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TWO-GRID METHODS FOR NONLINEAR MULTI-DIMENSIONAL WEAKLY SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. The convergence rate of the piecewise constant collocation method for the nonlinear weakly singular integral equation is investigated by G. Vainikko [**8**]. For this method, it is necessary to solve a large nonlinear algebraic system. This can be done straightforwardly only for comparatively rough discretizations. In this paper a two-grid iteration method is considered which enables us to find practically the solution of this system for fine discretizations. The main result is Theorem 3 about the convergence and the convergence rate of this method. This theorem generalizes for nonlinear equations the result proved in [**7, 8**] for linear equations.

1. Integral equation. In this paper we shall deal with the integral equation

$$u(x) = \int_G K(x,y,u(y))\,dy + f(x), \qquad x \in G, \tag{1}$$

where $G \subset \mathbf{R}^n$ is an open bounded set with a piecewise smooth boundary ∂G. The following assumptions (A1)–(A4) are made.

(A1) The kernel $K(x,y,u)$ is twice continuously differentiable with respect to x, y and u for $x \in G$, $y \in G$, $x \neq y$, $u \in (-\infty, \infty)$, whereby there exists a real number $\nu \in (-\infty, n)$ such that, for any nonnegative integer $k \leq 2$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{Z}_+^n$ with $k + |\alpha| + |\beta| \leq 2$ the following inequalities hold:

$$|D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} K(x,y,u)| \leq b_1(|u|)\gamma_{\nu+|\alpha|}(x,y),$$

$$\begin{aligned}|D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} K(x,y,u_1) - D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} K(x,y,u_2)|\\ \leq b_2(\max\{|u_1|,|u_2|\})|u_1-u_2|\gamma_{\nu+|\alpha|}(x,y).\end{aligned}$$

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Here

$$\gamma_\mu(x,y) = \begin{cases} 1, & \mu < 0, \\ 1 + |\log|x-y||, & \mu = 0, \\ |x-y|^{-\mu}, & \mu > 0. \end{cases}$$

and the functions $b_1 : \mathbf{R} \to \mathbf{R}^+$ and $b_2 : \mathbf{R} \to \mathbf{R}_+$ are assumed to be monotonically increasing. The following usual conventions are adopted:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n,$$
$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2} \quad \text{for } x = (x_1, \dots, x_n) \in \mathbf{R}^n,$$

$$D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$
$$D_{x+y}^\beta = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n}\right)^{\beta_n}.$$

(A2) Integral equation (1) has a solution $u_0 \in L^\infty(G)$ and the linearized integral equation

$$v(x) = \int_G K_0(x,y)v(y)\,dy, \qquad K_0(x,y) = \left[\frac{\partial K(x,y,u)}{\partial u}\right]_{u=u_0(y)},$$

has in $L^\infty(G)$ only the trivial solution $v = 0$.

(A3) $f \in C^{2,\nu}(G)$ with the same ν as in (A1), i.e., f is twice continuously differentiable on G and, for any multi-index $\alpha \in \mathbf{Z}_+^n$ with $|\alpha| \le 2$,

$$|D^\alpha f(x)| \le c_f \begin{cases} 1, & |\alpha| < n - \nu, \\ 1 + |\log \rho(x)|, & |\alpha| = n - \nu, \\ \rho(x)^{n-\nu-|\alpha|}, & |\alpha| > n - \nu, \end{cases}$$

where $c_f = \text{const}$ and $\rho(x) = \inf_{y \in \partial G}|x-y|$ is the distance from x to ∂G.

(A4) For any $x^1, x^2 \in G$ with the same ν as in (A1),

$$|f(x^1) - f(x^2)| \le c_f' \begin{cases} d_G(x^1,x^2), & \nu < n-1, \\ d_G(x^1,x^2)[1 + |\log d_G(x^1,x^2)|], & \nu = n-1, \\ d_G(x^1,x^2)^{n-\nu}, & \nu > n-1, \end{cases}$$

where $d_G(x^1, x^2)$ is defined as the infimum of the lengths of polygonal paths in G, joining points x^1 and x^2; if x^1 and x^2 belong to different connectivity components of G, then by definition $d_G(x^1, x^2) = \infty$.

In many cases (A4) is a consequence of (A3), e.g., if $\nu < n-1$ or if G consists of a finite number of connectivity components, whereby each of them is convex [**8**, pp. 19-20].

We note that the assumption (A1) holds, for example, for the kernel

$$K(x, y, u) = |x-y|^{-\nu} K_1(x, y, u)$$

where $\nu \in (0, n)$ and $K_1(x, y, u)$ is three times continuously differentiable with respect to x, y, u for $x, y \in \overline{G}$, $u \in (-\infty, \infty)$.

From (A1)–(A3) it follows that, for the solution u_0 of (1), we have $u_0 \in C^{2,\nu}(G)$ [**8**, pp. 137–138].

2. Piecewise constant collocation method. For any $h > 0$ we introduce an "approximate" partition of G into measurable sets (cells) $G_{j,h} \subset \mathbf{R}^n$, $j = 1, \dots, l_h$, so that

$$\text{diam}\, G_{j,h} \leq h, \qquad G^0_{i,h} \cap G^0_{j,h} = \varnothing \quad \text{for } i \neq j \tag{2}$$

($G^0_{j,h}$ is the interior of $G_{j,h}$),

$$(\overline{G} \backslash \overline{G}_h) \cup (\overline{G}_h \backslash \overline{G}) \subset S_h \tag{3}$$

where $S_h = \{x \in \mathbf{R}^n : \rho(x) \leq h^2\}$ and $G_h = \cup_{j=1}^{l_h} G_{j,h}$.

We call cell $G_{j,h}$ boundary-incident, if $\partial G \cap \text{co}\, G_{j,h} \neq \varnothing$, where $\text{co}\, G_{j,h}$ is the convex span of $G_{j,h}$. We assume that for any boundary-incident cell $G_{j,h}$ there exists a nonvoid measurable part $G'_{j,h} \subset G_{j,h} \cap G$ so that

$$d_G - \text{diam}\, G'_{j,h} \leq h, \qquad G_{j,h} \backslash G'_{j,h} \subset S_h \tag{4}$$

with

$$d_G - \text{diam}\, G'_{j,h} = \sup_{x,y \in G_{j,h}} d_G(x, y).$$

Note that for inner cells (i.e., for cells with $\partial G \cap \text{co}\, G_{j,h} = \varnothing$) one has

$$d_G - \text{diam}\, G_{j,h} = \text{diam}\, G_{j,h} \leq h.$$

In any cell $G_{j,h}$ we choose a collocation point $\xi_{j,h}$:

$$\xi_{j,h} = (\operatorname{meas} G_{j,h})^{-1} \int_{G_{j,h}} y\, dy \quad \text{if } \partial G \cap \operatorname{co} G_{j,h} = \varnothing, \tag{5}$$
$$\xi_{j,h} \in G'_{j,h} \text{ arbitrary} \quad \text{if } \partial G \cap \operatorname{co} G_{j,h} \neq \varnothing.$$

We assume that $\xi_{j,h}$ belongs to G_{jh}.

In the case of the piecewise constant collocation method, the approximate values $u_{i,h,0} = u_{h,0}(\xi_{i,h})$ of the solution u_0 of the integral equation (1) are calculated from the following systems of equations [**8**, p. 141]

$$u_{i,h} = \sum_{j=1}^{l_h} \int_{G_{j,h}} K(\xi_{i,h}, y, u_{j,h})\, dy + f(\xi_{i,h}), \qquad i = 1, \dots, l_h. \tag{6}$$

Let us define the following Banach spaces and operators:

$E = BC(G)$, the space of bounded continuous functions $u : G \to \mathbf{R}$, with the norm

$$||u|| = \sup_{x \in G} |u(x)|;$$

$E_h = C(\Xi_h)$, the space of grid functions $u_h : \Xi_h \to \mathbf{R}$, with the norm

$$||u_h|| = \max_{\xi_{j,h} \in \Xi_h} |u_h(\xi_{j,h})|, \qquad \Xi_h = \{\xi_{j,h}\}, \qquad j = 1, \dots, l_h;$$

$p_h \in L(E, E_h)$, the connection operator,

$$(p_h u)(\xi_{j,h}) = u(\xi_{j,h}) \quad \text{for } u \in E,\ \xi_{j,h} \in \Xi_h;$$

$T : E \to E$, the integral operator of equation (1),

$$(Tu)(x) = \int_G K(x, y, u(y))\, dy \quad \text{for } u \in E,\ x \in G;$$

$T_h : E_h \to E_h$, the approximation to T,

$$(T_h u_h)(\xi_{i,h}) = \sum_{j=1}^{l_h} \int_{G_{j,h}} K(\xi_{i,h}, y, u_h(\xi_{j,h}))\, dy$$
$$\text{for } u_h \in E_h,\ \xi_{i,h} \in \Xi_h.$$

Making use of these definitions, we can write system (6) in the form

$$u_h = T_h u_h + p_h f, \qquad u_h \in E_h. \tag{7}$$

Theorem 1 (Theorem 8.3 in [**8**]). *Let the assumptions* (A1)–(A4) *hold, and let the partition of G and collocation points satisfy* (2)–(5). *Then there exist $h_0 > 0$ and $\delta_0 > 0$ so that, for $h < h_0$, the system* (6) *has unique solution $u_{h,0} = (u_{i,h,0})$ in the ball $||u_h - p_h u_0|| < \delta_0$. The following error estimation holds*

$$||u_{h,0} - p_h u_0|| \le \text{const}\, (\varepsilon_{\nu,h})^2 \tag{8}$$

where

$$\varepsilon_{\nu,h} = \begin{cases} h, & \nu < n-1, \\ h(1 + |\log h|), & \nu = n-1, \\ h^{n-\nu}, & \nu > n-1. \end{cases}$$

3. Two-grid method. For the solution of the large nonlinear system (6), the two-grid iteration method can be used. Let $h \ll h_*$. Let us introduce approximate partitions of G into cells $G_{j,h}$, $j = 1, \dots, l_h$, and G_{j',h_*}, $j' = 1, \dots, l_{h_*}$, and choose corresponding collocation points $\xi_{j,h} \in G_{j,h} \cap G$ and $\xi_{j',h_*} \in G_{j,h_*} \cap G$ as in Section 2. For simplicity, we assume that the following compatibility conditions are fulfilled:

(i) every cell $G_{j,h}$, $j = 1, \dots, l_h$, is contained in some cell ("panel") G_{j',h_*}, $1 \le j' \le l_{h_*}$, and, conversely, every panel G_{j',h_*}, $j' = 1, \dots, l_{h_*}$, is a union of some cells $G_{j,h}$, $1 \le j \le l_h$;

(ii) every collocation point ξ_{j',h_*}, $j' = 1, \dots, l_{h_*}$, occurs as a collocation point for some cell $G_{j,h} \subset G_{j',h_*}$, i.e., $\Xi_{h_*} \subset \Xi_h$.

Let us introduce the connection operators between the spaces E_h and E_{h_*} as follows: $p_{h_*h} \in L(E_h, E_{h_*})$, the restriction operator,

$$(p_{h_*h} u_h)(\xi_{j,h_*}) = u_h(\xi_{j,h_*}) \quad \text{for } u_h \in E_h,\ \xi_{j,h_*} \in \Xi_{h_*};$$

$p_{hh_*} \in L(E_{h_*}, E_h)$, the piecewise constant prolongation operator,

$$(p_{hh_*} u_{h_*})(\xi_{j,h}) = u_{h_*}(\Pi_{h_*h} \xi_{j,h}) \quad \text{for } u_{h_*} \in E_{h_*},\ \xi_{j,h} \in \Xi_h$$

where $\Pi_{h_*h}\xi_{j,h} = \xi_{j',h_*}$ with j' such that $G_{j',h_*} \supset G_{j,h}$.

Let us use for the solution of equation (7) the two-grid iteration method in the following form

$$\begin{aligned} v_h^k &= T_h u_h^k + p_h f, \\ w_{h_*}^k - T_{h_*} w_{h_*}^k &= p_{h_*} f - T_{h_*}(p_{h_*h} v_h^k) + p_{h_*h} T_h v_h^k, \\ u_h^{k+1} &= v_h^k + p_{hh_*}(w_{h_*}^k - p_{h_*h} v_h^k), \qquad k = 0, 1, \dots , \end{aligned} \tag{9}$$

where u_h^0 is the initial guess of the solution. The two-grid methods of this type for integral equations originate from works of H. Brakhage [**3**] and K.E. Atkinson [**1**]. For the linear equation, method (9) coincides with the two-grid method in [**8**, p. 84]. For the nonlinear case similar methods are considered by W. Hackbusch [**4**] and C.T. Kelley [**5, 6**], for boundary integral equations by K.E. Atkinson [**2**].

To apply method (9), it is necessary, for every k, to solve the nonlinear equation in the form

$$(I_{h_*} - T_{h_*}) w_{h_*} = g_{h_*}, \qquad g_{h_*} = p_{h_*} f - T_{h_*}(p_{h_*h} v_h) + p_{h_*h} T_h v_h. \tag{10}$$

Note that, to compare with the equation (7) which corresponds to fine discretization, equation (10) corresponds to rough discretization and thus the dimension of this system is essentially less than the dimension of (7). About the solvability of (10) the following result holds.

Theorem 2. *Let the assumptions of Theorem* 1 *hold,* $v_h = T_h u_h + p_h f$ *and* $||u_h|| \leq \text{const}$. *Then there exists* $h_1 > 0$ *so that, for every* $h_* < h_1$, $h < h_*$, *the equation* (10) *has a unique solution* $w_{h_*} = (I_{h_*} - T_{h_*})^{-1} g_{h_*}$ *in the ball*

$$||u_{h_*} - p_{h_*} u_0|| < \delta_0 \tag{11}$$

where δ_0 *is the same as in Theorem* 1. *The following error estimation holds*:

$$||w_{h_*} - p_{h_*} u_0|| \leq \text{const}\, \varepsilon_{\nu,h_*}. \tag{12}$$

To prove this theorem, we shall use the following Lemmas 1 and 2.

Lemma 1. *Let the assumptions of Theorem* 1 *hold, and*

$$\sup_{y\in G\cup G_h} |u(y)| \le \text{const}\,.$$

Then

$$\int_{(\overline{G}\setminus\overline{G}_h)\cup(\overline{G}_h\setminus\overline{G})} |K(x,y,u(y))|\,dy \le \text{const}\,(\varepsilon_{\nu,h})^2$$

and, for any $x^1, x^2 \in G$,

$$|(Tu)(x^1) - (Tu)(x^2)| \le \text{const} \begin{cases} d_G(x^1,x^2), & \nu < n-1, \\ d_G(x^1,x^2)[1+|\log d_G(x^1,x^2)|], & \nu = n-1, \\ [d_G(x^1,x^2)]^{n-\nu}, & \nu > n-1. \end{cases}$$

Proof. The proof of Lemma 1 is analogous to the proof of Lemmas 2.3 and 5.1 in [**8**].

Lemma 2. *Let the assumptions of Theorem* 2 *hold. Then*

$$\max_{1\le i\le l_h} |v_h(\Pi_{h_*h}\xi_{i,h}) - v_h(\xi_{i,h})| \le \text{const}\,\varepsilon_{\nu,h_*}. \tag{13}$$

Proof. Due to (2), (4) and (5), we have

$$d_G(\Pi_{h_*h}\xi_{i,h}, \xi_{i,h}) \le h_*. \tag{14}$$

Let us assign to $u_h \in E_h$ the piecewise constant function

$$\bar{u}_h(x) = \sum_{j=1}^{l_h} u_h(\xi_{j,h})\chi_{j,h}(x),$$

$$\chi_{j,h}(x) = \begin{cases} 1, & x \in G_{j,h}, \\ 0, & x \notin G_{j,h}. \end{cases}$$

Making use of Lemma 1, we get

$$\begin{aligned}
&|(T_h u_h)(\Pi_{h_*h}\xi_{i,h}) - (T_h u_h)(\xi_{i,h})| \\
&= \left| \sum_{j=1}^{l_h} \int_{G_{j,h}} [K(\Pi_{h_*h}\xi_{i,h}, y, u_h(\xi_{j,h})) - K(\xi_{i,h}, y, u_h(\xi_{j,h}))]\, dy \right| \\
&= \left| \int_{G_h} [K(\Pi_{h_*h}\xi_{i,h}, y, \bar{u}_h(y)) - K(\xi_{i,h}, y, \bar{u}_h(y))]\, dy \right| \\
&\le |(T\bar{u}_h)(\Pi_{h_*h}\xi_{i,h}) - (T\bar{u}_h)(\xi_{i,h})| + \mathrm{const}\, (\varepsilon_{\nu,h})^2 \le \mathrm{const}\, \varepsilon_{\nu,h_*}.
\end{aligned}$$

From (A4) and (14) it follows that

$$|f(\Pi_{h_*h}\xi_{i,h}) - f(\xi_{i,h})| \le \mathrm{const}\, \varepsilon_{\nu,h_*}.$$

Therefore,

$$\begin{aligned}
|v_h(\Pi_{h_*h}\xi_{i,h}) - v_h(\xi_{i,h})| &\le |(T_h u_h)(\Pi_{h_*h}\xi_{i,h}) - (T_h u_h)(\xi_{i,h})| \\
&\quad + |f(\Pi_{h_*h}\xi_{i,h}) - f(\xi_{i,h})| \le \mathrm{const}\, \varepsilon_{\nu,h_*},
\end{aligned}$$

which proves the estimate (13). □

Proof of Theorem 2. We have

$$\begin{aligned}
||g_{h_*} - p_{h_*} f|| &= ||T_{h_*}(p_{h_*h} v_h) - p_{h_*h} T_h v_h|| \\
&= \max_{1\le i\le l_{h_*}} \Bigg| \sum_{j'=1}^{l_{h_*}} \int_{G_{j',h_*}} K(\xi_{i,h_*}, y, v_h(\xi_{j',h_*}))\, dy \\
&\qquad - \sum_{j=1}^{l_h} \int_{G_{j,h}} K(\xi_{i,h_*}, y, v_h(\xi_{j,h}))\, dy \Bigg| \\
&= \max_{1\le i\le l_{h_*}} \Bigg| \sum_{j=1}^{l_h} \int_{G_{j,h}} [K(\xi_{i,h_*}, y, v_h(\Pi_{h_*h}\xi_{j,h})) \\
&\qquad - K(\xi_{i,h_*}, y, v_h(\xi_{j,h}))]\, dy \Bigg| \\
&\le \mathrm{const} \max_{1\le j\le l_h} |v_h(\Pi_{h_*h}\xi_{j,h}) - v_h(\xi_{j,h})|.
\end{aligned}$$

By Lemma 2, now

$$||g_{h_*} - p_{h_*} f|| \leq \text{const}\, \varepsilon_{\nu, h_*}$$

and we can complete the proof similarly to the proof of Theorem 8.3 in [**8**]. □

4. Convergence rate. For convergence analysis, we rewrite the formulas (9) in the form

$$u_h^{k+1} = \Phi u_h^k, \qquad k = 0, 1, \dots, \tag{15}$$

where

$$\begin{aligned}\Phi u_h = (I_h - p_{hh_*} p_{h_* h})(T_h u_h + p_h f) \\ + p_{hh_*}(I_{h_*} - T_{h_*})^{-1}[p_{h_*} f - T_{h_*}(p_{h_* h} T_h u_h + p_{h_*} f) \\ + p_{h_* h} T_h (T_h u_h + p_h f)].\end{aligned}$$

Thus, the two-grid method (9) is considered here as an iterative method to solve the equation

$$u_h = \Phi u_h. \tag{16}$$

To study the convergence of the iterative method (15), the following well-known result is used.

Lemma 3. *Let equation* (16) *have a solution* $u_{h,0} \in E_h$, *and let* $Q_h = \{u_h : ||u_h - u_{h,0}|| \leq \delta\}$. *If*

$$||\phi'(u_h)|| \leq q < 1, \qquad \forall\, u_h \in Q_h,$$

then $u_{h,0}$ *is the unique solution of equation* (16) *in* Q_h. *For every initial guess* $u_h^0 \in Q_h$ *the iterative method* (15) *converges to* $u_{h,0}$ *with the rate*

$$||u_h^{k+1} - u_{h,0}|| \leq q||u_h^k - u_{h,0}||, \qquad k = 0, 1, \dots. \tag{17}$$

Theorem 3. *Let the assumptions of Theorem* 1 *hold. Then there exist* $h_0 > 0$ *and* $\delta_0 > 0$ *so that,for every* $h < h_0$, *the equation* (7) *has a unique solution* $u_{h,0}$ *in the ball*

$$||u_h - p_h u_0|| < \delta_0. \tag{18}$$

The two-grid iterative method (9) *with* $h_* < h_0$, $h < h_*$, *converges, for sufficiently good initial guess* u_h^0, *to this solution with the rate*

$$(19)\qquad ||u_h^{k+1} - u_{h,0}|| \le c\varepsilon_{\nu,h_*}||u_h^k - u_{h,0}||, \qquad k = 0, 1, \dots ,$$

where the constant c *is independent of* h *and* h_*.

Proof. By Theorem 1, there exist $h_1 > 0$ and $\delta_0 > 0$ so that, for $h < h_1$, equation (7) has a unique solution $u_{h,0}$ satisfying (18). It is easy to see that this solution $u_{h,0}$ is a solution of the equation (16) too.

We shall check up the assumptions of Lemma 3. The Fréchet derivative of Φ is

$$(20)\quad \begin{aligned}\Phi'(u_h)\Delta u_h = (I_h - p_{hh_*}p_{h_*h})T_h'(u_h)\Delta u_h \\ + p_{hh_*}[I_{h_*} - T_{h_*}'(w_{h_*})]^{-1}[p_{h_*h}T_h'(v_h) \\ - T_{h_*}'(p_{h_*h}v_h)p_{h_*h}]T_h'(u_h)\Delta u_h\end{aligned}$$

where

$$(T_h'(u_h)\Delta u_h)(\xi_{i,h}) = \sum_{j=1}^{l_h}\int_{G_{j,h}} \frac{\partial K(\xi_{j,h}, y, u_h(\xi_{j,h}))}{\partial u}\, dy \Delta u_{j,h},$$

w_{h_*} is the solution of equation (10) satisfying (11) and $v_h = T_h u_h + p_h f$. By Theorem 2, there exists $h_2 \le h_1$ so that, for $h_* < h_2$, equation (10) has the unique solution w_{h_*} satisfying (11) and, for this solution, estimation (12) holds. Thus,

$$\begin{aligned}||T_{h_*}'(w_{h_*}) - T_{h_*}'(p_{h_*}u_0)|| &\le \text{const}\, ||w_{h_*} - p_{h_*}u_0|| \\ &\le \text{const}\, \varepsilon_{\nu,h_*} \to 0, \qquad h_* \to 0.\end{aligned}$$

It is proved in [**8**, p. 142] that $T_{h_*}'(p_{h_*}u_0) \to T'(u_0)$ compactly. Therefore, $T_{h_*}'(w_{h_*}) \to T'(u_0)$ compactly, too. As a consequence of this (see [**8**, p. 54]), there exists $h_3 \le h_2$ so that, for $h_* < h_3$,

$$||[I_{h_*} - T_{h_*}'(w_{h_*})]^{-1}|| \le \text{const}\,.$$

Now from (20) for $h < h_* < h_3$ and for u_h satisfying (18), we get the following estimation

$$(21)\quad \begin{aligned}||\Phi'(u_h)|| \le ||(I_h - p_{hh_*}p_{h_*h})T_h'(u_h)|| \\ + \text{const}\, ||[p_{h_*h}T_h'(v_h) - T_{h_*}'(p_{h_*h}v_h)p_{h_*h}]T_h'(u_h)||.\end{aligned}$$

Let us estimate both terms on the right side of (21). Denote $z_h = T'_h(u_h)\Delta u_h$. For the first term,

$$\begin{aligned}[(I_h - p_{hh_*}p_{h_*h})T'_h(u_h)\Delta u_h](\xi_{i,h}) \\ = z_h(\xi_{i,h}) - z_h(\Pi_{h_*h}\xi_{i,h}) \\ = \sum_{j=1}^{l_h}\int_{G_{j,h}}\left[\frac{\partial K(\xi_{i,h},y,u_h(\xi_{j,h}))}{\partial u} - \frac{\partial K(\Pi_{h_*h}\xi_{i,h}y,u_h(\xi_{j,h}))}{\partial u}\right]dy\Delta u_{j,h}\end{aligned}$$

and, similarly, to the estimation of the analogical expression in the proof of Lemma 2, we get

$$\max_{1\le i\le l_h}|z_h(\xi_{i,h}) - z_h(\Pi_{h_*h}\xi_{i,h})| \le \operatorname{const}\varepsilon_{\nu,h_*}||\Delta u_h||, \tag{22}$$

i.e.,

$$||(I_h - p_{hh_*}p_{h_*h})T'_h(u_h)|| \le \operatorname{const}\varepsilon_{\nu,h_*}. \tag{23}$$

For the second term,

$$\begin{aligned}&\{[p_{h_*h}T'_h(v_h) - T'_{h_*}(p_{h_*h}v_h)p_{h_*h}]T'_h(u_h)\Delta u_h)\}(\xi_{i,h_*}) \\ &= \sum_{j=1}^{l_h}\int_{G_{j,h}}\frac{\partial K(\xi_{i,h_*},y,v_h(\xi_{j,h}))}{\partial u}\,dyz_h(\xi_{j,h}) \\ &\quad - \sum_{j'=1}^{l_{h_*}}\int_{G_{j',h_*}}\frac{\partial K(\xi_{i,h_*},y,v_h(\xi_{j',h_*}))}{\partial u}\,dyz_h(\xi_{j',h_*}) \\ &= \sum_{j=1}^{l_h}\int_{G_{j,h}}\left[\frac{\partial K(\xi_{i,h_*},y,v_h(\xi_{j,h}))}{\partial u}\right. \\ &\qquad\qquad \left. - \frac{\partial K(\xi_{i,h_*},y,v_h(\Pi_{h_*h}\xi_{j,h}))}{\partial u}\right]dyz_h(\xi_{j,h}) \\ &\quad + \sum_{j=1}^{l_h}\int_{G_{j,h}}\frac{\partial K(\xi_{i,h_*},y,v_h(\Pi_{h_*h}\xi_{j,h}))}{\partial u}\,dy[z_h(\xi_{j,h}) - z_h(\Pi_{h_*h}\xi_{j,h})]\end{aligned}$$

and, by (13)and (22), we get

$$||[p_{h_*h}T'_h(v_h) - T'_{h_*}(p_{h_*h}v_h)p_{h_*h}]T'_h(u_h)\Delta u_h|| \le \operatorname{const}\varepsilon_{\nu,h_*}||\Delta u_h||. \tag{24}$$

By the estimations (21), (23) and (24), we finally find

$$||\Phi'(u_h)|| \le c\varepsilon_{\nu,h_*}.$$

Further, let us choose $h_0 \le h_3$ so that $q = c\varepsilon_{\nu,h_*} < 1$ for $h_* < h_0$. Then estimation (19) follows from (17) and, for this h_0, all the assertions of the theorem are fulfilled. □

Note that, for the two-grid method (9), Theorem 3 gives for the nonlinear equation the same rate of convergence as Theorem 5.2 in [**8**] for the linear equation.

Remark. Making use of the one-node quadrature formula, the operator T_h in (7) can be approximated by the following operator $\tilde{T}_h$:

$$(\tilde{T}_h u_h)(\xi_{i,h}) = \sum_{\substack{j=1 \\ \text{dist}\,(\xi_{i,h},\text{co}\,G_{j,h})\ge h}}^{l_h} K(\xi_{i,h},\xi_{j,h},u_h(\xi_{j,h}))\,\text{meas}\,G_{j,h}.$$

This gives us the method

$$u_h = \tilde{T}_h u_h + p_h f, \qquad u_h \in E_h. \tag{25}$$

The convergence conditions and the rate of the convergence of the method (25) are studied in [**8**], Theorem 8.3. The system (25) can be solved by the two-grid method similarly to the case of the system (7). We get the following formulas instead of (9):

$$\begin{aligned} v_h^k &= \tilde{T}_k u_h^k + p_h f, \\ w_{h_*}^k - \tilde{T}_{h_*} w_{h_*}^k &= p_{h_*} f - \tilde{T}_{h_*}(p_{h_*h} v_h^k) + p_{h_*h}\tilde{T}_h v_h^k, \\ u_h^{k+1} &= v_h^k + p_{hh_*}(w_{h_*}^k - p_{h_*h} v_h^k), \qquad k = 0,1,\dots . \end{aligned} \tag{26}$$

All the assertions of Theorem 3 hold for the two-grid method (26). This can be proved analogically to Theorems 2 and 3, making use of the estimations

$$||T_h u_h - \tilde{T}_h u_h|| \le \text{const}\,\varepsilon_{\nu,h}, \qquad ||T'(u_h) - \tilde{T}'(u_h)|| \le \text{const}\,\varepsilon_{\nu,h}.$$

These estimations can be established in the same way as it is done for the linear case in [**8**, Lemma 5.5].

5. Linearized two-grid method. It is necessary, on every step of the two-grid method (9), to solve the nonlinear system (10). For that we can use some iterative method, for example, Newton's method. In the case if u_h^k is a sufficiently good approximation of $u_{h,0}$, we can use only one step of Newton's method and get the following linearized two-grid method

$$\begin{gathered} v_h^k = T_h u_h^k + p_h f, \\ [I_{h_*} - T'_{h_*}(p_{h_*h} v_h^k)](w_{h_*}^k - p_{h_*h} v_h^k) = p_{h_*h}(T_h v_h^k + p_h f - v_h^k), \\ u_h^{k+1} = v_h^k + p_{hh_*}(w_{h_*}^k - p_{h_*h} v_h^k). \end{gathered}$$

6. Numerical example. Consider the integral equation

$$u(x) = \int_0^1 |x-y|^{-1/4} u^2(y)\, dy + f(x)$$

where $f(x)$ is selected so that $u_0(x) = x^{3/2}$ is a solution. It is easy to see that for this equation the assumptions (A1)–(A4) hold with $\nu = 1/4$. For the collocation method, let us choose cells $G_{j,h} = (x_{j-1}, x_j)$ and collocation points $\xi_{j,h} = x_j - h/2$, $j = 1, \dots, l_h = N$, where $h = 1/N$ and $x_j = jh$.

In this case the system (6) has the form

$$u_{i,h} = \sum_{j=1}^{N} a_{ij,h} u_{j,h}^2 + f_{i,h}, \qquad i = 1, 2, \dots, N \tag{27}$$

where $f_{i,h} = f(\xi_{i,h})$ and the integrals

$$a_{ij,h} = \int_{x_{j-1}}^{x_j} |\xi_{i,h} - y|^{-1/4}\, dy$$

are easy to find.

We calculate the solution $u_{h_*,0}$ of the nonlinear system (27) for $h = h_* = 1/N_*$ by Newton's method. The same system for $h < h_*$ we solve by two-grid iteration method (9) where initial guess is $u_h^0 = p_{hh_*} u_{h_*,0}$. In this case it appears that, for the solution of the system (10), it is sufficient to make only one step of Newton's method, i.e., the linearized two-grid method is suitable.

Some results of the numerical experiments are presented in the following table where k is the number of steps of two-grid method and

$$\varepsilon_k = ||u_h^k - p_h u_0|| = \max_{1\le i\le N} |u_{i,h}^k - u_0(\xi_{i,h})|.$$

N_*	N	Number of steps k	Norm of the error ε_k	Time of solution (in seconds)
3	9	4	8.1E-3	0.10
3	27	6	9.3E-4	0.17
9	27	3	9.1E-4	0.15
3	81	7	1.1E-4	0.75
9	81	4	1.0E-4	0.50
27	81	3	1.0E-4	0.56
3	243	8	1.5E-5	6.5
9	243	5	1.3E-5	4.1
27	243	3	1.3E-5	2.7
81	243	2	1.4E-5	3.4
3	729	9	2.1E-6	63
9	729	5	2.4E-6	35
27	729	4	1.8E-6	28
81	729	3	2.4E-6	23

We see that a good strategy is $N_* \approx N^{1/2}$. Then it is sufficient to make four steps of two-grid method.

The experiment was carried out on the computer IBM 4381 (in double precision).

REFERENCES

1. K.E. Atkinson, *Iterative variants of the Nyström method for the numerical solution of integral equations*, Numer. Math. **22** (1973), 17–31.

2. ———, *Two-grid iteration methods for linear integral equations of the second kind on piecewise smooth surfaces in* R^3, SIAM J. Sci. Comput. **15** (1994), 1083–1104.

3. H. Brakhage, *Über die Numerische Behandlung von Integralgleichungen nach der Quadraturformelmethode*, Numer. Math. **2** (1960), 183–196.

4. W. Hackbusch, *Multi-grid methods and applications*, Springer-Verlag, Berlin-New York, 1985.

5. C.T. Kelley, *A fast two-grid method for matrix* H*-equations*, Transport Theory Statist. Phys. **18** (1989), 185–204.

6. C.T. Kelley and E.W. Sachs, *Multilevel algorithms for constrained compact fixed point problems*, SIAM J. Sci. Statist. Comput. **15** (1993), 645–667.

7. G. Vainikko, *Solution of large systems arising by discretization of multidimensional weakly singular integral equations*, Acta et Comm. Univ. Tartuensis **937** (1992), 3–14.

8. ———, *Multidimensional weakly singular integral equations*, Lecture Notes Math. **1549** (1993),

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