},\frac{i}{n}\right) a_{l-i}\right)\xi_l^2 \\
&= \sum_{\substack{l,i=1\\ i\not\equiv l \bmod 2}}^{n-1} h\left(\frac{l}{n},\frac{i}{n}\right)\frac{1}{|l-i|^2}(\xi_l-\xi_i)^2 + \sum_{l=1}^{n-1} R_l\xi_l^2
\end{aligned}$$

with

$$R_l := \frac{\pi^2}{2} - 2\sum_{\substack{i=1\\ i\not\equiv l \bmod 2}}^{n-1} h\left(\frac{l}{n},\frac{i}{n}\right)\frac{1}{|\, l-i\,|^2}.$$

It remains to show that $R_l \sim 1/l+1/(n-l)$. Because $h \leq 1$ for $\alpha \leq 4.54$ (see Remark 3.1,6), we have

$$\begin{aligned}
R_l &= \frac{\pi^2}{2} - 2\sum_{\substack{i=1\\ i\not\equiv l \bmod 2}}^{n-1} h\left(\frac{l}{n},\frac{i}{n}\right)\frac{1}{|l-i|^2} \\
&= \sum_{\substack{i=-\infty\\ i\not\equiv l \bmod 2}}^{0} \frac{2}{(l-i)^2} + \sum_{\substack{i=n\\ i\not\equiv l \bmod 2}}^{\infty} \frac{2}{(l-i)^2}
\end{aligned}$$

$$+ \sum_{\substack{i=1 \\ i \not\equiv l \bmod 2}}^{n-1} \frac{2}{(l-i)^2} \left(1 - h\left(\frac{l}{n}, \frac{i}{n} \right) \right) \geq 0.$$

Thus, there holds

$$R_l \sim \frac{1}{l} + \frac{1}{n-l} + \sum_{\substack{i=1 \\ i \not\equiv l \bmod 2}}^{n-1} \frac{2}{(l-i)^2} \left(1 - h\left(\frac{l}{n}, \frac{i}{n} \right) \right).$$

What remains to be shown is the estimate

$$\tilde{R}_l := \sum_{\substack{i=1 \\ i \not\equiv l \bmod 2}}^{n-1} 2\, r_{l,i} \leq C\frac{1}{l} + C\frac{1}{n-l},$$
$$r_{l,i} := \frac{1}{(l-i)^2} \left(1 - h\left(\frac{l}{n}, \frac{i}{n} \right) \right).$$

Using the definition of the function h and Taylor's theorem, simple technical considerations show that

$$r_{l,i} = \frac{1}{n^2} \frac{\tilde{r}_{l,i}}{\left(\int_0^1 \gamma'(l/n + h((i-l)/n))\, dh \right)^2} \tag{27}$$

with

$$\begin{aligned}
\tilde{r}_{l,i} := & \int_0^1 \gamma''\left(\frac{l}{n} + h\left(\frac{i-l}{n} \right) \right) dh \\
& \cdot \int_0^1 \int_0^1 \gamma''\left(\frac{i}{n} + uv\left(\frac{l-i}{n} \right) \right) dvu\, du \\
& + \gamma'\left(\frac{i}{n} \right) \int_0^1 \int_0^1 \int_0^1 \gamma'''\left(\frac{l}{n} + uvw\left(\frac{i-l}{n} \right) \right) dvu\, duw\, dw \\
& - \int_0^1 \int_0^1 \gamma''\left(\frac{i}{n} + uv\left(\frac{l-i}{n} \right) \right) dvu\, du \\
& \cdot \int_0^1 \int_0^1 \gamma''\left(\frac{l}{n} + uv\left(\frac{i-l}{n} \right) \right) dvu\, du.
\end{aligned}$$

Now we have to distinguish several cases. Choose a positive number $\varepsilon < 1/2$.

First let $l < n\varepsilon$. There holds $\gamma'(x) \sim x^{\alpha-1}$ if $x \in [0,1/2)$. Furthermore, for these l and arbitrary i, we have

$$\int_0^1 \gamma'\left(\frac{l}{n} + h\left(\frac{i-l}{n}\right)\right) dh \geq C\left(\frac{l+i}{n}\right)^{\alpha-1}. \tag{28}$$

Moreover there holds $\gamma''(x) \leq Cx^{\alpha-2}$, $\gamma'''(x) \leq Cx^{\alpha-3}$, and hence

$$\begin{aligned}\int_0^1 \gamma''\left(\frac{l}{n} + h\left(\frac{i-l}{n}\right)\right) dh &\leq C\int_0^1 \left(\frac{l}{n} + h\left(\frac{i-l}{n}\right)\right)^{\alpha-2} dh \\ &\leq C\left(\frac{l+i}{n}\right)^{\alpha-2}.\end{aligned}$$

Analogously, there holds

$$\int_0^1\int_0^1\int_0^1 \gamma'''\left(\frac{l}{n} + vuw\left(\frac{i-l}{n}\right)\right) dvu\,duw\,dw \leq C\left(\frac{l+i}{n}\right)^{\alpha-3}.$$

Using (27), (28) and the definitions of $r_{l,i}$ and $\tilde{R}_l$, we see that

$$r_{l,i} \leq \frac{C}{(l+i)^2}, \qquad \tilde{R}_l = 2\sum_{\substack{i=1\\ i\not\equiv l \bmod 2}}^{n-1} r_{l,i} \leq C\sum_{\substack{i=1\\ i\not\equiv l \bmod 2}}^{n-1} \frac{1}{(l+i)^2} \leq C\frac{1}{l}.$$

The second case $l > n(1-\varepsilon)$ can be reduced to the first case $l < n\varepsilon$ with the help of the relations

$$\begin{aligned}\gamma(x) &= 1-\gamma(1-x), &\qquad \gamma'(x) &= \gamma'(1-x),\\ \gamma''(x) &= -\gamma''(1-x), &\qquad \gamma'''(x) &= \gamma'''(1-x).\end{aligned}$$

In the third case $n\varepsilon < l < n(1-\varepsilon)$ it remains to show that $R_l \leq C/n$, because in this case $l \sim n, n-l \sim n$. The assertion $R_l \leq C/n$ is true if the function

$$\hat{R}_l(y) := \frac{(\gamma(x)-\gamma(y))^2 - \gamma'(x)\gamma'(y)(x-y)^2}{(\gamma(x)-\gamma(y))^2(x-y)^2}$$

is integrable for fixed $x = l/n$. Now $\hat{R}_l(y)$ can be transformed analogously to $r_{l,i}$ and we get

$$\hat{R}_l(y) := \frac{\tilde{R}_l(y)}{(\int_0^1 \gamma'(x+h(y-x))\,dh)^2}$$

with

$$\begin{aligned}\tilde{R}_l(y) := &\int_0^1 \gamma''(x+h(y-x))\,dh \\ &\cdot \int_0^1\int_0^1 \gamma''(y+uv(x-y))\,dvu\,du \\ &+ \gamma'(y)\int_0^1\int_0^1\int_0^1 \gamma'''(x+uvw(y-x))\,dvu\,duw\,dw \\ &- \int_0^1\int_0^1 \gamma'''(y+uv(x-y))\,dvu\,du \\ &\cdot \int_0^1\int_0^1 \gamma''(x+uv(y-x))\,dvu\,du.\end{aligned}$$

This term $\hat{R}_l(y)$ is integrable if the numerator $\tilde{R}_l(y)$ is integrable, because of

$$\int_0^1 \gamma'(x+h(y-x))\,dh \geq C, \qquad y \in [0,1].$$

For $y \in U_\delta(0)$, $U_\delta(z_1) := \{z : |z-z_1| < \delta$, there holds $\gamma''(y) \sim y^{\alpha-2}, \gamma'''(y) \sim y^{\alpha-3}$. We see that $\tilde{R}_l(y)$ is integrable if $y^{2\alpha-4}$ is integrable. Obviously, this is true if $\alpha > 3/2$. This completes the proof of Theorem 4.1. □

5. The convergence of the quadrature method. In this section we shall derive error estimates. We suppose $f \in C^\infty$ in order to get the explicit asymptotic behavior of the solution (cf. Lemma 2.2). First we remark that, due to the definition,

$$\|u-u_n\|_{1/2} = \|w-w_n\|_{1/2,\alpha}, \tag{29}$$

holds where u is the solution of (1), $w = u\circ\gamma$ is the solution of (7), w_n is the approximate solution of (7) defined by (6) and (11), and $u_n \circ \gamma = w_n$. Furthermore, $K_n^d = P_n^d p_n$ holds, where $K_n^d \in L(\tilde{H}_{1/2,\alpha})$ is the interpolation projector onto the piecewise polynomials of degree d, $p_n : \tilde{H}_{1/2,\alpha} \to E_n$ is the discretization operator defined by (15) and $P_n^d : E_n \to \tilde{H}_{1/2,\alpha}$ is the prolongation operator defined by $P_n^d\{\xi_j\}_{j=1}^{n-1} =$

$\sum_{j=1}^{n-1} \xi_j \phi_j^{(d)}$. So Equation (11) is equivalent to $w_n = P_n^d \tilde{w}_n$ with $\tilde{w}_n = \{\xi_j\}_{j=1}^{n-1}$ defined by (6). Using the triangle inequality, we obtain

$$\begin{aligned} \|u - u_n\|_{1/2} &= \|w - P_n^d \tilde{w}_n\|_{1/2,\alpha} \\ &\leq \|w - K_n^d w\|_{1/2,\alpha} + \|P_n^d(\tilde{w}_n - p_n w)\|_{1/2,\alpha}. \end{aligned} \tag{30}$$

First we estimate $\|K_n^d w - w\|_{1/2,\alpha}$. To this end, we use the following lemma.

Lemma 5.1. *For all $w \in \tilde{H}_{1/2}$, $\|w\|_{1/2,\alpha} \leq C\|w\|_{\frac{1}{2}}$ holds.*

Proof. If $u \in \tilde{H}_{1/2}$, $u(\gamma(t)) = w(t)$, then $\|w\|_{1/2,\alpha} = \|u\|_{1/2}$. Thus, we get

$$\begin{aligned} \|u\|_{1/2}^2 &= \int_0^1 \int_0^1 \frac{|u(x) - u(y)|^2}{|x-y|^2}\, dx\, dy + 2\int_0^1 \frac{|u(x)|^2}{x(1-x)}\, dx \\ &= \int_0^1 \int_0^1 \frac{|w(x) - w(y)|^2}{|x-y|^2}\, h(x,y)\, dx\, dy \\ &\quad + 2\int_0^1 \frac{|w(x)|^2}{x(1-x)} \frac{\gamma'(x) x(1-x)}{\gamma(x)(1-\gamma(x))}\, dx \\ &\leq C\left(\int_0^1 \int_0^1 \frac{|w(x) - w(y)|^2}{|x-y|^2}\, dx\, dy + 2\int_0^1 \frac{|w(x)|^2}{x(1-x)}\, dx \right) \\ &= C\|w\|_{1/2}, \end{aligned}$$

with the properties 3 and 6 of Remark 3.1. Thus,

$$\|w\|_{1/2,\alpha} = \|u\|_{1/2} \leq C\|w\|_{1/2}$$

and the lemma is proved. □

Due to [**22**, Section 5.31],

$$\|K_n^d w - w\|_{1/2,\alpha} \leq \|K_n^d w - w\|_{1/2} \leq C n^{1/2-s} \|w\|_s \tag{31}$$

holds if $w \in H^s$ and $1 \leq s \leq d+1$.

Now we consider the second term of the sum in (30).

Lemma 5.2. *There holds*

$$\|P_n^d \xi\|_{1/2,\alpha} \le C\|\xi\|_{E_n}$$

for all $\xi = \{\xi_j\} \in E_n$ *with a constant* C *independent of* n.

The proof runs analogously to the proof of the corresponding estimate in Theorem 3.1.

Using this lemma, we get

$$\|P_n^d(\tilde{w}_n - p_n w)\|_{1/2,\alpha} \le C\|\tilde{w}_n - p_n w\|_{E_n}. \tag{32}$$

Theorems 3.1 and 4.1 yield the discrete stability of the sequence $\{A_n\}_{n\in\mathbf{N}}$, $A_n \in L(E_n, F_n)$. Thus, we can estimate

$$\begin{aligned} \|\tilde{w}_n - p_n w\|_{E_n} &= \|A_n^{-1}(q_n A w - A_n p_n w)\|_{F_n} \\ &\le C\|A_n p_n w - q_n A w\|_{F_n}. \end{aligned} \tag{33}$$

The operators A and A_n can be represented in the form $A = D + L$ and $A_n = D_n + L_n$ with

$$\begin{aligned} Dw(s) &= \text{p.f.} \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2}\, d\sigma, \\ Lw(s) &= (A - D)w(s) = \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2}(h(s,\sigma) - 1)\, d\sigma \\ &= \int_0^1 w(\sigma) l(s,\sigma)\, d\sigma, \\ D_n &= n\,(a_{k-j})_{k,j=1}^{n-1}, \qquad L_n = A_n - D_n = n(a_{k-j}(h(t_k,t_j) - 1))_{k,j=1}^{n-1}. \end{aligned}$$

Obviously,

$$\|q_n A w - A_n p_n w\|_{F_n} \le \|q_n D w - D_n p_n w\|_{F_n} + \|q_n L w - L_n p_n w\|_{F_n}. \tag{34}$$

Because we have to estimate the norm $\|\cdot\|_{F_n}$, but no explicit formula for that norm is available, we shall use the following lemma.

Lemma 5.3. *Let* M *be an arbitrary, but fixed, real positive number, and let* $\alpha < M$. *Then there holds*

$$\|\psi\|_{F_n} \le C\|\psi\|_{l_\infty}$$

for all $\psi = \{\psi_j\}_{j=1}^{n-1} \in l_\infty$.

Proof. Due to Sobolev's embedding theorem, the mapping $E : H_{1/2} \to L_M$ is continuous. Thus by duality, the mapping $E : L_q \to H_{-1/2}$ is continuous for $q = M/(M-1)$. Let $g \in L_\infty$ be an arbitrary function. Using the definition of $H_{-1/2,\alpha}$, we get

$$\begin{aligned} \|g\|_{-1/2,\alpha} &= \left\| \frac{g \circ \gamma^{-1}}{\gamma' \circ \gamma^{-1}} \right\|_{-1/2} \\ &\le C \left\| \frac{g \circ \gamma^{-1}}{\gamma' \circ \gamma^{-1}} \right\|_{L_q} \\ &\le C \|g \circ \gamma^{-1}\|_{L_\infty} \left\| \frac{1}{\gamma' \circ \gamma^{-1}} \right\|_{L_q}. \end{aligned}$$

From the properties of γ we see that

$$\begin{aligned} \left\| \frac{1}{\gamma' \circ \gamma^{-1}} \right\|_{L_q}^q &= \int_0^1 \left| \frac{1}{\gamma'(\gamma^{-1}(t))} \right|^q dt \\ &= \int_0^1 \left| \frac{1}{\gamma'(t)} \right|^q \gamma'(t)\, dt \le C, \end{aligned}$$

since $\alpha < M$ and $q - 1 = 1/(M-1)$. We arrive at

$$\|g\|_{-1/2,\alpha} \le C \|g \circ \gamma^{-1}\|_{L_\infty} = C \|g\|_{L_\infty}.$$

By duality we get

$$\|f\|_{L_1} \le C \|f\|_{1/2,\alpha} \tag{35}$$

for arbitrary $f \in \tilde{H}_{1/2,\alpha}$. Using the norm equivalence of Theorem 3.1, we find

$$\begin{aligned} \left\| \sum_{j=1}^{n-1} \xi_j \phi_j \right\|_{L_1} &\le C \left\| \sum_{j=1}^{n-1} \xi_j \phi_j \right\|_{1/2,\alpha} = C \|\{\xi_j\}_{j=1}^{n-1}\|_{E_n} \\ &\le C \|\sqrt{B_n} \{\xi_j\}_{j=1}^{n-1}\|_{l_2(n)}. \end{aligned}$$

On the other hand,

$$\Big\| \sum_{j=1}^{n-1} \xi_j \phi_j \Big\|_{L_1} \sim \frac{1}{n} \sum_{j=1}^{n-1} |\xi_j| =: \|\{\xi_j\}_{j=1}^{n-1}\|_{l_1(n)}.$$

holds. Thus, we get

$$\|\{\xi_j\}_{j=1}^{n-1}\|_{l_1(n)} \le C \|\sqrt{B_n}\{\xi_j\}_{j=1}^{n-1}\|_{l_2(n)}. \tag{36}$$

Furthermore, we obtain

$$\begin{aligned}\|\psi\|_{F_n} &= \|\sqrt{B_n}^{\,-1}\psi\|_{l_2(n)} \le \sup_{\|\eta\|_{l_2(n)}\le 1} \langle \sqrt{B_n}^{\,-1}\psi, \eta\rangle_{l_2(n)} \\ &= \sup_{\|\sqrt{B_n}\zeta\|_{l_2(n)}\le 1} \langle \psi, \zeta\rangle_{l_2(n)} \le \sup_{\|\zeta\|_{l_1(n)}\le C} \langle \psi, \zeta\rangle_{l_2(n)} \le C\|\psi\|_{l_\infty},\end{aligned}$$

and the lemma is proved. □

Now choose a sufficiently large number M and assume $\alpha < M$. We can estimate

$$\begin{aligned}\|q_n Lw - L_n p_n w\|_{F_n} &\le C\|q_n Lw - L_n p_n w\|_{l_\infty} \\ &= C \Big\| \Big\{ \int_0^1 w(\sigma) l(\sigma, t_k)\, d\sigma \\ &\qquad - \frac{2}{n} \sum_{\substack{j=1 \\ j\equiv k+1 \bmod 2}}^{n-1} w(t_j) l(t_j, t_k) \Big\}_{k=1}^{n-1} \Big\|_{l_\infty} \\ &\le C \sup_{k=1,\dots,n-1} \Big| \int_0^1 w(\sigma) l(\sigma, t_k)\, d\sigma \\ &\qquad - \frac{2}{n} \sum_{\substack{j=1 \\ j\equiv k+1 \bmod 2}}^{n-1} w(t_j) l(t_j, t_k) \Big| \\ &\le C n^{-s} \sup_{k=1,\dots,n-1} \|l(\cdot, t_k) w\|_{W_1^s} \\ &\le C n^{-s} \sup_{0<t<1} \|l(\cdot, t) w\|_{W_1^s}\end{aligned} \tag{37}$$

if $0 \le s < \alpha/2$ and $w \in W_1^s$. Here W_1^s denotes the Sobolev space of power 1 and order s (cf. Triebel [**23**]). The last estimate is true, because w and its derivatives up to the order $\alpha/2$ are periodic, since the following lemma holds (see [**4**, pp. 109–110]).

Lemma 5.4. *Let $s > 0$. If $f \in W_1^s$ and $f^{(r)}(0) = f^{(r)}(1)$ for $r \le s$, then*

$$\left| \int_0^1 f(t)\, dt - \sum_{j=0}^{n-1} f(t_j) \frac{1}{n} \right| \le C n^{-s} \|f\|_{W_1^s}, \qquad t_j = \frac{j}{n}.$$

The function $l(t,\sigma)w(\sigma)$ and its derivatives up to the order s are periodic functions of σ if $s < \alpha/2$, because $(\partial/\partial\sigma)^k l(t,\cdot)$ is bounded for any fixed $t \in (0,1)$, $w(\sigma) \sim \sigma^{\alpha/2}$ in a neighborhood of 0, $w(\sigma) \sim (1-\sigma)^{\alpha/2}$ in a neighborhood of 1, and thus $w^{(s)}(0) = w^{(s)}(1)$ if $s < \alpha/2$. Consequently, we get Equation (37).

It remains to examine whether the norms $\|l(\cdot,t)w\|_{W_1^s}$ are uniformly bounded.

Lemma 5.5. *The mapping $\sigma \mapsto \sigma^{k+2}(1-\sigma)^{k+2}(\partial/\partial\sigma)^k l(\sigma,t)$ is uniformly bounded with respect to t for an arbitrary integer $k > 0$.*

Proof. First let $k = 0$. Then we have

$$l(\sigma,t) = \frac{\gamma'(\sigma)\gamma'(t)}{|\gamma(\sigma)-\gamma(t)|^2} - \frac{1}{|\sigma - t|^2}.$$

Using the Lagrange form of the remainder of the Taylor's series, we find

$$l(\sigma,t) \longrightarrow \frac{1}{6}\frac{\gamma'''(\sigma)}{\gamma'(\sigma)} - \frac{1}{4}\left(\frac{\gamma''(\sigma)}{\gamma'(\sigma)}\right)^2 \quad \text{if } t \longrightarrow \sigma \ne 0.$$

Obviously, the function $(1-\sigma)^2\sigma^2 l(\sigma,t)$ is uniformly bounded with respect to t if $\sigma \notin U_\varepsilon(0) \cup U_\varepsilon(1)$ and if $\sigma \in U_\varepsilon(0)$, $|t-\sigma| > \delta$. If $\sigma, t \in U_\varepsilon(0)$, $\sigma/a \ge t$ or $t \ge a\sigma$, $a > 1$, then $(1-\sigma)^2\sigma^2 l(\sigma,t) \le C$ follows from Remark 3.1.6. Using the Lagrange form of the remainder of the Taylor's series and the relations $\gamma'(\sigma) \sim \sigma^{\alpha-1}$, $\gamma''(\sigma) \sim \sigma^{\alpha-2}$, $\gamma'''(\sigma) \sim \sigma^{\alpha-3}$ in $U_\varepsilon(0)$, we get $(1-\sigma)^2\sigma^2 l(\sigma,t) \le C$ in the case $\sigma, t \in U_\varepsilon(0)$, $\sigma/a < t < a\sigma$. The number $a > 1$ have to be chosen in dependence of the constants in the mentioned equivalence relations. If $\sigma \in U_\varepsilon(1)$, then the uniform boundedness of $(1-\sigma)^2\sigma^2 l(\sigma,t)$ can be derived analogously.

Now let $k=1$. Then

$$\frac{\partial}{\partial\sigma} l(\sigma,t) = \frac{\gamma''(\sigma)\gamma'(t)}{|\gamma(\sigma)-\gamma(t)|^2} - 2\frac{\gamma'(\sigma)^2\gamma'(t)}{|\gamma(\sigma)-\gamma(t)|^3} + \frac{2}{|\sigma-t|^3}$$

holds. Using the Lagrange form of the remainder of Taylor's series again, we find

$$\frac{\partial}{\partial\sigma} l(\sigma,t) \to \frac{1}{3}\frac{\gamma^{(4)}(\sigma)}{\gamma'(\sigma)} - \frac{1}{2}\left(\frac{\gamma''(\sigma)}{\gamma'(\sigma)}\right)^2 \quad \text{if } t \to \sigma \neq 0.$$

Combining this assertion with $\gamma'(\sigma) \sim \sigma^{\alpha-1}$, $\gamma''(\sigma) \sim \sigma^{\alpha-2}$, $\gamma'''(\sigma) \sim \sigma^{\alpha-3}$ and $\gamma^{(4)}(\sigma) \sim \sigma^{\alpha-4}$ for $\sigma \in U_\varepsilon(0)$, we obtain the uniform boundedness of $(1-\sigma)^3\sigma^3(\partial/\partial\sigma)l(\sigma,t)$ analogously to the proof of the uniform boundedness of $(1-\sigma)^2\sigma^2 l(\sigma,t)$. By further differentiations of the formula for $l(\sigma,t)$ we get the assertion of the lemma. □

On the other hand, the solution $w(\sigma)$ of Equation (7) has an end point behavior like $w(\sigma) = \sigma^{\alpha/2}(1-\sigma)^{\alpha/2}\hat{g}(\sigma)$ with smooth $\hat{g}$ since the solution $u(\sigma)$ of Equation (1) can be written in the form $u(\sigma) = \sigma^{1/2}(1-\sigma)^{1/2}\tilde{g}(\sigma)$ (see Remark 2.2) and $w(\sigma) = u(\gamma(\sigma))$. This fact, together with Lemma 5.5, implies that

$$\sup_{0<t<1} \|l(t,\cdot)w\|_{W_1^s} < \infty \tag{38}$$

for any s satisfying $0 \le s < \alpha/2 - 1$. Indeed, consider, for example, the function $w(\sigma) = \sigma^{\alpha/2}$, and suppose s is an integer. Then

$$\begin{aligned}\left(\frac{\partial}{\partial\sigma}\right)^s (l(\sigma,t)\sigma^{\alpha/2}) &= \sum_{j=0}^{s} C_j \left(\frac{\partial}{\partial\sigma}\right)^j l(\sigma,t)\sigma^{\alpha/2-(s-j)} \\ &= \left(\sum_{j=0}^{s} C_j \sigma^{j+2}\left(\frac{\partial}{\partial\sigma}\right)^j l(\sigma,t)\right)\sigma^{\alpha/2-s-2}.\end{aligned}$$

The last function is integrable if $\alpha/2 - s - 2 > -1$, i.e., $s < \alpha/2 - 1$. If $s > 0$ is not an integer, then another straightforward argument including the special definition of the norm in W_1^s leads to the same result. Together with the estimate (37), we arrive at

$$(39) \qquad \|q_n Lw - L_n p_n w\|_{F_n} \le C n^{-s}, \qquad 0 \le s < \alpha/2 - 1.$$

It remains to estimate $\|q_n Dw - D_n p_n w\|_{F_n}$. Using Sobolev's embedding theorem and Lemma 5.3, we get

$$(40) \qquad \begin{aligned} \|q_n Dw - D_n p_n w\|_{F_n} &\le C\|q_n Dw - D_n p_n w\|_{l_\infty} \\ &\le C\|P_n q_n Dw - P_n D_n p_n w\|_{L_\infty} \\ &= C\|K_n Dw - P_n D_n p_n w\|_{L_\infty} \\ &\le C\|K_n^R D^R w - P_n^R D_n^R p_n^R w\|_{H_{1/2+\varepsilon}} \end{aligned}$$

with the following definitions. We set

$$P_n(\{\xi_j\}_{j=1}^{n-1}) := \sum_{j=1}^{n-1} \xi_j \psi_j; \qquad t_j = j/n,$$

where ψ_j is the smoothest spline on $\mathbf{R}$ of order $\tilde{d}$ with $\psi_j(k/n) = \delta_{j,k}$, $k \in \mathbf{Z}$, i.e., ψ_j is $(\tilde{d}-1)$-times continuously differentiable and $\psi_j|_{(t_k, t_{k+1})}$ is a polynomial of degree $\tilde{d}$,

$$P_n^R(\{\xi_j\}_{j=-\infty}^{\infty}) := \sum_{j=-\infty}^{\infty} \xi_j \psi_j, \qquad p_n^R f = q_n^R f = \{f(t_j)\}_{j=-\infty}^{\infty},$$

$$K_n := P_n p_n = P_n q_n, \qquad K_n^R := P_n^R p_n^R = P_n^R q_n^R,$$

$$D^R f(t) := p.f. \int_{-\infty}^{\infty} \frac{f(\sigma)}{|\sigma - t|^2}\, d\sigma, \qquad D_n^R := n(a_{k-j})_{k,j=-\infty}^{\infty} = nC(a),$$

with

$$a_k = \begin{cases} -\pi^2/2 & k = 0, \\ 2/|k|^2 & k \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases} \qquad a(t) = a(e^{is}) = -\pi|s|; \quad -\pi < s \le \pi.$$

Here $C(a)$ denotes the convolution matrix generated by the Fourier coefficients of a. The spline order $\tilde{d}$ is only of technical importance in the proof. It has to be chosen large enough. Here the estimate (40) is true for arbitrary $\tilde{d}$. The operators K_n and K_n^R are the interpolation projectors onto the space of smoothest splines of order $\tilde{d}$. Note that the last estimate in (40) is true, because $K_n Dw - P_n D_n p_n w$ is a restriction of $K_n^R D^R w - P_n^R D_n^R p_n^R w$.

Lemma 5.6. *If $3/2 < s \le \tilde{d} + 2$, and if ε is arbitrarily small, but fixed, then there is a constant $C > 0$ such that*

$$\|K_n^R D^R f - P_n^R D_n^R p_n^R f\|_{H_{1/2+\varepsilon}} \le C n^{3/2+\varepsilon-s} \|f\|_{H_s}$$

for any $f \in H_s(R)$.

Proof. It is well known that the vector $(e^{-i\xi j})_{j=-\infty}^{\infty}$, $-\pi < \xi \le \pi$, is an eigenvector of the convolution operator $C(a) = (a_{k-j})_{k,j=-\infty}^{\infty}$ corresponding to the eigenvalue $a(e^{i\xi}) = -\pi|\xi|$. Furthermore, for $g^\xi(t) := e^{-i\xi t}$, we obtain

$$p_n^R g^\xi = \{e^{-i\xi \frac{j}{n}}\}_{j=-\infty}^{\infty}, \qquad C(a) p_n^R g^\xi = a(e^{i\frac{\xi}{n}}) p_n^R g^\xi.$$

Obviously, we also have

$$P_n^R n C(a) p_n^R g^\xi = n a(e^{i\frac{\xi}{n}}) P_n^R p_n^R g^\xi = n a(e^{i\frac{\xi}{n}}) K_n^R g^\xi.$$

Let F denote the usual Fourier transform

$$(Ff)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{it\xi}\, dt.$$

Then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ff)(\xi) g^\xi(t)\, d\xi$$

holds, and therefore

$$\begin{aligned} P_n^R n C(a) p_n^R f &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ff)(\xi) (P_n^R n C(a) p_n^R g^\xi)\, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ff)(\xi) n a(e^{i\frac{\xi}{n}}) K_n^R g^\xi\, d\xi. \end{aligned}$$

On the other hand,

$$(D^R f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\pi\xi \text{ sign}(\xi)(Ff)(\xi) g^{\xi}(t)\, d\xi$$

(cf. [**17**, Section 1.6]), and thus

$$(K_n^R D^R f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\pi\xi \text{ sign}(\xi)(Ff)(\xi)(K_n^R g^{\xi})(t)\, d\xi.$$

Consequently, we get

$$P_n^R n C(a) p_n^R f - K_n^R D^R f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi)(K_n^R g^{\xi})(t)\, d\xi,$$

with $r_n(\xi) = na(e^{i\xi/n}) + \pi|\xi|$. With the definition of a we see that

$$|r_n(\xi)| \le C|\xi|^l/n^{l-1}, \qquad l \ge 1. \tag{41}$$

Obviously,

$$\begin{aligned} &\|P_n^R n C(a) p_n^R f - K_n^R D^R f(t)\|_{H_{1/2+\varepsilon}} \\ &\quad \le \|(K_n^R - L_n^R)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi) g^{\xi}(t)\, d\xi\|_{H_{1/2+\varepsilon}} \\ &\qquad + \left\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi) g^{\xi}(t)\, d\xi \right\|_{H_{1/2+\varepsilon}}, \end{aligned} \tag{42}$$

holds, where L_n^R denotes the orthoprojection onto the spline space $\text{lin}\,\{\psi_j\}_{j=-\infty}^{\infty}$. Using (41), we get the following estimate for the second term of the sum

$$\begin{aligned} &\left\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi) g^{\xi}\, d\xi \right\|_{H_{1/2+\varepsilon}} \\ &\qquad \le \|(Ff)(\xi) r_n(\xi) \sqrt{1+\xi^2}^{1/2+\varepsilon}\|_{L_2} \\ &\qquad \le C n^{1-l} \|(Ff)(\xi) \sqrt{1+\xi^2}^{l+1/2+\varepsilon}\|_{L_2} \\ &\qquad \le C n^{1-l} \|f\|_{H_{l+1/2+\varepsilon}} \\ &\qquad \le C n^{3/2+\varepsilon-s} \|f\|_{H_s} \end{aligned} \tag{43}$$

with $l = s - 1/2 - \varepsilon$.

From [**22**, Section 2], we see that, for $1/2 + \varepsilon \le s_1 \le \tilde{d} + 1$,

$$\begin{aligned}(44)\quad \Big\|(K_n^R - L_n^R)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi)g^{\xi}\,d\xi\Big\|_{H_{1/2+\varepsilon}} &\le Cn^{1/2+\varepsilon-s_1}\Big\|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi)g^{\xi}\,d\xi\Big\|_{H_{s_1}} \\ &\le Cn^{1/2+\varepsilon-s_1}\|(Ff)(\xi)r_n(\xi)\sqrt{1+\xi^2}^{s_1}\|_{L_2} \\ &\le Cn^{1/2+\varepsilon-s_1}\|(Ff)(\xi)\sqrt{1+\xi^2}^{s_1+1}\|_{L_2} \\ &\le Cn^{1/2+\varepsilon-s_1}\|f\|_{H_{s_1+1}} = Cn^{3/2+\varepsilon-s}\|f\|_{H_s},\end{aligned}$$

holds for all $s = s_1 + 1$ with $3/2 < s \le \tilde{d} + 2$. This completes the proof of the lemma. □

Note that the estimate (40) is true for arbitrary $\tilde{d}$ and using Lemma 5.6, we arrive at

$$(45)\qquad \|q_n D w - D_n p_n w\|_{F_n} \le Cn^{3/2+\varepsilon-s}\|w\|_{H_s},$$

if $3/2 < s$ and $w \in H_s$. From the estimates (30), (31), (32), (33), (34), (39) and (45) we obtain the following theorem.

Theorem 5.1. *Let $0 < s < \min\{d, \alpha/2 - 1\}$. If u is the solution of Equation (1), w the solution of (7), w_n the solution of the quadrature equation (6) extended by (11) and $u_n := w_n \circ \gamma^{-1}$, then there holds*

$$(46)\qquad \|u - u_n\|_{1/2} = \|w - w_n\|_{1/2,\alpha} \le Cn^{-s},$$

if $w \in H_{s+3/2+\varepsilon}$ for some positive ε.

Remark 5.1. If f is sufficiently smooth, then there holds $w \in H_{s+3/2+\varepsilon}$ for $s < \alpha/2 - 1$.

This is an easy consequence of the definition of w and Remark 2.2.

6. Another quadrature method for the hypersingular integral equation. In this section, we shall propose another quadrature method for the Equation (1), using a cos-transformation. We shall proceed as in Section 3.8 of [**22**], where the numerical solution of first-kind integral equations with logarithmic kernel is treated.

Consider the hypersingular integral equation (1). Now we change the variables by using another transformation function γ_1:

$$\gamma_1(s) := \frac{1-\cos\pi s}{2}; \qquad s \in [0,1].$$

As in Section 2, we have that Equation (1) is equivalent to

$$Aw(s) := \text{p.f.} \int_0^1 \frac{\gamma_1'(s)\gamma_1'(\sigma)}{|\gamma_1(\sigma)-\gamma_1(s)|^2}\, w(\sigma)\, d\sigma = g(s) \tag{47}$$

with

$$w(s) := u(\gamma_1(s)), \qquad g(s) := f(\gamma_1(s))\gamma_1'(s).$$

There holds

$$\begin{aligned}\frac{\gamma_1'(s)\gamma_1'(\sigma)}{|\gamma_1(\sigma)-\gamma_1(s)|^2} &= \frac{\pi^2}{4}\frac{\sin\pi\sigma\sin\pi s}{((\cos\pi s-\cos\pi\sigma)/2)^2}\\ &= \frac{\pi^2}{4}\left(\frac{1}{\sin^2\pi(s-\sigma)/2} - \frac{1}{\sin^2\pi(s+\sigma)/2}\right).\end{aligned}$$

Thus,

$$Aw(s) = \text{p.f.} \int_0^1 \frac{\pi^2}{4}\left(\frac{1}{\sin^2\pi(s-\sigma)/2} - \frac{1}{\sin^2\pi(s+\sigma)/2}\right) w(\sigma)\, d\sigma = g(s). \tag{48}$$

Analogously to Section 2, we can derive the quadrature method for n even

$$g(t_k) = \sum_{\substack{j=1\\ j\equiv k+1 \bmod 2}}^{n-1} \frac{\pi^2}{2n}\left(\frac{1}{\sin^2\pi(t_k-t_j)/2} - \frac{1}{\sin^2\pi(t_k+t_j)/2}\right)\xi_j - \frac{n\pi^2}{2}\xi_k, \tag{49}$$

with $k = 1, \ldots, n-1$.

The kernel function $(\gamma_1'(s)\gamma_1'(\sigma))/|\gamma_1(\sigma)-\gamma_1(s)|^2$ of A is 2-periodic and odd with respect to each variable over the interval $[-1,1]$. For real t, let H^t denote the Sobolev space of 2-periodic functions (distributions). We will especially be interested in the subspace H_o^t of odd functions, $H_o^t = \{f \in H^t : f(-s) = -f(s)\}$. There holds

$$(A_o w)|_{[0,1]} = A(w|_{[0,1]}), \qquad w \in H_o^t, \tag{50}$$

where

$$A_o w(s) := \frac{\pi^2}{4} \text{p.f.} \int_{-1}^{1} \frac{w(\sigma)}{\sin^2 \pi(s-\sigma)/2}\, d\sigma. \tag{51}$$

An easy computation shows that A_o maps odd functions into odd functions and even functions into even functions.

We continue g to an odd function on $[-1,1]$ by $g(-s) := -g(s)$ and set $\xi_{-k} := -\xi_k; k = 1, \dots, n-1$, which corresponds to an odd continuation of w. Then the quadrature method (49) is equivalent to

$$g(t_k) = \sum_{\substack{j=1-n \\ j \equiv k+1 \bmod 2}}^{n-1} \frac{\pi^2}{2n} \frac{1}{\sin^2 \pi(t_k - t_j)/2} \xi_j - \frac{n\pi^2}{2} \xi_k, \qquad 0 = \xi_0, \tag{52}$$

with $k = 1-n, \dots, n-1$, $k \neq 0$. The restriction $k \neq 0$ can be omitted, because $g(0) = f(\gamma_1(0))\gamma_1'(0) = 0$ and the sum of the righthand side of (52) is zero if $k = 0$, because of $\xi_j = -\xi_{-j}$.

Furthermore, A_o is the hypersingular integral operator on the unit circle (cf. [**13**]). Stability and error estimates for the following quadrature method applied to this hypersingular integral are proved in [**13**]:

$$\begin{aligned} g(t_k) &= \sum_{\substack{j=1-n \\ j \equiv k+1 \bmod 2}}^{n-1} \frac{\pi^2}{2n} \frac{1}{\sin^2 \pi(t_k - t_j)/2} \xi_j - \frac{n\pi^2}{2} \xi_k, \\ 0 &= \sum_{j=1-n}^{n-1} \xi_j \end{aligned} \tag{53}$$

with $k = 1-n, \dots, n-1$. In the papers of Kress [**13**] and Prošsdorf and Saranen [**21**], it is shown that the product integration formula leads to the same quadrature method. The quadrature methods (52) and (53) differ from each other only by the one-dimensional functional which guarantees the uniqueness of the solution. So we get the stability of (52) and of (49) by perturbation theorems (cf. [**17**]). For the quadrature method (53), the following convergence estimate is proved in [**13**]

$$(54) \qquad \|w - w_n\|_r \le Cn^{r-s}\|w\|_s, \qquad \|w - w_n\|_{1/2} \le Cn^{1-s}\|w\|_s,$$

provided $w \in H_s$ and $s > 3/2$, $s \ge r \ge 1$. Repeating the arguments of [**13**], we get the same convergence estimate for (52) and for (49). Thus, cf. Lemma 5.1,

$$\|u - u_n\|_{1/2} = \|w - w_n\|_{1/2,\alpha} \le \|w - w_n\|_{1/2} \le Cn^{1-s}\|w\|_s,$$

if $w \in H_s$ and $s > 3/2$.

7. Numerical results. We consider numerical tests for the functions $f = f_i$, $i = 1, 2, 3$, with $f_1(t) = -2\pi$, $f_2(t) = -8\pi(1-2t)$ and $f_3(t) = (0.00001 + t^3)^{1/2}$. Clearly, we get $Du = f$, $u = u_i$, $i = 1, 2$ with $u_1(t) = 2t^{1/2}(1-t)^{1/2}$ and $u_2(t) = 4t^{1/2}(1-t)^{1/2}(1-2t)$. Because we do not know the exact solution in the case $i = 3$, we compute the error by setting $u \sim u_{2048}$, where u_{2048} is the solution of (6) with $n = 2048$. With the corresponding function $g(s) = f(\gamma(s))\gamma'(s)$, we have computed u_n by using the quadrature method (6). To solve the equation system of (6) we use the conjugate-gradient-method (cf., e.g., [**10**]).

In Tables 1–3, we give the supremum-error

$$\mathrm{err}_n := \sup_{j=1,\dots,n-1} |u(\gamma(t_j)) - u_n(\gamma(t_j))|$$

as well as the associated experimental convergence orders

$$\mathrm{ord}_n := -\frac{\log(\mathrm{err}_n) - \log(\mathrm{err}_{n/2})}{\log(n) - \log(n/2)}$$

for $\alpha = 2$, $\alpha = 3$, $\alpha = 4$ and for the cos-transformation. All computations are performed with a maximum accuracy of $9 \cdot 10^{-11}$. When the error err_n is close to 10^{-10}, then ord_n decreases with increasing n. But this is a numerical effect, and we only list the computations, where $\mathrm{err}_n > 10^{-9}$.

Theorem 4.1 guarantees a convergence of order s with $0 < s < \alpha/2-1$. During the estimation, we have used Sobolev's embedding theorem and hence we expect that our convergence order is not optimal. In fact, we expect an order $\alpha/2$ instead of $\alpha/2-1$. The numerical results seem to confirm this hypothesis. In all tests the numerical convergence rate for $\alpha = 2$ is better than expected; indeed, ord_n is approximately 2. For $\alpha = 2$, the transformed function w is smooth, because the end-point behavior of u is $t^{1/2}(1-t)^{\frac{1}{2}}$, and thus the better convergence order in this case is not surprising.

In the case of the cos-transformation, an arbitrarily high convergence order is expected. The numerical tests in the case of f_1 and f_2 show an error close to the accuracy $\mathrm{err}_n \approx 10^{-11}$ already for $n = 4$. This results from the fact that our quadrature method is equivalent to the method for the unit circle (see [**13**]), and the function w belongs to the set of trial functions. Thus, the error cannot be improved and we get a meaningless value for the convergence order. The numerical results in the case f_3 show that the order is increasing.

Finally we give an example for the hypersingular integral equation on a smooth open curve Γ. Let β denote the parametrization of Γ. Then

$$\begin{aligned} f(t) = D_\Gamma u(t) &= \text{p.f.} \int_\Gamma \frac{u(\tau)}{|t-\tau|^2}\, d\tau \\ &= \text{p.f.} \int_0^1 \frac{u(\beta(\sigma))}{|\beta(s)-\beta(\sigma)|^2} |\beta'(\sigma)|\, d\sigma, \end{aligned}$$

TABLE 1. $u_1(t) = 2t^{1/2}(1-t)^{1/2}, \quad f_1(t) = -2\pi.$

	$\alpha = 2$		$\alpha = 3$		$\alpha = 4$		cos-trans.	
n	err_n	ord_n	err_n	ord_n	err_n	ord_n	err_n	ord_n
4	$1.22 \cdot 10^{-2}$		$5.71 \cdot 10^{-2}$		$7.75 \cdot 10^{-2}$		$7.05 \cdot 10^{-11}$	
8	$2.94 \cdot 10^{-3}$	2.05	$1.51 \cdot 10^{-2}$	1.92	$1.29 \cdot 10^{-2}$	2.28		
16	$7.05 \cdot 10^{-4}$	2.06	$5.03 \cdot 10^{-3}$	1.59	$3.03 \cdot 10^{-3}$	2.09		
32	$1.74 \cdot 10^{-4}$	2.02	$1.73 \cdot 10^{-3}$	1.53	$7.42 \cdot 10^{-4}$	2.02		
64	$4.31 \cdot 10^{-5}$	2.01	$6.06 \cdot 10^{-4}$	1.51	$1.84 \cdot 10^{-4}$	2.01		
128	$1.07 \cdot 10^{-5}$	2.00	$2.13 \cdot 10^{-4}$	1.50	$4.59 \cdot 10^{-5}$	2.00		
256	$2.68 \cdot 10^{-6}$	2.00	$7.51 \cdot 10^{-5}$	1.50	$1.14 \cdot 10^{-5}$	2.00		
512	$6.69 \cdot 10^{-7}$	2.00	$2.65 \cdot 10^{-5}$	1.50	$2.86 \cdot 10^{-6}$	2.00		
1024	$1.67 \cdot 10^{-7}$	2.00	$9.37 \cdot 10^{-6}$	1.50	$7.14 \cdot 10^{-7}$	2.00		
2048	$5.10 \cdot 10^{-8}$	1.71	$3.31 \cdot 10^{-6}$	1.50	$1.79 \cdot 10^{-7}$	2.00		

TABLE 2. $u_2(t) = 4t^{1/2}(1-t)^{1/2}(1-2t)$, $f_2(t) = -8\pi(1-2t)$.

	$\alpha = 2$		$\alpha = 3$		$\alpha = 4$		cos-trans.	
n	err_n	ord_n	err_n	ord_n	err_n	ord_n	err_n	ord_n
4	$1.78 \cdot 10^{-2}$		$1.31 \cdot 10^{-1}$		$2.28 \cdot 10^{-1}$		$0.00 \cdot 10^{0}$	
8	$5.90 \cdot 10^{-3}$	1.60	$3.01 \cdot 10^{-2}$	2.12	$2.56 \cdot 10^{-2}$	3.16	$7.04 \cdot 10^{-11}$	
16	$1.41 \cdot 10^{-3}$	2.06	$1.00 \cdot 10^{-2}$	1.58	$6.05 \cdot 10^{-3}$	2.07		
32	$3.47 \cdot 10^{-4}$	2.02	$3.47 \cdot 10^{-3}$	1.53	$1.48 \cdot 10^{-3}$	2.03		
64	$8.61 \cdot 10^{-5}$	2.01	$1.21 \cdot 10^{-3}$	1.52	$3.68 \cdot 10^{-4}$	2.01		
128	$2.15 \cdot 10^{-5}$	2.00	$4.26 \cdot 10^{-4}$	1.51	$9.17 \cdot 10^{-5}$	2.00		
256	$5.36 \cdot 10^{-6}$	2.00	$1.50 \cdot 10^{-4}$	1.50	$2.29 \cdot 10^{-5}$	2.00		
512	$1.34 \cdot 10^{-6}$	2.00	$5.30 \cdot 10^{-5}$	1.50	$5.72 \cdot 10^{-6}$	2.00		
1024	$3.35 \cdot 10^{-7}$	2.00	$1.87 \cdot 10^{-5}$	1.50	$1.43 \cdot 10^{-6}$	2.00		
2048	$8.37 \cdot 10^{-8}$	2.00	$6.62 \cdot 10^{-6}$	1.50	$3.75 \cdot 10^{-7}$	2.00		

TABLE 3. $f_3(t) = -(0.00001 + t^3)^{1/2}$.

	$\alpha = 2$		$\alpha = 3$		$\alpha = 4$		cos-trans.	
n	err_n	ord_n	err_n	ord_n	err_n	ord_n	err_n	ord_n
4	$1.27 \cdot 10^{-3}$		$6.82 \cdot 10^{-3}$		$1.05 \cdot 10^{-2}$		$4.38 \cdot 10^{-5}$	
8	$3.20 \cdot 10^{-4}$	1.99	$1.63 \cdot 10^{-3}$	2.06	$1.39 \cdot 10^{-3}$	2.92	$3.24 \cdot 10^{-6}$	3.74
16	$7.64 \cdot 10^{-5}$	2.07	$5.43 \cdot 10^{-4}$	1.58	$3.27 \cdot 10^{-4}$	2.09	$1.62 \cdot 10^{-7}$	4.31
32	$1.88 \cdot 10^{-5}$	2.02	$1.87 \cdot 10^{-4}$	1.53	$8.02 \cdot 10^{-5}$	2.03	$3.90 \cdot 10^{-9}$	5.38
64	$4.65 \cdot 10^{-6}$	2.01	$6.55 \cdot 10^{-5}$	1.52	$1.99 \cdot 10^{-5}$	2.01		
128	$1.16 \cdot 10^{-6}$	2.00	$2.30 \cdot 10^{-5}$	1.51	$4.96 \cdot 10^{-6}$	2.00		
256	$2.89 \cdot 10^{-7}$	2.00	$8.11 \cdot 10^{-6}$	1.50	$1.23 \cdot 10^{-6}$	2.00		
512	$7.16 \cdot 10^{-8}$	2.01	$2.83 \cdot 10^{-6}$	1.52	$3.08 \cdot 10^{-7}$	2.00		
1024	$1.69 \cdot 10^{-8}$	2.08	$9.34 \cdot 10^{-7}$	1.60	$7.12 \cdot 10^{-8}$	2.11		

holds, and

$$f(\beta(s))|\beta'(s)| = \text{p.f.} \int_0^1 \frac{u(\beta(\sigma))}{|s-\sigma|^2}\, d\sigma + \int_0^1 \kappa(s,\sigma) u(\beta(\sigma))\, d\sigma$$

with

$$\kappa(s,\sigma) = \frac{|\beta'(\sigma)\beta'(s)|}{|\beta(s)-\beta(\sigma)|^2} - \frac{1}{|s-\sigma|^2}.$$

Transforming this equation with the help of γ, we get

$$g(s) = \text{p.f.} \int_0^1 \frac{w(\sigma)}{|s-\sigma|} h(s,\sigma)\, d\sigma + \int_0^1 k(s,\sigma) w(\sigma)\, d\sigma,$$

with

$$w(s) = u(\beta(\gamma(s))), \qquad g(s) = f(\beta(\gamma(s)))|\beta'(\gamma(s))|\gamma'(s)$$
$$h(s,\sigma) = \frac{|s-\sigma|^2 \gamma'(s)\gamma'(\sigma)}{|\gamma(s)-\gamma(\sigma)|^2}$$
$$k(s,\sigma) = \frac{|\beta'(\gamma(\sigma))\beta'(\gamma(s))|\gamma'(s)\gamma'(\sigma)}{|\beta(\gamma(s))-\beta(\gamma(\sigma))|^2} - \frac{\gamma'(s)\gamma'(\sigma)}{|\gamma(s)-\gamma(\sigma)|^2}.$$

We consider the example where Γ is a circular arc, $\beta(s) = e^{i\pi s}$, $0 \le s \le 1$. Then k takes the form

$$k(s,\sigma) = \left(\frac{\pi}{4\sin^2 \pi(\gamma(s)-\gamma(\sigma)/2)} - \frac{1}{|\gamma(s)-\gamma(\sigma)|^2} \right) \gamma'(s)\gamma'(\sigma).$$

In Table 4 we give the supremum error and the associated experimental convergence orders for the example function $f_4(t) = -2\pi$. Because we do not know the exact solution in this case, we compute the error again by setting $u \sim u_{2048}$. The results in Table 4 show the same behavior as for the case of a straightline segment.

TABLE 4. $f_4(t) = -2 * \pi$. The case of the circle arc.

	$\alpha = 2$		$\alpha = 3$		$\alpha = 4$		cos-trans.	
n	err_n	ord_n	err_n	ord_n	err_n	ord_n	err_n	ord_n
4	$1.85 \cdot 10^{-2}$		$7.12 \cdot 10^{-2}$		$9.13 \cdot 10^{-2}$		$1.58 \cdot 10^{-3}$	
8	$3.37 \cdot 10^{-3}$	2.46	$1.72 \cdot 10^{-2}$	2.05	$1.47 \cdot 10^{-2}$	2.63	$7.69 \cdot 10^{-7}$	1.10
16	$7.99 \cdot 10^{-4}$	2.08	$5.68 \cdot 10^{-3}$	1.60	$3.42 \cdot 10^{-3}$	2.10	$7.66 \cdot 10^{-8}$	3.33
32	$1.96 \cdot 10^{-4}$	2.03	$1.96 \cdot 10^{-3}$	1.54	$8.37 \cdot 10^{-4}$	2.03		
64	$4.86 \cdot 10^{-5}$	2.01	$6.84 \cdot 10^{-4}$	1.52	$2.08 \cdot 10^{-4}$	2.01		
128	$1.21 \cdot 10^{-5}$	2.00	$2.40 \cdot 10^{-4}$	1.51	$5.17 \cdot 10^{-5}$	2.00		
256	$3.02 \cdot 10^{-6}$	2.00	$8.46 \cdot 10^{-5}$	1.50	$1.29 \cdot 10^{-5}$	2.00		
512	$7.47 \cdot 10^{-7}$	2.01	$2.95 \cdot 10^{-5}$	1.51	$3.21 \cdot 10^{-6}$	2.00		
1024	$1.76 \cdot 10^{-7}$	2.08	$9.75 \cdot 10^{-6}$	1.60	$7.44 \cdot 10^{-7}$	2.11		

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