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ON THE EXISTENCE OF A GLOBAL MILD SOLUTION FOR A NONLINEAR INTEGRODIFFERENTIAL EQUATION WITH A SINGULAR KERNEL

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ABSTRACT. We study the equation

$$u_t(t,x) = \int_0^t b(t-s)g(u_x(s,x))_x\,ds + f(t,x),$$

where u is the unknown and the kernel b, the nonlinear function g, and the forcing function f are given. We assume that the kernel $b:(0,\infty)\to R$ is nonnegative, nonincreasing, convex and singular.

Making use of energy estimates and the theory of maximal monotone operators, we prove the global existence of a mild solution.

1. Introduction. We consider the equation

$$u_t(t,x) = \int_0^t b(t-s)g(u_x(s,x))_x\,ds + f(t,x); \qquad t>0,\ 0\le x\le 1, \tag{1.1}$$

with the boundary and initial conditions

$$u(t,0)=u(t,1)=0; \qquad t>0,$$
$$u(0,x)=u_0(x); \qquad 0\le x\le 1.$$

Here u is the unknown and the kernel b, the nonlinear function g, the forcing function f, and the initial data u_0 are given. Under certain assumptions, the equation (1.1) models the behavior of a thin viscoelastic body (see [**8, 11, 15**]).

We study the equation (1.1) with a singular kernel, that is, we assume that $b\in L^1_{\text{loc}}([0,\infty))$ and that $\lim_{t\to 0+} b(t)=\infty$. While the singularity

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of the kernel should make solutions smoother, it also makes the analysis of the equation harder. One of the major research interests in this field is to prove the existence of a global strong solution to (1.1) with arbitrary initial data.

So far this has been achieved only partially: one has to assume that the kernel is more singular at 0 than for example $t^{-1/2}$ (see [**6**]), or $t^{-2/3}$ (see [**4**]). These are only technical restrictions that do not have any physical meaning and it is reasonable to expect that a strong solution exists for a general singular kernel. As a step towards the above mentioned goal we prove the existence of a global mild solution with arbitrary initial data.

The method of proof is based on two regularized versions of the original equation and energy estimates. The primary regularized equation is obtained by adding a viscosity term to the original equation (1.1). The energy estimates for the primary regularized equation are established via the secondary regularized equation that has a smoother kernel in addition to the viscosity term. A sequence of solutions of the primary regularized equation is shown to converge to a solution of the equation (1.1). The theory of maximal monotone operators is applied to handle the nonlinearity and weak convergence.

2. Theorem and comments. We now give the assumptions and our result followed by some comments on the earlier work and the notation.

We assume that the kernel b satisfies

i) $b \in L^1_{\mathrm{loc}}([0,\infty))$,

ii) b is nonnegative, nonincreasing, and convex, and

(iii) $\int_0^\delta b'(s)\,ds = -\infty$, for $\delta > 0$.

For the nonlinear function $g \in C^1(-\infty,\infty)$, we require that $g(0) = 0$, and that there are constants α and β such that $0 < \alpha \le g'(t) \le \beta < \infty$, for all $t \in R$. Moreover, we assume that the forcing function f and the initial value u_0 satisfy

$$f \in W^{1,2}_{\mathrm{loc}}([0,\infty); L^2(0,1)),$$
$$u_0 \in W^{2,2}_0(0,1),$$

respectively.

Then the following holds.

Theorem. *Let b, g, f, and u_0 be as above. Then there exists a function u such that*

$$\begin{aligned} u &\in L^\infty_{\mathrm{loc}}([0,\infty); W_0^{1,2}(0,1)) \cap W^{1,\infty}_{\mathrm{loc}}([0,\infty); L^2(0,1)), \\ g(u_x)_x &\in L^\infty_{\mathrm{loc}}([0,\infty); W^{-1,2}(0,1)), \\ b * g(u_x)_x &\in L^\infty_{\mathrm{loc}}([0,\infty); L^2(0,1)), \end{aligned}$$

and the equation (1.1) *holds for almost every $t \in (0,\infty)$. Moreover, u is weakly continuous on $[0,\infty)$ with values in $W_0^{1,2}(0,1)$ and the initial condition is attained in this sense.*

We say that the function u given by the theorem is a mild solution because $g(u_x)_x$ has its values in $W^{-1,2}(0,1)$, which is a space of distributions.

This is an extension of an earlier result by Londen (see [**14**]). He assumes that the kernel is more singular than $t^{-1/2}$ and then proves the existence of a solution u that satisfies

$$\begin{aligned} u &\in L^\infty(0,T; W_0^{1,2}(0,1)) \cap C([0,T]; L^2(0,1)), \\ g(u_x)_x &\in L^\infty(0,T; W^{-1,2}(0,1)), \end{aligned}$$

and

$$\begin{aligned} u(t) - u_0 = \int_0^t b(t-\tau) \int_0^\tau g(u_x(s))_x \, ds \, d\tau \\ + \int_0^t f(\tau)\, d\tau, \end{aligned}$$

for $t \in [0,T]$, where T appears in a condition for the kernel b. That is, the equation holds in a weaker sense.

As we mentioned earlier, Gripenberg proves in [**6**] that if the kernel is more singular than $t^{-1/2}$, then there exists a function u such that

$$u \in W^{1,\infty}_{\mathrm{loc}}([0,\infty); L^2(0,1)) \cap L^2_{\mathrm{loc}}([0,\infty); W_0^{2,2}(0,1))$$

and the equation (1.1) holds. In particular, $g(u_x)_x \in L^2_{\mathrm{loc}}([0,\infty); L^2(0,1))$, so that the function u is a strong solution.

Finally, we introduce some notation. The duality between $W_0^{1,2}(0,1)$ and $W^{-1,2}(0,1)$ as well as the innerproduct on $L^2(0,1)$ is denoted by $\langle \cdot, \cdot \rangle$. The corresponding norms are denoted by $\|\cdot\|_1$, $\|\cdot\|_{-1}$ and $\|\cdot\|_0$. Let E be one of the above mentioned Hilbert spaces. For $T > 0$ and $1 \le p \le \infty$, we define

$$W^{1,p}(0,T;E) = \left\{ \varphi \in AC([0,T];E) : \frac{d}{dt}\varphi \in L^p(0,T;E) \right\}.$$

3. On the regularized equations and their solutions. The primary and secondary regularized equations are

$$\begin{aligned} u_t^\varepsilon(t,x) &= \int_0^t b(t-s) g(u_x^\varepsilon(s,x))_x \, ds + \varepsilon u_{xx}^\varepsilon(t,x) + f(t,x), \\ u^\varepsilon(t,0) &= u^\varepsilon(t,1) = 0, \qquad u^\varepsilon(0,x) = u_0(x), \end{aligned} \tag{3.1}$$

and

(3.2)

$$\begin{aligned} u_t^{\varepsilon,n}(t,x) &= \int_0^t b_n(t-s) g(u_x^{\varepsilon,n}(s,x))_x \, ds + \varepsilon u_{xx}^{\varepsilon,n}(t,x) + f^n(t,x), \\ u^{\varepsilon,n}(t,0) &= u^{\varepsilon,n}(t,1) = 0, \qquad u^{\varepsilon,n}(0,x) = u_0^n(x), \end{aligned}$$

respectively. The viscosity terms $\varepsilon u_{xx}^\varepsilon$ and $\varepsilon u_{xx}^{\varepsilon,n}$, where $\varepsilon > 0$ allow us to treat (3.1) and (3.2) like perturbed linear diffusion equations.

The kernels b_n, $n = 1, 2, 3 \ldots$, whose construction is described in Appendix B, satisfy

$$\begin{aligned} b_n &\in C^2([0,\infty)) \quad \forall\, n, \\ (-1)^i b_n^{(i)} &\ge 0; \qquad i = 0,1,2, \ \forall\, n, \\ b_n &\to b \quad \text{in } L^1_{\text{loc}}([0,\infty)), \quad \text{as } n \to \infty, \\ b_n &\to b \quad \text{in } C_{\text{loc}}((0,\infty)), \quad \text{as } n \to \infty, \\ b_n' &\to b' \quad \text{in } L^1_{\text{loc}}((0,\infty)), \quad \text{as } n \to \infty \text{ and} \\ b_n &\le b + 1 \quad \forall\, n. \end{aligned}$$

The sequence of the approximating forcing functions $f^n \in W^{1,2}_{\text{loc}}([0,\infty); L^2(0,1))$ is defined as follows:

$$f^n = f + (f_0^n - f(0)),$$

where the functions $f_0^n \in W_0^{1,2}(0,1)$ are chosen so that $\|f_0^n - f(0)\|_0 \to 0$, as $n \to \infty$.

Finally, the approximating initial functions $u_0^n \in W_0^{3,2}(0,1)$ are chosen so that

$$\|u_0 - u_0^n\|_{W_0^{2,2}(0,1)} \to 0,$$

as n tends to infinity.

To simplify the notation, we define

$$G : W_0^{1,2}(0,1) \to [0,\infty);\ G(\varphi) = \int_0^1 \int_0^{\varphi_x(x)} g(\tau)\, d\tau\, dx.$$

Then $g(\varphi_x)_x = \partial G(\varphi)$, where ∂ is the subdifferential operator.

We have the following basic result.

Lemma 3.1. *For every $T > 0$, the problems* (3.1) *and* (3.2) *have unique solutions $u^\varepsilon, u^{\varepsilon,n} \in W^{1,2}(0,T;L^2(0,1))$ such that $u^\varepsilon(t), u^{\varepsilon,n}(t) \in W_0^{1,2}(0,1) \cap W^{2,2}(0,1)$ and the equations are satisfied for almost every $t \in (0,T)$. Moreover,*

$$\sup_{0\le t\le T} \|u^\varepsilon(t)\|_1,\ \sup_{0\le t\le T} \|u^{\varepsilon,n}(t)\|_1 < \infty,$$

$$u^\varepsilon_{xx}, g(u^\varepsilon_x)_x, u^{\varepsilon,n}_{xx}, g(u^{\varepsilon,n}_x)_x \in L^2(0,T;L^2(0,1)),$$

$$G(u^\varepsilon), G(u^{\varepsilon,n}) \in AC([0,T]),$$

and for all $\varepsilon > 0$,

$$\sup_n \int_0^T \|g(u_x^{\varepsilon,n}(t))_x\|_0^2\, dt < \infty.$$

The proof of the existence of solutions uses Yosida approximations and the theory of maximal monotone operators and is similar to the proof of Lemma 1 from [**13**] (see also Corollary 2.1 in [**3**]). The proof of the uniqueness of the solutions is straightforward. A detailed version of the proof of Lemma 3.1 can be found in [**10**].

To establish some estimates for the solution of the primary regularized equation (3.1) we would like to be able to differentiate the secondary regularized equation (3.2) with respect to time. Lemma 3.1 does not guarantee enough regularity for this and hence we need an additional regularity result.

Lemma 3.2. *The solution* $u^{\varepsilon,n}$ *of the secondary regularized equation* (3.2) *satisfies* $u_t^{\varepsilon,n}, u_{xx}^{\varepsilon,n} \in W^{1,2}(0,T;L^2(0,1))$, $u_t^{\varepsilon,n}(t) \in W_0^{1,2}(0,1) \cap W^{2,2}(0,1)$, *and*

$$u_{tt}^{\varepsilon,n}(t) - \varepsilon u_{xxt}^{\varepsilon,n}(t) = \frac{d}{dt}\int_0^t b_n(t-s)g(u_x^{\varepsilon,n}(s))_x\,ds + f_t^n(t)$$

for almost every $t \in [0,T]$.

Outline of the proof. By differentiating the equation (3.2) formally with respect to t and defining $v = u_t^{\varepsilon,n}$ we obtain the equation

$$v_t(t) - \varepsilon v_{xx}(t) = \frac{d}{dt}\int_0^t b_n(t-s)g(u_x^{\varepsilon,n}(s))_x\,ds + f_t^n(t)$$

with the initial condition

$$v(0) = \varepsilon u_{0,xx}^n + f_0^n \in W_0^{1,2}(0,1).$$

We can apply [**2**, Théorème 3.6] to prove the existence of a solution to this problem. It is easy to see that

$$w(t) = \int_0^t v(\tau)\,d\tau - (u^{\varepsilon,n}(t) - u_0^n)$$

satisfies $w_t = \varepsilon w_{xx}$ with $w(0) = 0$. Hence $w = 0$ and

$$u^{\varepsilon,n}(t) = u_0^n + \int_0^t v(\tau)\,d\tau$$

which leads to the desired conclusion. The complete proof of this lemma is given in [**10**]. □

Next we present an important lemma that allows us to derive energy estimates for the solutions of the primary regularized equation from the energy estimates of the solutions of the secondary regularized equation. More precisely, we prove that the solution of the secondary regularized equation converges to the solution of the primary regularized equation as the smoother kernels tend to the original kernel.

Lemma 3.3. *Let $\varepsilon > 0$. If u^ε and $u^{\varepsilon,n}$ are the solutions of the equations* (3.1) *and* (3.2), *respectively, then* $\sup_{0\le t\le T} \|u^\varepsilon(t)-u^{\varepsilon,n}(t)\|_0$, $\|u^\varepsilon - u^{\varepsilon,n}\|_{L^2(0,T;W_0^{1,2}(0,1))}$, $\|g(u_x^\varepsilon) - g(u_x^{\varepsilon,n})\|_{L^2(0,T;L^2(0,1))} \to 0$, *as* $n \to \infty$.

Outline of the proof. We subtract the equation (3.2) from the equation (3.1) and obtain

$$u_t^\varepsilon(t) - u_t^{\varepsilon,n}(t) - \int_0^t b(t-s)[g(u_x^\varepsilon(s))_x - g(u_x^{\varepsilon,n}(s))_x]\,ds - \int_0^t [b(t-s) - b_n(t-s)]g(u_x^{\varepsilon,n}(s))_x\,ds - \varepsilon[u_{xx}^\varepsilon(t) - u_{xx}^{\varepsilon,n}(t)] = f(t) - f^n(t).$$

By manipulating the previous equation and choosing $\widehat{T} \in (0,T]$ to be small enough one can show that for $t \in [0,\widehat{T}]$

$$\frac{1}{2}\|u^\varepsilon(t) - u^{\varepsilon,n}(t)\|_0^2 + \frac{\varepsilon}{2}\|u^\varepsilon - u^{\varepsilon,n}\|^2_{L^2(0,t;W_0^{1,2}(0,1))} \le A_n\|u^\varepsilon - u^{\varepsilon,n}\|_{L^2(0,t;L^2(0,1))} + \frac{1}{2}\|u_0 - u_0^n\|_0^2,$$

where $\{An\}_n$ is a sequence of positive constants that converges to zero due to the convergence of approximating kernels and forcing functions. It then follows that the lemma holds on the interval $[0,\widehat{T}]$ and it is easy to extend this result to the interval $[0,T]$ and complete the proof of the lemma. A detailed version of this proof can be found in [**10**].

4. Proof of the theorem. We first derive uniform bounds for the solutions of the primary regularized equation and then use the weak compactness of the spaces involved to extract weakly convergent subsequences of solutions. We finish the proof by proving a strong convergence result and applying the theory of maximal monotone operators to the nonlinear convolution term.

The uniform boundedness results are given in three lemmas. In all cases, the main idea is to first establish bounds for the solutions of the secondary regularized equation, which is easier to manipulate due to the smoothness of the kernel, and then use the convergence results of Lemma 3.3 to derive bounds for the solutions of the primary regularized equation. Three crucial technical lemmas are presented in Appendix A.

Lemma 4.1. *Let* u^ε *and* $u^{\varepsilon,n}$ *be the solutions of the regularized equations* (3.1) *and* (3.2), *respectively, on the interval* $[0,T]$. *Then*

$$\sup_{n,\varepsilon>0,0\le t\le T} \|u^{\varepsilon,n}(t)\|_1 < \infty, \tag{4.1}$$

$$\sup_{n,\varepsilon>0,0\le t\le T} \left\| \int_0^t g(u_x^{\varepsilon,n}(\tau))_x \, d\tau \right\|_0 < \infty, \tag{4.2}$$

$$\sup_{n,\varepsilon>0} \varepsilon \int_0^T \|g(u_x^{\varepsilon,n}(\tau))_x\|_0^2 \, d\tau < \infty, \tag{4.3}$$

and

$$\sup_{\varepsilon>0,0\le t\le T} \|u^{\varepsilon}(t)\|_1 < \infty, \tag{4.4}$$

$$\sup_{\varepsilon>0,0\le t\le T} \left\| \int_0^t g(u_x^{\varepsilon}(\tau))_x \, d\tau \right\|_0 < \infty, \tag{4.5}$$

$$\sup_{\varepsilon>0} \varepsilon \int_0^T \|g(u_x^{\varepsilon}(\tau))_x\|_0^2 \, d\tau < \infty. \tag{4.6}$$

Proof. Let $t \in [0,T]$. Form the scalar product of (3.2) and $-g(u_x^{\varepsilon,n})_x$ and integrate over $(0,t)$. With [**2**, Lemme 3.3], this yields

$$\begin{aligned} G(u^{\varepsilon,n}(t)) + \varepsilon \int_0^t \langle u_{xx}^{\varepsilon,n}(\tau), g'(u_x^{\varepsilon,n}(\tau)) u_{xx}^{\varepsilon,n}(\tau)\rangle \, d\tau \\ + \int_0^t \left\langle g(u_x^{\varepsilon,n}(\tau))_x, \int_0^\tau b_n(\tau-s) g(u_x^{\varepsilon,n}(s))_x \, ds \right\rangle d\tau \\ = G(u_0^n) + \int_0^t \langle f^n(\tau), -g(u_x^{\varepsilon,n}(\tau))_x\rangle \, d\tau. \end{aligned}$$

Next, we use the relation (3.26) of [**12**] on the convolution term and integrate the last term on the righthand side by parts. After some straightforward estimates we obtain

$$
\begin{aligned}
(4.7)\quad & \frac{1}{2}\alpha\|u^{\varepsilon,n}(t)\|_1^2+\frac{\varepsilon}{\beta}\int_0^t\|g(u_x^{\varepsilon,n}(\tau))_x\|_0^2\,d\tau \\
&\qquad+\frac{b_n(t)}{2}\bigg\|\int_0^t g(u_x^{\varepsilon,n}(\tau))_x\,d\tau\bigg\|_0^2 \\
&\qquad-\frac{1}{2}\int_0^t b_n'(\tau)\bigg\|\int_0^\tau g(u_x^{\varepsilon,n}(s))_x\,ds\bigg\|_0^2\,d\tau \\
&\qquad-\frac{1}{2}\int_0^t b_n'(t-\tau)\bigg\|\int_\tau^t g(u_x^{\varepsilon,n}(s))_x\,ds\bigg\|_0^2\,d\tau \\
&\qquad+\frac{1}{2}\int_0^t\int_0^\tau b_n''(\tau-s)\bigg\|\int_s^\tau g(u_x^{\varepsilon,n}(v))_x\,dv\bigg\|_0^2\,ds\,d\tau \\
&\quad\le\frac{1}{2}\beta\|u_0^n\|_1^2+\|f_n(t)\|_0\bigg\|\int_0^t g(u_x^{\varepsilon,n}(\tau))_x\,d\tau\bigg\|_0 \\
&\qquad+\max_{0\le s\le t}\bigg\|\int_0^s g(u_x^{\varepsilon,n}(\tau))_x\,d\tau\bigg\|_0\int_0^t\|f_t^n(\tau)\|\,d\tau.
\end{aligned}
$$

Note that all the terms on the lefthand side are positive.

Let $\widehat{T}>0$ be such that $c=\inf\{b_n(t): n>0, 0\le t\le\widehat{T}\}>0$. By continuity there exists a $t^*=t^*(\varepsilon,n)\in[0,\widehat{T}]$ such that

$$
\bigg\|\int_0^{t^*} g(u_x^{\varepsilon,n}(s))_x\,ds\bigg\|_0=\max_{0\le\tau\le\widehat{T}}\bigg\|\int_0^\tau g(u_x^{\varepsilon,n}(s))_x s\bigg\|_0.
$$

Hence (4.7) with $t=t^*$ leads to

$$
\begin{aligned}
\frac{c}{2}\bigg\|\int_0^{t^*} g(u_x^{\varepsilon,n}(\tau))_x\,d\tau\bigg\|_0^2 &\le\frac{b_n(t^*)}{2}\bigg\|\int_0^{t^*} g(u_x^{\varepsilon,n}(\tau))_x\,d\tau\bigg\|_0^2 \\
&\le A\bigg\|\int_0^{t^*} g(u_x^{\varepsilon,n}(\tau))_x\,d\tau\bigg\|_0+B,
\end{aligned}
$$

where the constants A and B are finite, since the forcing functions f_n and the initial values u_0^n converge. Thus it follows from the previous inequality that

$$K = \sup_{\varepsilon>0,n,0\le t\le \widehat{T}} \left\| \int_0^t g(u_x^{\varepsilon,n}(\tau))_x \, d\tau \right\|_0 \le \sup_{\varepsilon>0,n} \left\| \int_0^{t^*} g(u_x^{\varepsilon,n}(\tau))_x \, d\tau \right\|_0 < \infty.$$

Consequently,

$$\sup_{\varepsilon>0,n} \varepsilon \int_0^T \|g(u_x^{\varepsilon,n}(\tau))_x\|_0^2 \, d\tau < \infty,$$

and

$$\sup_{\varepsilon>0,n,0\le t\le \hat{T}} \|u^{\varepsilon,n}(t)\|_1 < \infty.$$

We use the previously obtained bounds as a starting point to obtain bounds valid on the whole interval $[0,T]$. Write $z(t) = g(u_x^{\varepsilon,n}(t))_x$ and consider the inequality (4.7) for $t > \hat{T}$. In particular, using only the fifth term on the lefthand side, we have for all $0 \le \delta \le \widehat{T}$

$$\int_0^{t-\delta} (-b_n')(t-\tau) \left\| \int_0^t z(\sigma)\, d\sigma - \int_0^\tau z(\sigma)\, d\sigma \right\|_0^2 d\tau \le A \left\| \int_0^t z(\sigma) d\sigma \right\|_0 + B.$$

After applying the Triangle Inequality and rearranging the terms, we have

$$\left(\int_0^{t-\delta} (-b_n')(t-\tau)\, d\tau \right) \left\| \int_0^t z(\sigma) d\sigma \right\|_0^2 \le \left(A + 2\int_0^{t-\delta} (-b_n')(t-\tau) \left\| \int_0^\tau z(\sigma)\, d\sigma \right\|_0 d\tau \right) \left\| \int_0^t z(\sigma)\, d\sigma \right\|_0 + \int_0^{t-\delta} (-b_n')(t-\tau) \left\| \int_0^\tau z(\sigma)\, d\sigma \right\|_0^2 d\tau + B. \tag{4.8}$$

Now

$$\int_0^{t-\delta} (-b_n')(t-\tau)\, d\tau = b_n(\delta) - b_n(t) \ge b_n(\delta) - b_n(\widehat{T}),$$

where the inequality follows from the fact that b_n is nonincreasing. By using the singularity of b and the convergence of the sequence $\{b_n\}_n$ it is easy to see that we can choose a $\delta^* \in (0, \widehat{T})$ and N so that

$$c = \inf_{t \geq \hat{T}, n \geq N} \int_0^{t-\delta^*} (-b_n')(t-\tau)\, d\tau > 0.$$

We also define

$$C = b(\delta^*) + 1 \geq b_n(\delta^*) \geq \int_0^{t-\delta^*} (-b_n')(t-\tau)\, d\tau.$$

Then it follows from the inequality (4.8) with $\delta = \delta^*$ that for $t \in [0, \widehat{T} + \delta^*]$

$$\left\| \int_0^t z(\sigma)\, d\sigma \right\|_0^2 \leq \frac{1}{c}\,(A + 2KC) \left\| \int_0^t z(\sigma)\, d\sigma \right\|_0 + \frac{1}{c}\left(B + CK^2\right).$$

Thus

$$\sup_{\varepsilon > 0, 0 \leq t \leq \widehat{T} + \delta^*, n} \left\| \int_0^t g(u_x^{\varepsilon,n}(\tau))_x\, d\tau \right\|_0 < \infty,$$

and it is easy to see that we can continue in the same way so that we finally have

$$\sup_{\varepsilon > 0, 0 \leq t \leq T, n} \left\| \int_0^t g(u_x^{\varepsilon,n}(\tau))_x\, d\tau \right\|_0 < \infty. \tag{4.9}$$

By using the previous estimate in the inequality (4.7), we obtain

$$\sup_{\varepsilon > 0, n} \varepsilon \int_0^T \|g(u_x^{\varepsilon,n}(\tau))_x\|_0^2\, d\tau < \infty,$$

and

$$\sup_{\varepsilon > 0, 0 \leq t \leq T, n} \|u^{\varepsilon,n}(t)\|_1 < \infty.$$

This completes the proof of the first part of the Lemma.

By (4.9) and Lemma 3.3, for each $t \in [0,T]$, there exists a subsequence $\{u^{\varepsilon,n}\}_n$ such that

$$\int_0^t g(u_x^{\varepsilon,n}(\tau))_x \, d\tau \rightharpoonup \int_0^t g(u_x^{\varepsilon}(\tau))_x \, d\tau, \quad \text{weakly in } L^2(0,1),$$

as $n \to \infty$. Then it follows from (4.9) that

$$\sup_{\varepsilon>0, 0\le t\le T} \left\| \int_0^t g(u_x^{\varepsilon}(\tau))_x \, d\tau \right\|_0 < \infty. \tag{4.10}$$

From (3.1) we obtain

$$\begin{aligned} G(u^\varepsilon(t)) + \varepsilon \int_0^t \langle u_{xx}^\varepsilon(\tau), g'(u_x^\varepsilon(\tau)) u_{xx}^\varepsilon(\tau)\rangle \, d\tau \\ + \int_0^t \Big\langle -g(u_x^\varepsilon(\tau))_x, \int_0^\tau b(\tau-s)(-g(u_x^\varepsilon(s))_x) \, ds \Big\rangle \, d\tau \\ = G(u_0) + \Big\langle f(t), \int_0^t -g(u_x^\varepsilon(s))_x \, ds \Big\rangle \\ - \int_0^t \Big\langle f_t(\tau), \int_0^\tau -g(u_x^\varepsilon(s))_x \, ds \Big\rangle \, d\tau. \end{aligned}$$

Since b is of positive type, the term containing the convolution is positive (see Chapter 16 of [**7**]). Thus, using (4.10) and some straightforward estimates, we get

$$\frac{1}{2}\alpha \|u^\varepsilon(t)\|_1^2 + \frac{\varepsilon}{\beta} \int_0^t \|g(u_x^\varepsilon(\tau))_x\|_0^2 \, d\tau \le \frac{1}{2}\beta \|u_0\|_1^2 + C,$$

where the constant C is determined by f and (4.10). Thus,

$$\sup_{\varepsilon>0, 0\le t\le T} \|u_\varepsilon(t)\|_1 < \infty,$$

and

$$\sup_{\varepsilon>0} \varepsilon \int_0^T \|g(u_x^\varepsilon(\tau))_x\|_0^2 \, d\tau < \infty.$$

This completes the proof of Lemma 4.1. $\square$

Lemma 4.2. *Let u^ε and $u^{\varepsilon,n}$ be the solutions of the regularized equations* (3.1) *and* (3.2), *respectively, on the interval* $[0,T]$. *Then*

$$\sup_{n,\varepsilon>0,0\le t\le T} \|u_t^{n,\varepsilon}(t)\|_0 < \infty, \tag{4.11}$$

$$\sup_{n,\varepsilon>0} \varepsilon \int_0^T \|u_t^{\varepsilon,n}(\tau)\|_1^2 \, d\tau < \infty, \tag{4.12}$$

and

$$\sup_{\varepsilon>0} \|u_t^\varepsilon\|_{L^\infty(0,T;L^2(0,1))} < \infty. \tag{4.13}$$

Proof. Differentiate the equation (3.2) with respect to t, form the scalar product with $u_t^{\varepsilon,n}$, and integrate over $(0,t)$. This yields

$$\begin{aligned}\int_0^t \langle u_{tt}^{\varepsilon,n}(\tau), u_t^{\varepsilon,n}(\tau)\rangle \, d\tau - \varepsilon \int_0^t \langle u_{xxt}^{\varepsilon,n}(\tau), u_t^{\varepsilon,n}(\tau)\rangle \, d\tau \\ + \int_0^t \Big\langle \frac{d}{d\tau} \int_0^\tau b_n(\tau-s)(-g(u_x^{\varepsilon,n}(s))_x) \, ds, u_t^{\varepsilon,n}(\tau)\Big\rangle \, d\tau \\ = \int_0^t \langle f_t^n(\tau), u_t^{\varepsilon,n}(\tau)\rangle \, d\tau.\end{aligned}$$

We wish to apply a result from Appendix A to the convolution term. For this purpose, the term must be slightly manipulated. Write

$$\tilde g(u^{\varepsilon,n}(t)) = -g(u_x^{\varepsilon,n}(t))_x + g(u_{0,x}^n)_x,$$

so that

$$\begin{aligned}\int_0^t \Big\langle \frac{d}{d\tau} \int_0^\tau b_n(\tau-s)(-g(u_x^{\varepsilon,n}(s)))_x \, ds, u_t^{\varepsilon,n}(\tau)\Big\rangle \, d\tau \\ = \int_0^t \Big\langle \frac{d}{d\tau} \int_0^\tau b_n(\tau-s)\tilde g(u^{\varepsilon,n}(s)) \, ds, u_t^{\varepsilon,n}(\tau)\Big\rangle \, d\tau \\ - \int_0^t \Big\langle \frac{d}{d\tau} \int_0^\tau b_n(\tau-s) g(u_{0,x}^n)_x \, ds, u_t^{\varepsilon,n}(\tau)\Big\rangle \, d\tau.\end{aligned}$$

Now,

$$\int_0^t \left\langle \frac{d}{d\tau} \int_0^\tau b_n(\tau - s) g(u_{0,x}^n)_x \, ds, u_t^{\varepsilon,n}(\tau) \right\rangle d\tau \\ \leq \int_0^t b_n(\tau) \beta \|u_{0,xx}^n\|_0 \|u_t^{\varepsilon,n}(\tau)\|_0 \, d\tau,$$

and Lemma A.1 (use $H = L^2(0,1)$, $\varphi(\cdot) = G(\cdot) - G(u_0^n) + \langle g(u_{0,x}^n)_x, \cdot - u_0^n \rangle$) gives

$$\int_0^t \left\langle \frac{d}{d\tau} \int_0^\tau b_n(\tau - s) \tilde{g}(u^{\varepsilon,n}(s)) \, ds, u_t^{\varepsilon,n}(\tau) \right\rangle d\tau \geq 0.$$

Hence, for all $t \in [0, T]$,

$$\frac{1}{2} \|u_t^{\varepsilon,n}(t)\|_0^2 + \varepsilon \int_0^t \|u_t^{\varepsilon,n}(\tau)\|_1^2 \, d\tau \\ \leq c_1 + \int_0^t \left(c_2 \left(b(\tau) + 1\right) + \|f_t(\tau)\|_0\right) \|u_t^{\varepsilon,n}(\tau)\|_0 \, d\tau,$$

where c_1 and c_2 are constants independent of ε and n. They are finite due to the convergence of the initial values and the forcing functions.

With [**2**, Lemme A.5] we obtain from the previous inequality the estimate

$$\|u_t^{\varepsilon,n}(t)\|_0 \leq c_1 + \int_0^t \left(c_2 \left(b(\tau) + 1\right) + \|f_t(\tau)\|_0\right) \, dt,$$

which holds for all $t \in [0, T]$. Then

$$\sup_{n,\varepsilon > 0, 0 \leq t \leq T} \|u_t^{n,\varepsilon}(t)\|_0 < \infty \tag{4.14}$$

and

$$\sup_{n,\varepsilon > 0} \varepsilon \int_0^T \|u_t^{n,\varepsilon}(\tau)\|_1^2 \, d\tau < \infty.$$

From (4.14) and Lemma 3.3 it follows that there is a subsequence $\{u^{\varepsilon,n}\}_n$ such that

$$u_t^{\varepsilon,n} \overset{*}{\rightharpoonup} u_t^\varepsilon, \quad \text{weakly-}* \text{ in } L^\infty(0, T; L^2(0,1)).$$

Then it is easy to show that

$$\sup_{\varepsilon>0} \|u_t^\varepsilon\|_{L^\infty(0,T;L^2(0,1))} < \infty,$$

which completes the proof of Lemma 4.2. $\square$

We now give an important result that allows us to prove a strong convergence result.

Lemma 4.3. *Let* u^ε *and* $u^{\varepsilon,n}$ *be the solutions of the regularized equations* (3.1) *and* (3.2)*, respectively, on the interval* $[0,T]$*. Then*

$$\sup_{0<\varepsilon<1,n} \int_0^T \int_0^t (-b_n')(t-s)\|g(u_x^{\varepsilon,n}(t))-g(u_x^{\varepsilon,n}(s))\|_0^2\, ds\, dt < \infty, \tag{4.15}$$

and

$$\sup_{0<\varepsilon<1} \int_0^T \int_0^t (-b')(t-s)\|g(u_x^\varepsilon(t)) - g(u_x^\varepsilon(s))\|_0^2\, ds\, dt < \infty. \tag{4.16}$$

Note that since b' is not integrable at zero the latter inequality implies that $\|g(u_x^\varepsilon(t) - g(u_x^\varepsilon(s)\|_0^2$ must be small (uniformly in ε) when $t-s$ is close to zero. Hence the estimate gives information about the continuity of the limit of $g(u_x^\varepsilon)$.

Proof. We differentiate the equation (3.2) with respect to t, form the scalar product with $u^{\varepsilon,n}$ and integrate over $(0,T)$. This yields, after a partial integration,

$$\begin{aligned}\int_0^T &\langle u_{tt}^{\varepsilon,n}(t), u^{\varepsilon,n}(t)\rangle\, dt \\ &+ \int_0^T \Big\langle \frac{d}{dt}\int_0^t b_n(t-s)(-g(u_x^{\varepsilon,n}(s))_x)\, ds, u^{\varepsilon,n}(t)\Big\rangle\, dt \\ &= -\varepsilon \int_0^T \langle u_x^{\varepsilon,n}(t), u_{xt}^{\varepsilon,n}(t)\rangle\, dt + \int_0^T \langle u^{\varepsilon,n}(t), f_t^n(t)\rangle\, dt.\end{aligned} \tag{4.17}$$

To handle the convolution term, we apply Lemma A.2 with $Au^{\varepsilon,n} = -g(u_x^{\varepsilon,n})_x$, $\varphi(u^{\varepsilon,n}) = \int_0^1 \int_0^{u_x^{\varepsilon,n}} g(\tau)\,d\tau\,dx$ and integrate partially twice, which results in

$$\begin{aligned}
\int_0^T \Big\langle \frac{d}{dt}\int_0^t b_n(t-s)(-g(u_x^{\varepsilon,n}(s))_x)\,ds, u^{\varepsilon,n}(t)\Big\rangle\,dt \\
= \int_0^T b_n(t)\int_0^1 \int_0^{u_x^{\varepsilon,n}(t)} g(\tau)\,d\tau\,dx\,dt \\
- \int_0^T b_n(T-s)\int_0^1 \int_0^{u_x^{\varepsilon,n}(s)} g(\tau)\,d\tau\,dx\,ds \\
+ \int_0^T b_n(T-s)\langle g(u_x^{\varepsilon,n}(s)), u_x^{\varepsilon,n}(s)\rangle\,ds \\
- \int_0^T\int_0^t b_n'(t-s)\int_0^1\int_0^{u_x^{\varepsilon,n}(t)} g(\tau)\,d\tau\,dx\,ds\,dt \\
+ \int_0^T\int_0^t b_n'(t-s)\int_0^1\int_0^{u_x^{\varepsilon,n}(s)} g(\tau)\,d\tau\,dx\,ds\,dt \\
+ \int_0^T\int_0^t b_n'(t-s)\langle g(u_x^{\varepsilon,n}(s)), u_x^{\varepsilon,n}(t) - u_x^{\varepsilon,n}(s)\rangle\,ds\,dt.
\end{aligned}$$

By using this and combining terms we can rewrite the equation (4.17) in the following way:

$$\begin{aligned}
\int_0^T\int_0^t (-b_n')(t-s)\int_0^1\int_{u_x^{\varepsilon,n}(s)}^{u_x^{\varepsilon,n}(t)} [g(\tau) - g(u_x^{\varepsilon,n}(s))]\,d\tau\,dx\,ds\,dt \\
+ \int_0^T b_n(t)\int_0^1\int_0^{u_x^{\varepsilon,n}(t)} g(\tau)\,d\tau\,dx\,dt \\
+ \int_0^T b_n(T-s)\langle g(u_x^{\varepsilon,n}(s)), u_x^{\varepsilon,n}(s)\rangle\,ds \\
= \int_0^T b_n(T-s)\int_0^1\int_0^{u_x^{\varepsilon,n}(s)} g(\tau)\,d\tau\,dx\,ds \\
- \langle u^{\varepsilon,n}(T), u_t^{\varepsilon,n}(T)\rangle \\
+ \langle u_0^n, \varepsilon u_{0,xx}^n + f_0^n\rangle + \int_0^T \|u_t^{\varepsilon,n}(t)\|_0^2\,dt \\
- \varepsilon\int_0^T \langle u_{xt}^{\varepsilon,n}(t), u_x^{\varepsilon,n}(t)\rangle\,dt + \int_0^T \langle f_t(t), u^{\varepsilon,n}(t)\rangle\,dt.
\end{aligned}$$

Note that the second and the third term on the lefthand side are positive and that the righthand side is uniformly bounded in ε and n by Lemma 4.1 and Lemma 4.2. Since

$$\int_0^T \int_0^t (-b_n')(t-s) \int_0^1 \int_{u_x^{\varepsilon,n}(s)}^{u_x^{\varepsilon,n}(t)} [g(\tau) - g(u_x^{\varepsilon,n}(s))]\, d\tau\, dx\, ds\, dt \geq \frac{1}{2\beta} \int_0^T \int_0^t (-b_n')(t-s) \|g(u_x^{\varepsilon,n}(t)) - g(u_x^{\varepsilon,n}(s))\|_0^2\, ds\, dt,$$

it then follows that

$$\sup_{0<\varepsilon<1,n} \int_0^T \int_0^t (-b_n')(t-s) \|g(u_x^{\varepsilon,n}(t)) - g(u_x^{\varepsilon,n}(s))\|_0^2\, ds\, dt < \infty. \tag{4.18}$$

Next, let $\tau \in (0,T)$. We have

$$\begin{aligned} &\int_\tau^T \int_0^{t-\tau} (-b')(t-s) \|g(u_x^{\varepsilon}(t)) - g(u_x^{\varepsilon}(s))\|_0^2\, ds\, dt \\ &\leq \limsup_{n\to\infty} \int_\tau^T \int_0^{t-\tau} (-b_n')(t-s) \|g(u_x^{\varepsilon}(t)) - g(u_x^{\varepsilon}(s))\|_0^2\, ds\, dt \\ &\quad + \limsup_{n\to\infty} \left| \int_\tau^T \int_0^{t-\tau} [(-b') - (-b_n')](t-s) \|g(u_x^{\varepsilon}(t)) - g(u_x^{\varepsilon}(s))\|_0^2\, ds\, dt \right|. \end{aligned} \tag{4.19}$$

To investigate the first term on the righthand side of (4.19), use the triangle inequality to get

$$\begin{aligned} &\int_\tau^T \int_0^{t-\tau} (-b_n')(t-s) \|g(u_x^{\varepsilon}(t)) - g(u_x^{\varepsilon}(s))\|_0^2\, ds\, dt \\ &\leq 3 \int_\tau^T \int_0^{t-\tau} (-b_n')(t-s) \|g(u_x^{\varepsilon}(t)) - g(u_x^{\varepsilon,n}(t))\|_0^2\, ds\, dt \\ &\quad + 3 \int_\tau^T \int_0^{t-\tau} (-b_n')(t-s) \|g(u_x^{\varepsilon,n}(t)) - g(u_x^{\varepsilon,n}(s))\|_0^2\, ds\, dt \\ &\quad + 3 \int_\tau^T \int_0^{t-\tau} (-b_n')(t-s) \|g(u_x^{\varepsilon,n}(s)) - g(u_x^{\varepsilon}(s))\|_0^2\, ds\, dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

The sequence of approximating kernels is constructed so that for $t-s \geq \tau > 0$ there exist constants C_τ such that $(-b_n')(t-s) \leq (-b_n')(\tau) \leq C_\tau$ uniformly in n. Thus by Lemma 3.3,

$$I_1 \leq 3TC_\tau \int_0^T \|g(u_x^\varepsilon(t)) - g(u_x^{\varepsilon,n}(t))\|_0^2 \, dt \to 0,$$

and

$$I_3 \leq 3TC_\tau \int_0^T \|g(u_x^{\varepsilon,n}(s)) - g(u_x^\varepsilon(s))\|_0^2 \, ds \to 0,$$

as $n \to \infty$. By the estimate (4.18), $\sup_{0<\tau<T, 0<\varepsilon<1, n} I_2 < \infty$, so that the first term on the righthand side of (4.19) is uniformly bounded in ε and τ.

By Lemma 4.1 and the L^1 convergence of $\{b_n'\}_n$ on compact subsets of $(0,\infty)$, the second term on the righthand side of (4.19) vanishes for every $0 < \tau < T$.

Hence the lefthand side of (4.19) is uniformly bounded in τ and ε and

$$\begin{aligned}
&\sup_\varepsilon \left(\int_0^T \int_0^t (-b')(t-s) \|g(u_x^\varepsilon(t)) - g(u_x^\varepsilon(s))\|_0^2 \, ds \, dt \right) \\
&\quad = \sup_\varepsilon \left(\lim_{\tau \to 0} \int_\tau^T \int_0^{t-\tau} (-b')(t-s) \|g(u_x^\varepsilon(t)) - g(u_x^\varepsilon(s))\|_0^2 \, ds \, dt \right) \\
&\quad < \infty.
\end{aligned}$$

This completes the proof of Lemma 4.3. $\square$

We are now ready to finish the proof of the Theorem. By Lemma 4.1, there exists a subsequence $\{u^\varepsilon\}_\varepsilon$ and functions $u^* \in L^\infty(0,T; W_0^{1,2}(0,1))$ and $g^* \in L^\infty(0,T; L^2(0,1))$ such that

$$u_x^\varepsilon \overset{*}{\rightharpoonup} u_x^*, \quad \text{weakly-}* \text{ in } L^\infty(0,T; L^2(0,1)), \tag{4.20}$$

and

$$g(u_x^\varepsilon) \overset{*}{\rightharpoonup} g^*, \quad \text{weakly-}* \text{ in } L^\infty(0,T; L^2(0,1)). \tag{4.21}$$

Moreover, by using Lemma 4.2, the compact imbedding $W_0^{1,2}(0,1) \hookrightarrow L^2(0,1)$, and the Arzela-Ascoli Theorem, it is easy to show that $u^* \in W^{1,\infty}(0,T; L^2(0,1))$ and that

$$u_t^\varepsilon \overset{*}{\rightharpoonup} u_t^*, \quad \text{weakly-}* \text{ in } L^\infty(0,T; L^2(0,1)). \tag{4.22}$$

We now apply the method of proof of Theorem 1 from [**5**] to prove a strong convergence result for the nonlinear term in the convolution integral. Only an outline without the rather mechanical intermediate steps is given below.

With the contraction semigroup $\{J_s\}_{s\geq 0} : L^2(0,T;L^2(0,1)) \to L^2(0,T;L^2(0,1))$ given by

$$(J_s\varphi)(t) = \begin{cases} \varphi(t+s), & \text{if } 0 \leq t \leq T-s \\ 0, & \text{if } T-s < t \leq T \end{cases}$$

we define

$$g^{\varepsilon,h} = \frac{1}{h}\int_0^h J_s g(u_x^\varepsilon)\, ds$$

for $h \in (0,1]$. The function $g^{*,h}$ is defined in a similar way. We use (4.5) to prove that

$$\sup_{\varepsilon>0, 0\leq t\leq T} \|g^{\varepsilon,h}(t)\|_1 \leq C/h, \tag{4.23}$$

where C is a constant. From (4.4) and (4.16) it follows that we can apply Lemma A.3 and hence there exists a continuous function $\omega : [0,T] \to [0,\infty)$ such that $\omega(0) = 0$ and

$$\sup_{0<\varepsilon<1} \|J_s g(u_x^\varepsilon) - g(u_x^\varepsilon)\|_{L^2(0,T;L^2(0,1))} \leq \omega(s). \tag{4.24}$$

A similar estimate can be proved for g^*. Now it is easy to show that

$$\|g^{\varepsilon,h}(t+\tau) - g^{\varepsilon,h}(t)\|_0^2 \leq \frac{1}{h}[\omega(\tau)]^2, \tag{4.25}$$

uniformly in t and ε. For every fixed h, (4.23) and (4.25) together with the Arzela-Ascoli Theorem guarantee the existence of a subsequence $g^{\varepsilon,h}$ such that

$$g^{\varepsilon,h} \to g^{*,h}, \quad \text{strongly in } C([0,T];L^2(0,1)),$$

as ε tends to zero. By using (4.24) and the corresponding estimate for g^* it can be shown that

$$\sup_{\varepsilon>0} \|g^{\varepsilon,h} - g(u_x^\varepsilon)\|^2_{L^2(0,T;L^2(0,1))}\, , \|g^{*,h} - g^*\|^2_{L^2(0,T;L^2(0,1))} \leq \frac{1}{h}\int_0^h [\omega(s)]^2\, ds.$$

Hence

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\|g(u_x^\varepsilon)-g^*\|_{L^2(0,T;L^2(0,1))}\\&\le \limsup_{\varepsilon\to 0}\|g(u_x^\varepsilon)-g^{\varepsilon,h}\|_{L^2(0,T;L^2(0,1))}\\&\quad+\limsup_{\varepsilon\to 0}\|g^{\varepsilon,h}-g^{*,h}\|_{L^2(0,T;L^2(0,1))}\\&\quad+\limsup_{\varepsilon\to 0}\|g^{*,h}-g^*\|_{L^2(0,T;L^2(0,1))}\\&\le 2\bigg(\frac{1}{h}\int_0^h[\omega(s)]^2\,ds\bigg)^{1/2}.\end{aligned}$$

Since the right-most side of the previous inequality can be made arbitrarily small by letting $h\to 0+$,

$$g(u_x^\varepsilon)\to g^*,\quad \text{strongly in } L^2(0,T;L^2(0,1)). \tag{4.26}$$

A complete proof of this result can be found in [**10**].

From the strong convergence of $g(u_x^\varepsilon)$ and the weak-$*$ convergence of u_x^ε it follows that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}(g(u_x^\varepsilon),u_x^\varepsilon)_{L^2(0,T;L^2(0,1))}-(g^*,u_x^*)_{L^2(0,T;L^2(0,1))}\\&=\limsup_{\varepsilon\to 0}(g(u_x^\varepsilon)-g^*,u_x^\varepsilon-u_x^*)_{L^2(0,T;L^2(0,1))}\\&\le\limsup_{\varepsilon\to 0}\|g(u_x^\varepsilon)-g^*\|_{L^2(0,T;L^2(0,1))}\|u_x^\varepsilon-u_x^*\|_{L^2(0,T;L^2(0,1))}\\&=0.\end{aligned}\tag{4.27}$$

Since the function g is maximal monotone in $L^2(0,1)$, it defines a maximal monotone operator in $L^2(0,T;L^2(0,1))$ (see [**9**, Proposition 12]). We can then apply Proposition 2.5 from [**2**] with (4.27) to conclude that

$$g^*=g(u_x^*). \tag{4.28}$$

Now we let ε tend to zero in the primary regularized equation. Since $L^\infty(0,T;L^2(0,1))$ is compactly imbedded in $L^2(0,T;W^{-1,2}(0,1))$, the result (4.22) implies that

$$u_t^\varepsilon\to u_t^*,\quad \text{strongly in } L^2(0,T;W^{-1,2}(0,1)).$$

By (4.26) and (4.28), $\int_0^t b(t-s)g(u_x^\varepsilon(s))_x\,ds \to \int_0^t b(t-s)g(u_x^*(s))_x\,ds$, strongly in $L^2(0,T;W^{-1,2}(0,1))$. From (4.6) it follows that

$$\varepsilon u_{xx}^\varepsilon \to 0, \quad \text{strongly in } L^2(0,T;W^{-1,2}(0,1)).$$

Hence, by letting ε tend to zero in (3.1), we obtain

$$u_t^*(t) = \int_0^t b(t-s)g(u_x^*(s))_x\,ds + f(t)$$

for almost every $t \in (0,T]$. From $u_t^*, f \in L^\infty(0,T;L^2(0,1))$ we conclude that $b * g(u_x^*)_x \in L^\infty(0,T;L^2(0,1))$. Clearly $g(u_x^*)_x \in L^\infty(0,T;W^{-1,2}(0,1))$. Finally, it is easy to show that u^* is weakly continuous with values in $W_0^{1,2}(0,1)$ and the initial condition is attained in this sense.

Having obtained a subsequence of approximating solutions that converges to a solution on the interval $[0,T]$ we can select a further subsequence that works on the interval $[0,2T]$. Continuing in this way we finally get a solution that is valid on $[0,\infty)$ and has the properties stated in the theorem.

Appendix

A. Auxiliary lemmas. Throughout this section we assume that H is a Hilbert space and we denote the corresponding inner product by $\langle\cdot,\cdot\rangle$.

The following lemma is an infinite dimensional version of Lemma 20.5.3 from [**7**].

Lemma A.1. *Let $T' > 0$ and assume that $a \in C^2([0,T'])$ is such that $(-1)^i a^{(i)} \geq 0$ for $i = 0,1,2$. Let $\varphi : H \to [-\infty,\infty]$ be a proper, l.s.c., and convex function and let A be the subdifferential of φ:*

$$A = \partial\varphi.$$

Assume that the function u satisfies $u \in W^{1,2}(0,T';H)$, $u(t) \in \mathcal{D}(A)$ for almost every $t \in [0,T']$, and $Au \in L^2(0,T';H)$. Moreover, assume that

$$\varphi(u(0)) = 0, \qquad \varphi(u(t)) \geq 0 \quad \forall\, t \in [0,T'].$$

Then

$$\int_0^T \Big\langle \frac{d}{dt}\int_0^t a(t-s)Au(s)\,ds, u'(t)\Big\rangle\,dt \geq 0$$

for $0 \leq T \leq T'$.

Proof. By using [**2**, Lemme 3.3], it is easy to show that

$$\begin{aligned}\text{(A.1)}\quad \int_0^T \Big\langle \frac{d}{dt}\int_0^t a(t-s)Au(s)\,ds, u'(t)\Big\rangle\,dt &\\ = a(T)\varphi(u(T)) - \int_0^T a'(t)\varphi(u(t))\,dt &\\ - \int_0^T a'(T-s)\varphi(u(T))\,ds &\\ + \int_0^T a'(T-s)\varphi(u(s))\,ds &\\ + \int_0^T a'(T-s)\langle Au(s), u(T)-u(s)\rangle\,ds &\\ + \int_0^T \int_0^t a''(t-s)\varphi(u(t))\,ds\,dt &\\ - \int_0^T \int_0^t a''(t-s)\varphi(u(s))\,ds\,dt &\\ - \int_0^T \int_0^t a''(t-s)\langle Au(s), u(t)-u(s)\rangle\,ds\,dt, &\end{aligned}$$

for $0 \leq T \leq T'$. It remains to show that the righthand side of (A.1) is nonnegative. The first two terms on the righthand side of (A.1) are nonnegative by the nonnegativity of $\varphi(u)$ and the properties of the function a. Furthermore, we employ the definition of the subdifferential and find out that

$$
\begin{aligned}
-\int_0^T a'(T-s)\varphi(u(T))\,ds &+ \int_0^T a'(T-s)\varphi(u(s))\,ds \\
&+ \int_0^T a'(T-s)\langle Au(s), u(T)-u(s)\rangle\,ds \\
&= \int_0^T (-a')(T-s)[-(\varphi(u(s))-\varphi(u(T)) \\
&\quad - \langle Au(s), u(s)-u(T)\rangle)]\,ds \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T \int_0^t a''(t-s)\varphi(u(t))\,ds\,dt &- \int_0^T \int_0^t a''(t-s)\varphi(u(s))\,ds\,dt \\
&- \int_0^T \int_0^t a''(t-s)\langle Au(s), u(t)-u(s)\rangle\,ds\,dt \\
&= \int_0^T \int_0^t a''(t-s)[-(\varphi(u(s))-\varphi(u(t)) \\
&\quad - \langle Au(s), u(s)-u(t)\rangle)]\,ds\,dt \geq 0.
\end{aligned}
$$

Hence the righthand side of (A.1) is nonnegative and the proof is complete. □

The following lemma is an infinite dimensional version of Lemma 20.5.1 from [**7**], and the proof is a trivial modification of the proof presented there.

Lemma A.2. *Let $T' > 0$ and $a \in C^1([0,T'])$. Let $A : H \supset \mathcal{D}(A) \to H$ and $\varphi : H \supset \mathcal{D}(\varphi) \to [-\infty,\infty]$. Assume that the function u satisfies*

$$u \in L^2(0,T';H),$$

$u \in \mathcal{D}(A) \cap \mathcal{D}(\varphi)$ for almost every $t \in [0,T']$ and

$$Au \in L^2(0,T';H), \qquad \varphi(u) \in L^1(0,T').$$

Then

$$\begin{aligned}\int_0^T \Big\langle \frac{d}{dt}\int_0^t a(t-s)Au(s)\,ds, u(t)\Big\rangle\,dt \\ &= \int_0^T a(t)\varphi(u(t))\,dt \\ &\quad - \int_0^T a(T-s)\varphi(u(s))\,ds \\ &\quad + \int_0^T a(T-s)\langle Au(s), u(s)\rangle\,ds \\ &\quad - \int_0^T\int_0^t a'(t-s)\varphi(u(t))\,ds\,dt \\ &\quad + \int_0^T\int_0^t a'(t-s)\varphi(u(s))\,ds\,dt \\ &\quad + \int_0^T\int_0^t a'(t-s)\langle Au(s), u(t)-u(s)\rangle\,ds\,dt\end{aligned}$$

for $0 \le T \le T'$.

The following lemma is a further development of Lemma A.2 from [**5**] applied to a special case.

Lemma A.3. *Let* $\{J_s\}_{s\ge 0} : L^2(0,T;H) \to L^2(0,T;H)$ *be the translation semigroup defined by*

$$(J_s\varphi)(t) = \begin{cases} \varphi(t+s), & \text{if } 0 \le t \le T-s \\ 0, & \text{if } T-s < t \le T \end{cases}$$

and let the nonnegative function $a \in L^1_{\text{loc}}((0,T];R)$ *be such that*

$$\lim_{\alpha\to 0+}\int_\alpha^T a(\tau)\,d\tau = \infty.$$

Assume that $v_\varepsilon \in L^2(0,T;H)$ *for* $\varepsilon > 0$, *and that the family* $\{v_\varepsilon\}_{\varepsilon>0}$ *satisfies*

$$\sup_{\varepsilon>0}\int_0^T a(s)\|J_s v_\varepsilon - v_\varepsilon\|^2_{L^2(0,T;H)}\,ds < \infty, \tag{A.2}$$

and

$$\sup_{\varepsilon>0} \|v_\varepsilon\|_{L^2(0,T;H)} < \infty. \tag{A.3}$$

Then there exists a nonnegative continuous function ω *defined on* $[0,T]$ *such that* $\omega(0) = 0$*, and*

$$\sup_{\varepsilon>0} \|J_s v_\varepsilon - v_\varepsilon\|_{L^2(0,T;H)} \le \omega(s)$$

for $s \in [0,T]$.

Outline of the proof. Let $h > 0$. Choose $\delta(h) \in (0,h)$ so that $\int_{\delta(h)}^h a(\tau)\, d\tau = 1/h$, and define a function $d_h \in L^1((0,T);R)$ by $d_h = h\chi_{[\delta(h),h]}a$, where $\chi_{[\delta(h),h]}$ is the characteristic function of the interval $[\delta(h),h]$. Note that $0 \le d_h(s) \le ha(s)$ for all $s \in (0,T]$ and

$$\int_0^T d_h(\tau)\, d\tau = h\int_{\delta(h)}^h a(\tau)\, d\tau = 1. \tag{A.4}$$

Let $e_h \in C_0^\infty((0,T);R)$ be a nonnegative function such that

$$\int_0^T |e_h(\tau) - d_h(\tau)|\, d\tau \le \sqrt{h} \tag{A.5}$$

and define

$$v_\varepsilon^h = \int_0^T e_h(s) J_s v_\varepsilon\, ds \in L^2(0,T;H).$$

For any $s \in [0,T]$

$$\|J_s v_\varepsilon - v_\varepsilon\|_{L^2(0,T;H)} \le 2\|v_\varepsilon^h - v_\varepsilon\|_{L^2(0,T;H)} + \|J_s v_\varepsilon^h - v_\varepsilon^h\|_{L^2(0,T;H)},$$

where we used the fact that J_s is a contraction semigroup. With (A.2)-(A.5) and the contraction property of J_s one can show that

$$\|v_\varepsilon^h - v_\varepsilon\|^2_{L^2(0,T;H)} \le Ch,$$

where C is a constant independent of ε. Let A be the generator of the semigroup J_s. From the theory of semigroups it then follows that

$$\|J_s v_\varepsilon^h - v_\varepsilon^h\|_{L^2(0,T;H)} \leq \int_0^T |e_h'(\tau)| \|J_\tau A^{-1}(J_s v_\varepsilon - v_\varepsilon)\|_{L^2(0,T;H)}\, d\tau,$$

where

$$A^{-1}v = -\int_t^T v(\tau)\, d\tau$$

for $v \in L^2(0,T;H)$. Define $C(h) = \|e_h'\|_{L^\infty(0,T)}$. Then

$$\begin{aligned} \|J_s v_\varepsilon^h - v_\varepsilon^h\|_{L^2(0,T;H)} &\leq C(h)T\|A^{-1}(J_s v_\varepsilon - v_\varepsilon)\|_{L^2(0,T;H)} \\ &\leq C(h)T\sqrt{2sT}\|v_\varepsilon\|_{L^2(0,T;H)}. \end{aligned}$$

Thus we find that the lemma follows if we set

$$\omega(s) = \inf_{h>0}\left(2\sqrt{Ch} + C(h)T\sqrt{2sT}\|v_\varepsilon\|_{L^2(0,T;H)}\right). \qquad \square$$

B. Construction of the approximating kernels. Since b is convex it is absolutely continuous on each closed subinterval of $(0,\infty)$ and its right derivative b_+' is a nondecreasing right-continuous function. The derivative exists everywhere except on a countable subset of $(0,\infty)$.

The approximating kernels $b_n : [0,\infty) \to R$; $n \geq 1$ are required to satisfy:

(B.1) $\quad b_n \in C^2([0,\infty)) \quad \forall\, n,$

(B.2) $\quad (-1)^i b_n^{(i)} \geq 0; \qquad i = 0,1,2 \quad \forall\, n,$

(B.3) $\quad b_n \to b \quad \text{in } L^1_{\text{loc}}([0,\infty)), \quad \text{as } n \to \infty,$

(B.4) $\quad b_n \to b \quad \text{in } C_{\text{loc}}((0,\infty)), \quad \text{as } n \to \infty,$

(B.5) $\quad b_n' \to b' \quad \text{in } L^1_{\text{loc}}((0,\infty)), \quad \text{as } n \to \infty,$

(B.6) $\quad b_n \leq b + 1 \quad \forall\, n.$

The functions b_n will be constructed by approximating b_+'. Let $T > 0$ be fixed and consider intervals $[1/n, T\sqrt{n}]$ for $n \geq 1$. By the uniform continuity of b on $[1/n, T\sqrt{n}]$, there exists a $\delta_n > 0$ such that

$|b(t) - b(s)| \leq 1/n$ if $|s - t| \leq \delta_n$. Let $\{t_j\}$ be a division of the interval $[1/n, T\sqrt{n}]$ and $\tilde{b}_n \in C^2([1/n, T\sqrt{n}])$ a nonnegative, nonincreasing and convex function such that $t_{j+1} - t_j \leq \delta_n$, $\int_{1/n}^{T\sqrt{n}} |b'(s) - \tilde{b}'_n(s)|\, ds = \int_{1/n}^{T\sqrt{n}} |b'_+(s) - \tilde{b}'_n(s)|\, ds \leq 1/n$, and $\tilde{b}_n(t_j) = b(t_j)$. The function $\tilde{b}'_n$ can be constructed by suitably smoothing a piecewise constant function that approximates b'_+ and $\tilde{b}_n$ is then obtained by integrating.

Since $t_{j+1} - t_j \leq \delta_n$, one has $\sup_{t_j \leq t \leq t_{j+1}} |\tilde{b}_n(t) - b(t)| \leq 1/n$ for all j, so that

$$\sup_{1/n \leq t \leq T\sqrt{n}} |\tilde{b}_n(t) - b(t)| \leq 1/n.$$

Finally, let b_n be an extension of $\tilde{b}_n$ to the interval $[0, \infty)$ such that $b_n \in C^2([0, \infty))$, b_n is nonnegative, nonincreasing and convex and $b_n \leq b + 1$. It is then easy to see that the sequence $\{b_n\}_n$ satisfies all the requirements (B.1)–(B.6).

REFERENCES

1. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, Leyden, 1976.

2. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam; American Elsevier, New York, 1973. MR **50** #1060.

3. M.G. Crandall, S.-O. Londen and J.A. Nohel, *An abstract nonlinear Volterra integrodifferential equation*, J. Math. Anal. Appl. **64** (1978), 701–735.

4. H. Engler, *Smooth solutions to a one-dimensional problem with large data*, Lecture given at Workshop on Mathematical Problems in Viscoelasticity, February 4-6 1993, Carnegie Mellon University, Center for Nonlinear Analysis, Pittsburgh, Pennsylvania, USA.

5. ———, *Weak solutions of a class of quasilinear hyperbolic integro-differential equations describing viscoelastic materials*, Arch. Rational Mech. Anal. **113** (1991), 1–38.

6. G. Gripenberg, *Nonlinear Volterra equations of parabolic type due to singular kernels*, J. Differential Equations, to appear.

7. G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra integral and functional equations*, Cambridge University Press, Cambridge, 1990.

8. M.E. Gurtin and W.J. Hrusa, *On the thermodynamics of viscoelastic materials of single-integral type*, Quarterly Appl. Math. **49** 1991, 67–85.

9. A. Haraux, *Nonlinear evolution equations-global behavior of solutions*, Lecture Notes in Mathematics 841, Springer-Verlag, Berlin, 1981.

10. J. Heikonen, *On the existence of a global mild solution for a nonlinear parabolic integrodifferential equation*, Licentiate's thesis, Helsinki University of Technology, Finland, 1993.

11. W.J. Hrusa, J.A. Nohel and M. Renardy, *Initial value problems in viscoelasticity*, Appl. Mech. Rev. **41** (1988), 371–378.

12. S.-O. Londen, *On an integral equation in a Hilbert space*, SIAM J. Math. Anal. **8** (1977), 950–970.

13. ———, *An existence result on a Volterra equation in a Banach space*, Trans. Amer. Math. Soc. **235** (1978), 285–304.

14. ———, *On an integrodifferential Volterra equation with a maximal monotone mapping*, J. Differential Equations **27** (1978), 405–420.

15. M. Renardy, W.J. Hrusa and J.A. Nohel, *Mathematical problems in viscoelasticity*, Longman, London, 1987.

16. F. Treves, *Basic linear differential equations*, Academic Press, San Diego, 1975.

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