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EXISTENCE, UNIQUENESS AND SMOOTHNESS RESULTS FOR SECOND-KIND VOLTERRA EQUATIONS WITH WEAKLY SINGULAR KERNELS

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ABSTRACT. We consider second-kind Volterra equations with weakly singular kernels. When the kernels assume simple forms, we find analytic solution expressions and prove existence, uniqueness and smoothness properties. Similar results for some general cases are then proved by using an idea of Professor Kendall Atkinson [**1**].

1. Introduction. The purpose of this paper is to study second-kind Volterra equations with weakly singular kernels of the forms

$$(1.1)\qquad y(t)+\int_0^t K(t,s)p(t,s)y(s)\,ds=f(t),\qquad t\in(0,T]$$

and

$$(1.2)\qquad y(t)-\int_0^t K(t,s)q(t,s)y(s)\,ds=g(t),\qquad t\in(0,T].$$

In these equations the weakly singular kernels are expressed as the product of a smooth part, $K(t,s)$, and a singular part, $p(t,s)$ or $q(t,s)$, with

$$p(t,s)=\frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{\ln(t/s)}}\left(\frac{s}{t}\right)^{\mu}\frac{1}{s}$$
$$q(t,s)=\left(\frac{s}{t}\right)^{\mu}\frac{1}{s}$$

for some $\mu>0$.

Special cases of the equations of the types (1.1) and (1.2) arise from certain practical applications (cf. [**2, 6**]). The case when the smooth

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part of the kernels $K(t,s)=1$ has been considered by several authors. In particular, solutions in weighted L_p-spaces are considered in [**5**] and [**4**].

We confine ourselves to the space of real-valued continuous functions for solutions of integral equations (1.1) or (1.2). In other words, a function is a solution of an integral equation on $(0,T]$ if the function is continuous on $[0,T]$ and satisfies the integral equation. For $T>0$ and m a nonnegative integer, $C^m[0,T]$ denotes the space of the real-valued continuous functions whose derivatives of order up to m are continuously extendable to the end points $t=0$ and T. For $y\in C^m[0,T]$, we use the usual norm

$$||y||_m = \max_{0\le j\le m}\max_{0\le t\le T}|y^{(j)}(t)|.$$

When $m=0$, we will simply write $C[0,T]$ instead of $C^0[0,T]$.

We first mention some results on simpler integral equations corresponding to the case when $K(t,s)=1$ in (1.1) and (1.2) for $\mu>1$.

Theorem 1.1. *Assume that $\mu>1$. For any $f\in C[0,T]$, the integral equation*

$$y(t)+\int_0^t p(t,s)y(s)\,ds = f(t),\qquad t\in(0,T] \tag{1.3}$$

has a unique solution $y\in C[0,T]$. Furthermore, if for an integer $m\ge 1$, $f\in C^m[0,T]$, then $y\in C^m[0,T]$.

For any $g\in C[0,T]$, the integral equation

$$y(t)-\int_0^t q(t,s)y(s)\,ds = g(t),\qquad t\in(0,T] \tag{1.4}$$

has a unique solution $y\in C[0,T]$. Furthermore, if for an integer $m\ge 1$, $g\in C^m[0,T]$, then $y\in C^m[0,T]$.

Moreover, if g is defined in terms of f through the relation

$$g(t)=f(t)-\int_0^t p(t,s)f(s)\,ds \tag{1.5}$$

then the solution of (1.3) *is also the solution of* (1.4).

Proof. In [**3**] it is proved that for any nonnegative integer m, if $f \in C^m[0,T]$, the equation (1.3) has a unique solution $y(t)$, $y \in C^m[0,T]$, which also solves the equation (1.4) if g and f are related by (1.5). Since $f \in C^m[0,T]$ implies $g \in C^m[0,T]$ for the function g defined by (1.5) (cf. Lemma 2.3 below), it remains to prove that for an arbitrary nonnegative integer m, (1.4) has a unique solution $y \in C^m[0,T]$ for any given $g \in C^m[0,T]$. We follow the proof method used in [**3**].

For any $v \in C^m[0,T]$ we define $u = \mathcal{S}(v)$ through the relation

$$u(t) = \int_0^t q(t,s)v(s)\,ds + g(t).$$

By a change of variables $s = \lambda t$, we have

$$\int_0^t q(t,s)v(s)\,ds = \int_0^1 \lambda^{\mu-1}v(\lambda t)\,d\lambda.$$

Thus,

$$u^{(j)}(t) = \int_0^1 \lambda^{\mu-1+j}v^{(j)}(\lambda t)\,d\lambda + g^{(j)}(t), \qquad 0 \le j \le m.$$

Now if $v_1, v_2 \in C^m[0,T]$ and $u_1 = \mathcal{S}(v_1)$, $u_2 = \mathcal{S}(v_2)$, then

$$\begin{aligned} |u_1^{(j)}(t) - u_2^{(j)}(t)| &\le \int_0^1 \lambda^{\mu-1+j}|v_1^{(j)}(\lambda t) - v_2^{(j)}(\lambda t)|\,d\lambda \\ &\le \frac{1}{\mu+j}||v_1 - v_2||_m, \qquad 0 \le j \le m. \end{aligned}$$

Hence,

$$||u_1 - u_2||_m \le \frac{1}{\mu}||v_1 - v_2||_m.$$

Since $\mu > 1$, the above inequality implies that $\mathcal{S}$ is a contraction mapping on the Banach space $C^m[0,T]$. Therefore, $\mathcal{S}$ has a unique fixed point y on $C^m[0,T]$. Obviously, y is the solution of (1.4). $\square$

Smoothness property of solutions is important for theoretical analysis of integral equations, as well as for error analysis when numerical methods are used to solve the equations. The smoothness property of solutions $y(t)$ of (1.3) and (1.4) for the case $0 < \mu \le 1$ is not available from [**3**].

We will provide the smoothness result for solutions $y(t)$ of (1.3) and (1.4), covering all the cases for $\mu > 0$ in the next section. When $\mu \le 0$, to make the integration meaningful, a solution of (1.3) or (1.4) must, together with its certain derivatives, vanish at $t = 0$. This requirement in turn can be used to reduce the case of $\mu \le 0$ to that of $\mu > 0$. Hence, in this paper, we will always assume $\mu > 0$. We will discuss the relationship between (1.3) and (1.4) for the case of $\mu \in (0, 1]$. We will look at the equation (1.4), find an analytic expression for its solution, and draw conclusions on the smoothness property of the solution $y(t)$. In Section 3 we employ a technique of Professor Atkinson [**1**] to study the existence, uniqueness and smoothness of a solution to the general problem (1.2) or (1.1), when K satisfies $K(t, t) = 1$. In the last section we will remark on various generalizations of the results of the previous section. We notice that, unlike the special case when $K(t, s) = 1$, in general (1.1) and (1.2) are no longer equivalent even when $\mu > 1$, no matter how the right side $g(t)$ of (1.2) is adjusted.

2. The special case when $K(t, s) = 1$. As a first step, we consider the equation (1.4) for both cases when $0 < \mu \le 1$ and $\mu > 1$.

Lemma 2.1. (a) *In the case* $0 < \mu \le 1$, *assume* $g \in C^1[0, T]$. *Then the solutions of* (1.4) *are*

$$y(t) = c_0 t^{1-\mu} + g(t) + \frac{1}{\mu - 1} g(0) + t^{1-\mu} \int_0^t s^{\mu-2}(g(s) - g(0))\, ds, \qquad c_0 \in R. \tag{2.1}$$

(b) *In the case* $\mu = 1$, *assume* $g \in C^1[0, T]$ *and* $g(0) = 0$. *Then the solutions of* (1.4) *are*

$$y(t) = c_0 + g(t) + \int_0^t s^{-1} g(s)\, ds, \qquad c_0 \in R. \tag{2.2}$$

(c) *In the case* $\mu > 1$, *assume* $g \in C[0,T]$. *Then the unique solution of* (1.4) *is*

$$y(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s)\, ds \tag{2.3}$$

and we have the regularity estimate

$$||y||_0 \le c||g||_0. \tag{2.4}$$

Proof. First we find a relation between $g(0)$ and $y(0)$ for a solution $y \in C[0,T]$ of (1.4). By a change of variables $s = t\tau$, we have

$$\int_0^t q(t,s)y(s)\, ds = \int_0^1 \tau^{\mu-1} y(t\tau)\, d\tau \to \frac{1}{\mu} y(0) \qquad \text{as } t \to 0+\,.$$

Letting $t \to 0+$ in (1.4), we obtain

$$y(0) - \frac{1}{\mu} y(0) = g(0).$$

Hence,

$$y(0) = \frac{\mu}{\mu - 1} g(0) \qquad \text{if } \mu \ne 1, \tag{2.5}$$

and, when $\mu = 1$, $g(0) = 0$ is a necessary condition for the existence of a solution to (1.4).

Let us prove the solution formulas (2.1), (2.2) and (2.3) under the assumptions $g \in C^1[0,T]$ and $g(0) = 0$. Since a solution $y \in C[0,T]$, $g \in C^1[0,T]$, it is easy to verify from

$$y(t) = \int_0^t q(t,s)y(s)\, ds + g(t), \qquad t \in (0,T]$$

that $y' \in C(0,T]$. We can differentiate (1.4) with respect to $t \in (0,T)$ to find

$$y'(t) + \frac{\mu - 1}{t} y(t) = g'(t) + \frac{\mu}{t} g(t).$$

Multiply the equation by $t^{\mu-1}$ to obtain

$$(t^{\mu-1}y(t))' = t^{\mu-1}g'(t) + \mu t^{\mu-2}g(t). \tag{2.6}$$

Integrate (2.6) from t_0 to t for some $t_0 \in (0,T)$,

$$t^{\mu-1}y(t) = t_0^{\mu-1}y(t_0) + \int_{t_0}^t (s^{\mu-1}g'(s) + \mu s^{\mu-2}g(s))\,ds \equiv z(t).$$

Obviously, $z(0) = \lim_{t\to 0+} z(t)$ exists. Denote $c_0 = z(0) = \lim_{t\to 0+} t^{\mu-1}y(t)$. Now we integrate (2.6) from 0 to t to obtain

$$\begin{aligned} t^{\mu-1}y(t) &= c_0 + t^{\mu-1}g(t) + \int_0^t s^{\mu-2}g(s)\,ds, \\ c_0 &= \lim_{t\to 0+} t^{\mu-1}y(t), \end{aligned} \tag{2.7}$$

from which part (b) of the lemma follows.

Now we prove part (a) of the lemma. For $g \in C^1[0,T]$, let us write

$$g(t) = g(0) + h(t)$$

and

$$y(t) = \frac{\mu}{\mu-1}g(0) + z(t).$$

Then $z \in C[0,T]$ satisfies

$$z(t) - \int_0^t q(t,s)z(s)\,ds = h(t), \qquad t \in (0,T]$$

with $h \in C^1[0,T]$ and $h(0) = 0$. Using the formula (2.7), we find

$$z(t) = c_0 t^{1-\mu} + h(t) + t^{1-\mu}\int_0^t s^{\mu-2}h(s)\,ds.$$

Back to the variables y and g, we get the formula (2.1).

When $\mu > 1$ the derivation of (2.7) is valid without assuming $g(0) = 0$. Also, $c_0 = \lim_{t\to 0+} t^{\mu-1}y(t) = 0$. So we have the solution formula (2.3) and, as a simple consequence of (2.3), we have the estimate (2.4). Since $C^1[0,T]$ is dense in $C[0,T]$, by a standard approximation argument, we know from (2.4) that (2.3) is true under the mere assumption $g \in C[0,T]$, and once again we have the estimate (2.4). □

Based on the solution formulas given in Lemma 2.1, we get the following main result for the equation (1.4).

Theorem 2.2. *When $\mu > 1$ for any $g \in C^m[0,T]$ ($m \geq 0$ an integer), the integral equation* (1.4) *has a unique solution y, $y \in C^m[0,T]$ and $||y||_m \leq c||g||_m$.*

When $0 < \mu \leq 1$, for any $g \in C^m[0,T]$ ($m \geq 1$ an integer), with $g(0) = 0$ if $\mu = 1$, the integral equation (1.4) *has a family of solutions depending on a parameter. Out of the family of solutions, there is one particular solution y with C^1-continuity. Such a solution is unique, and $||y||_m \leq c||g||_m$.*

Proof. By Lemma 2.1 we only need to prove the regularity estimate $||y||_m \leq c||g||_m$, which follows from the solution formulas

$$y(t) = g(t) + \int_0^1 \tau^{\mu-2} g(t\tau)\,d\tau, \qquad \mu \geq 1$$

and

$$y(t) = g(t) + \frac{1}{\mu - 1} g(0) + \int_0^1 (g(t\tau) - g(0))\,d\tau, \qquad 0 < \mu < 1$$

for the C^1 solution in the case $0 < \mu \leq 1$. □

In order to apply the results on (1.4) to the integral equation (1.3), we need to use (1.5) to study the relations for properties between g and f.

Lemma 2.3. *Let g be defined by f through* (1.5). *Let $m \geq 0$ be an integer. Then $f \in C^m[0,T]$ implies $g \in C^m[0,T]$, and*

$$||g||_m \leq c||f||_m \tag{2.8}$$

for some constant c independent of f. For $\mu \neq 1$, $g(0) = (1 - 1/\sqrt{\mu})f(0)$. For $\mu = 1$, $g(0) = 0$.

Proof. The first property is a simple consequence of the equality

$$g(t) = f(t) - \int_0^1 \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\ln(1/\tau)}} \tau^{\mu-1} f(t\tau)\, d\tau \tag{2.9}$$

obtained from (1.5) by the change of variables $s = t\tau$.

To prove the other two properties, we use the integral identity

$$\int_0^1 \frac{\tau^{\mu-1}}{\sqrt{\ln(1/\tau)}}\, d\tau = \sqrt{\frac{\pi}{\mu}}.$$

Letting $t \to 0+$ in (2.9), we obtain

$$\begin{aligned} g(0) &= f(0) - f(0)\frac{1}{\sqrt{\pi}} \int_0^1 \frac{\tau^{\mu-1}}{\sqrt{\ln(1/\tau)}}\, d\tau \\ &= \left(1 - \frac{1}{\sqrt{\mu}}\right) f(0). \qquad \square \end{aligned}$$

We then consider equation (1.3).

Lemma 2.4. *Assume* $f \in C[0,T]$. *In the case* $0 < \mu \le 1$, *we further assume* $f \in C^1[0,T]$ *and* $f(0) = 0$. *Then* (1.3) *has a unique solution* $y \in C[0,T]$,

$$y(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s)\, ds, \tag{2.10}$$

where $g(t)$ *is defined in terms of* $f(t)$ *through* (1.5). *The following regularity estimates hold*:

$$||y||_0 \le c||f||_0 \quad \text{if } \mu > 1 \tag{2.11}$$

$$||y||_1 \le c||f||_1 \quad \text{if } 0 < \mu \le 1. \tag{2.12}$$

Proof. The part of the result for the case when $\mu > 1$ follows from Theorem 1.1 and Lemma 2.1. So we need only to consider the case when $\mu \in (0,1]$. We will use a unique solvability result of (1.3) from [**4**]. Noticing that the equations studied in [**4**] are defined for $t \in (0,\infty)$, we extend f by zero for $t > T$,

$$\tilde{f}(t) = \begin{cases} f(t), & 0 \le t \le T, \\ 0, & t > T, \end{cases}$$

and consider the equation

$$\tilde{y}(t) + \int_0^t p(t,s)\tilde{y}(s)\,ds = \tilde{f}(t), \qquad t > 0. \tag{2.13}$$

By Theorem 4.1 (a) in [**4**], for $p > 1/(\mu+1)$, if $\int_0^\infty |t^{-1}\tilde{f}(t)|^p\,dt < \infty$, then there is a unique function $\tilde{y}$ satisfying (2.13) and $\int_0^\infty |t^{-1}\tilde{y}(t)|^p\,dt < \infty$. From the assumptions on f, $\int_0^\infty |t^{-1}\tilde{f}(t)|^p\,dt < \infty$ for all $p > 0$. Obviously, $y(t) = \tilde{y}(t)$ for $t \le T$. Thus, there is a unique function y satisfying the equation (1.3) and

$$\int_0^T |t^{-1}y(t)|^p\,dt < \infty, \qquad \forall\, p > \frac{1}{\mu+1}. \tag{2.14}$$

From

$$y(t) = f(t) - \int_0^t p(t,s)y(s)\,ds, \qquad t \in (0,T]$$

we have $y \in C^1(0,T]$. Now let us choose $p > \max\{1, 1/\mu\}$ and denote by $q > 1$ the conjugate of p defined by $1/q + 1/p = 1$. Since

$$\begin{aligned} &\left| t^{\mu-1}\int_0^t p(t,s)y(s)\,ds \right| \\ &\qquad \le t^{\mu-1}\biggl\{\int_0^t |s^{-1}y(s)|^p\,ds\biggr\}^{1/p}\biggl\{\int_0^t |sp(t,s)|^q\,ds\biggr\}^{1/q} \\ &\qquad = t^{\mu-1/p}\biggl\{\int_0^t |s^{-1}y(s)|^p\,ds\biggr\}^{1/p}\frac{1}{\sqrt{\pi}}\biggl\{\int_0^1 \frac{\lambda^{q\mu}}{(\ln(1/\lambda))^{q/2}}\,d\lambda\biggr\}^{1/q} \\ &\qquad \to 0 \qquad \text{as } t \to 0+ \end{aligned}$$

and

$$t^{\mu-1}f(t) \to 0 \quad \text{as } t \to 0+$$

we find that

$$\lim_{t\to 0+}(t^{\mu-1}y(t)) = 0.$$

From the proof of Lemma 2 in [**3**], the function y satisfies (1.4). Hence, $y(t)$ satisfies the differential equation (2.6). Since $c_0 = \lim_{t\to 0+} t^{\mu-1}y(t) = 0$, following the proof of Lemma 2.1, we find

$$t^{\mu-1}y(t) = t^{\mu-1}g(t) + \int_0^t s^{\mu-2}g(s)\,ds, \qquad t \in (0,T],$$

where g is defined by (1.5). Hence, we have the solution formula (2.10). In particular, $y \in C[0,T]$ and is the solution of (1.3).

When $\mu = 1$, we rewrite (2.10) in the form

$$y(t) = g(t) + \int_0^t \ln(t/s)g'(s)\,ds.$$

When $0 < \mu < 1$, we rewrite (2.10) in the form

$$y(t) = g(t) + \frac{t^{1-\mu}}{1-\mu}\int_0^t (s^{\mu-1} - t^{\mu-1})g'(s)\,ds.$$

From these latter formulas and Lemma 2.3, the estimate (2.12) is obviously seen to be true. □

Similar to the proof of Theorem 2.2, we then have the following main result for the problem (1.3).

Theorem 2.5. *Assume $f \in C[0,T]$ if $\mu > 1$, $f \in C^1[0,T]$ and $f(0) = 0$ if $0 < \mu \le 1$. Then the integral equation* (1.3) *has a unique solution $y \in C[0,T]$ which satisfies*

$$||y||_0 \le c||f||_0 \quad \text{if } \mu > 1$$
$$||y||_1 \le c||f||_1 \quad \text{if } 0 < \mu \le 1.$$

If, for some integer $m \geq 1$, $f \in C^m[0,T]$, *then the solution* $y \in C^m[0,T]$, *and*

$$||y||_m \leq c||f||_m.$$

Now we provide several remarks concerning the results on the equation (1.3). Similar remarks can be stated for equation (1.4).

Remark 2.6. For the case $0 < \mu < 1$, the assumption $f \in C^1[0,T]$ can be replaced by a weaker one, namely, $f \in C^{0,1-\mu+\varepsilon}[0,T]$ for any $\varepsilon > 0$. We will then still have the analytic expression (2.10) for the solution, and the estimate (2.12) will be replaced by

$$||y||_0 \leq c||f||_{1-\mu+\varepsilon} \qquad \text{if } 0 < \mu < 1,$$

where the constant c depends only on μ and ε. We will not, however, delve into such minimal possible smoothness assumptions on right sides of integral equations.

Remark 2.7. When $0 < \mu \leq 1$, the estimate (2.12) can be modified into the following form

$$||y||_0 \leq c_1||f||_{C^1[0,\varepsilon]} + c_2||f||_{C[\varepsilon,T]}$$

for any $\varepsilon > 0$. A simple consequence of the above estimate is that, for the case $0 < \mu \leq 1$, the assumption $f \in C^1[0,T]$ can be replaced by the condition that $f(t)$ is continuously differentiable in a neighborhood of $t = 0$ and $f(t)$ is continuous away from $t = 0$. However, we will not pursue this kind of improvement.

Remark 2.8. We observe that (1.3) and (1.4) are equivalent in the following sense: If $\mu > 1$, then a $C[0,T]$ function is a solution of (1.3) if and only if it is a solution of (1.4); and such a solution is unique.

If $0 < \mu \leq 1$, then a $C^1[0,T]$ function is a solution of (1.3) if and only if it is a solution of (1.4); and such a $C^1[0,T]$ solution is unique.

3. The general case with $K(t,t) = 1$**.** We now consider the solvability of (1.1) and (1.2) and the smoothness of solutions. We will give a detailed proof only for the results concerning the equation (1.2), since the results for the equation (1.1) can be proved similarly.

Assume $K(t,t) = 1$, $K(t,s)$ is twice continuously differentiable for $0 \le s \le t \le T$, i.e., K and its derivatives up to order 2 are continuous over the set $\Omega = \{(t,s) \in R^2 : 0 < s < t < T\}$ and are continuously extendable to $\partial\Omega$, the boundary of Ω. In the next section, we will discuss various cases where $K(t,t) \ne 1$. We define

$$H(t,s) = \begin{cases} \dfrac{K(t,s) - K(s,s)}{t-s}, & \text{if } s < t, \\ \dfrac{\partial K(t,s)}{\partial t}, & \text{if } s = t. \end{cases} \tag{3.1}$$

Then $H(t,s)$ is continuously differentiable for $0 \le s \le t \le T$.

The equation (1.2) can be rewritten in the equivalent form

$$\begin{aligned} y(t) &- \int_0^t q(t,s) y(s)\, ds \\ &- \int_0^t H(t,s)(t-s) q(t,s) y(s)\, ds = g(t), \qquad t \in (0,T]. \end{aligned} \tag{3.2}$$

Let us define an operator

$$\mathcal{A}\, y(t) = y(t) - \int_0^t q(t,s) y(s)\, ds. \tag{3.3}$$

By Theorem 2.2, $\mathcal{A}$ is an isomorphism from $E[0,T]$ to $E[0,T]$,

$$||\mathcal{A}\,||_{E[0,T] \to E[0,T]} \le c,$$

where

$$E[0,T] = \begin{cases} C[0,T], ||y||_{E[0,T]} = ||y||_0, & \text{if } \mu > 1, \\ \{y \in C^1[0,T] \mid y(0) = 0\}, ||y||_{E[0,T]} = ||y||_1, & \text{if } \mu = 1, \\ C^1[0,T], ||y||_{E[0,T]} = ||y||_1, & \text{if } 0 < \mu < 1. \end{cases}$$

We assume for the right side of (1.2), $g \in E[0,T]$.

We remark that we seek a solution y of (1.2) in $E[0,T]$. So, in the case of $\mu \in (0,1]$, we only consider $C^1[0,T]$ a solution for (1.4), and such a solution is unique ensuring the existence of $\mathcal{A}^{-1}$.

Define

$$\mathcal{H}\, y(t) = \int_0^t H(t,s)(t-s)q(t,s)y(s)\, ds. \tag{3.4}$$

We observe that

(3.5)

$$\begin{aligned}\frac{d}{dt}\mathcal{H}\, y(t) &= \int_0^t \left(\frac{\partial K(t,s)}{\partial t} + (-\mu)H(t,s)\frac{t-s}{t}\right)q(t,s)y(s)\, ds \\ &= \int_0^1 \left(\frac{\partial K(t,s)}{\partial t}\bigg|_{s=tz} + (-\mu)H(t,tz)(1-z)\right)z^{\mu-1}y(tz)\, dz.\end{aligned}$$

Obviously,

$$\mathcal{H} : C[0,T] \to E[0,T], \quad \text{and} \quad \mathcal{H}\, y(0) = 0. \tag{3.6}$$

Now (3.2) can be written in the equivalent form

$$(I - \mathcal{A}^{-1}\mathcal{H})y = \mathcal{A}^{-1}g, \tag{3.7}$$

where the equation at $t = 0$ is defined by continuity.

Following the idea of [**1**], we use the Neumann series to solve the above equation. We define

$$y_j = (\mathcal{A}^{-1}\mathcal{H})^j \mathcal{A}^{-1}g, \qquad j = 0, 1, 2, \dots . \tag{3.8}$$

Obviously, $y_j \in E[0,T]$. We will prove that

$$y = \sum_{j=0}^{\infty} y_j \tag{3.9}$$

is the solution of (3.7). First, let us inductively bound the sequence $\{y_j(t)\}$. Denote

$$\begin{aligned} M &= \max_{0\le s\le t\le T} |H(t,s)| \\ G &= \max_{0\le t\le T} |\mathcal{A}^{-1}g(t)| \le c||g||_{E[0,T]}. \end{aligned}$$

We have

$$y_0(t) = \mathcal{A}^{-1} g(t)$$
$$|y_0(t)| \le G, \qquad 0 \le t \le T.$$

Assume, for $i \le j$,

$$|y_i(t)| \le G d_i t^i, \qquad 0 \le t \le T$$

for some constants d_i, $i \le j$, $d_0 = 1$. Note that

$$y_{j+1}(t) = \mathcal{A}^{-1} \mathcal{H} y_j(t) \in E[0,T]$$

and

$$\begin{aligned} |\mathcal{H} y_j(t)| &= \left| \int_0^t H(t,s)(t-s) q(t,s) y_j(s)\, ds \right| \\ &\le M \int_0^t (t-s) q(t,s) G d_j s^j \, ds \\ &= M G d_j \frac{t^{j+1}}{(\mu+j)(\mu+j+1)}. \end{aligned}$$

Thus, since $\mathcal{H} y_j(0) = 0$, applying Lemma 2.1, we obtain

$$y_{j+1}(t) = \mathcal{H} y_j(t) + t^{1-\mu} \int_0^t s^{\mu-2} \mathcal{H} y_j(s)\, ds,$$

and so

$$\begin{aligned} |y_{j+1}(t)| \le{}& M G d_j \frac{t^{j+1}}{(\mu+j)(\mu+j+1)} \\ &+ t^{1-\mu} \int_0^t s^{\mu-2} M G d_j \frac{s^{j+1}}{(\mu+j)(\mu+j+1)} \, ds \\ ={}& G \frac{M d_j}{(\mu+j)^2} t^{j+1}, \qquad 0 \le t \le T. \end{aligned}$$

Therefore, if we define a sequence $\{d_j\}$ by

$$d_0 = 1, \qquad d_{j+1} = \frac{M d_j}{(\mu+j)^2}, \qquad j = 0, 1, \dots, \tag{3.10}$$

then we have the estimates

$$(3.11) \qquad |y_j(t)| \le G d_j t^j, \qquad 0 \le t \le T,\ j = 0, 1, \dots .$$

From (3.10) and (3.11), we see that the series (3.9) converges uniformly on $[0, T]$. Hence, by a standard argument, $y = \sum_{j=0}^{\infty} y_j \in C[0, T]$ and satisfies equation (3.7). We further have $y = \mathcal{A}^{-1}\mathcal{H}\, y + \mathcal{A}^{-1} g \in E[0, T]$. Hence, y is a solution of (3.7).

For the uniqueness of a solution of (3.7), it is equivalent to prove that the homogeneous equation admits only the trivial solution. Let $y \in E[0, T]$ solve

$$y - \mathcal{A}^{-1}\mathcal{H}\, y = 0.$$

Then

$$y = (\mathcal{A}^{-1}\mathcal{H})^j y$$

for any positive integer j. Using the same technique as above, we then have the estimate $|y(t)| \le G d_j t^j$ for any positive integer j. Since $d_j \to 0$ as $j \to \infty$, we conclude immediately that $y(t) = 0$.

Therefore, we have the existence and uniqueness of a solution $y \in E[0, T]$ of the integral equation (1.2).

For higher order regularity of the solution, y, we assume for some integer $m \ge 1$,

$$g \in E[0, T] \cap C^m[0, T], \qquad K \in C^{m+1},\ 0 \le s \le t \le T.$$

From (3.5) it is immediate that $\mathcal{H}$ is a continuous mapping from $C^j[0, T]$ to $C^{j+1}[0, T]$, $0 \le j \le m - 1$. Using the property that $\mathcal{A}^{-1}$ maps $C^j[0, T]$ to $C^j[0, T]$, $1 \le j \le m$, and a simple induction argument on $y = \mathcal{A}^{-1}\, (\mathcal{H}\, y + g)$, we then have the regularity $y \in C^m[0, T]$.

In conclusion, we have proved

Theorem 3.1. *Assume $K(t, s)$ is twice continuously differentiable for $0 \le s \le t \le T$, $K(t, t) = 1$ and $g \in E[0, T]$. Then* (1.2) *admits a unique solution $y \in E[0, T]$, and we have*

$$||y||_{E[0,T]} \le c||g||_{E[0,T]}.$$

If we further assume, for some integer $m \geq 1$, $g \in C^m[0,T]$, $K(t,s)$ is $(m+1)$ times continuously differentiable for $0 \leq s \leq t \leq T$, then $y \in C^m[0,T]$, and

$$||y||_m \leq c||g||_m.$$

Similar results are true for the integral equation (1.1). The proof is also similar, and hence is omitted here. We notice that, in general, the equations (1.1) and (1.2) (with a suitably chosen right side) are not equivalent.

4. Some remarks. In the previous section we considered the equations (1.1) and (1.2) under the restriction $K(t,t) = 1$. As a first attempt to remove the restriction, we consider the equation

$$y(t) - \beta \int_0^t q(t,s)y(s)\,ds = g(t), \qquad t \in (0,T]. \tag{4.1}$$

We note that when the parameter $\beta = 1$, we get back to the equation (1.4). We have the following generalization of Lemma 2.1.

Lemma 4.1. *Let $k_{\beta,\mu}$ be the smallest integer greater than $\beta - \mu$, $l_{\beta,\mu} = \max\{0, k_{\beta,\mu}\}$. Assume that $g \in C^{l_{\beta,\mu}}[0,T]$ and $g^{(i)}(0) = 0$, $0 \leq i \leq k_{\beta,\mu} - 1$ when $k_{\beta,\mu} \geq 1$. When $\mu > \beta$, (4.1) has a unique solution $y \in C[0,T]$, given by*

$$y(t) = g(t) + \beta t^{\beta-\mu} \int_0^t s^{\mu-\beta-1} g(s)\,ds \tag{4.2}$$

which satisfies

$$y \in C^{l_{\beta,\mu}}[0,T]$$

and

$$||y||_{l_{\beta,\mu}} \leq c||g||_{l_{\beta,\mu}}.$$

When $\mu \leq \beta$, (4.1) has a family of $C[0,T]$ solutions given by

$$y(t) = c_0 t^{\beta-\mu} + g(t) + \beta t^{\beta-\mu} \int_0^t s^{\mu-\beta-1} g(s)\,ds, \qquad c_0 \in R.$$

To remove the restriction on the initial values of g, we use Taylor's expansions

$$y(t) = \sum_{j=0}^{k_{\beta,\mu}-1} \frac{y^{(j)}(0)}{j!} t^j + \bar{y}(t)$$

$$g(t) = \sum_{j=0}^{k_{\beta,\mu}-1} \frac{g^{(j)}(0)}{j!} t^j + \bar{g}(t),$$

where $y^{(j)}(0)$ and $g^{(j)}(0)$ are related by

$$y^{(j)}(0)\left(1 - \frac{\beta}{\mu + j}\right) = g^{(j)}(0), \qquad 0 \le j \le k_{\beta,\mu} - 1. \tag{4.3}$$

Then $\bar{y}$ satisfies

$$\bar{y}(t) - \beta \int_0^t q(t,s)\bar{y}(s)\, ds = \bar{g}(t), \qquad t \in (0,T].$$

We have an analytic expression for $\bar{y}(t)$ by using Lemma 4.1. Back to the original variable y, we then get the following generalization of Theorem 2.2.

Theorem 4.2. *Assume that $g \in C^{l_{\beta,\mu}}[0,T]$. If $\beta-\mu$ is a nonnegative integer, we also assume that $g^{(\beta-\mu)}(0) = 0$. Then, when $\mu > \beta$, the integral equation* (4.1) *has a unique $C[0,T]$ solution $y(t)$, $y \in C^{l_{\beta,\mu}}[0,T]$ and satisfies*

$$||y||_{l_{\beta,\mu}} \le c||g||_{l_{\beta,\mu}}.$$

If, furthermore, for some $m > l_{\beta,\mu}$, $g \in C^m[0,T]$, then the solution $y \in C^m[0,T]$, and

$$||y||_m \le c||g||_m.$$

When $\mu \le \beta$, (4.1) *has a family of $C[0,T]$ solutions depending on a parameter.*

Now we turn our attention to a generalization of the equation (1.3). We consider

$$y(t) + \alpha \int_0^t p(t,s)y(s)\, ds = f(t), \qquad t \in (0,T]. \tag{4.4}$$

It is easy to verify that (4.4) is transformed into (4.1) with

$$\beta = \alpha^2 \tag{4.5}$$

and

$$g(t) = f(t) - \alpha \int_0^t p(t,s) f(s)\, ds. \tag{4.6}$$

From (4.6), one can prove that

$$g^{(j)}(0) = f^{(j)}(0)\left(1 - \frac{\alpha}{\sqrt{\mu + j}}\right), \qquad j \geq 0.$$

A simple consequence is that if $\alpha^2 - \mu$ is a nonnegative integer, then automatically we have $g^{(\alpha^2-\mu)}(0) = 0$. In the case where $\mu \leq \beta$, one can similarly prove that $c_0 = \lim_{t\to 0+} t^{\mu-\beta} y(t) = 0$. Thus, from Theorem 4.2, we get the following result for the equation (4.4).

Theorem 4.3. *Let $\beta = \alpha^2$. Assume that $f \in C^{l_{\beta,\mu}}[0,T]$. Then the integral equation* (4.4) *has a unique solution $y \in C^{l_{\beta,\mu}}[0,T]$, which satisfies*

$$||y||_{l_{\beta,\mu}} \leq c||f||_{l_{\beta,\mu}}.$$

If, furthermore, for some $m > l_{\beta,\mu}$, $f \in C^m[0,T]$, then the solution $y \in C^m[0,T]$, and

$$||y||_m \leq c||f||_m.$$

Remark 4.4. With Theorems 4.2 and 4.3 and the technique used in Section 3, it is straightforward to generalize the results of Theorem 3.1 for the existence, uniqueness and smoothness of solutions of the integral equations (1.2) (and (1.1)), under the less restrictive assumption that $K(t,t) =$ constant. Since a presentation of the results and the proof are similar to what we have and what we do in the previous section, we omit the detail here.

In principle, one can discuss the integral equations (1.1) and (1.2) for the most general case when $K(t,s)$ is smooth, and $K(t,t) = h(t)$ is a smooth function of t. As noted in the above remark, any results for such general cases can be derived in a straightforward manner from corresponding results on integral equations of forms

$$y(t) + h(t) \int_0^t p(t,s) y(s)\, ds = f(t), \qquad t \in (0,T] \tag{4.7}$$

and

$$y(t) - h(t) \int_0^t q(t,s) y(s)\, ds = g(t), \qquad t \in (0,T]. \tag{4.8}$$

Let us consider (4.8), for example. If we follow the derivation presented in the proof of Lemma 2.1, we find that a solution $y(t)$ (if it exists and makes the derivation meaningful) solves

$$y'(t) - \left(\frac{h'(t)}{h(t)} + \frac{h(t)-\mu}{t}\right) y(t) = g'(t) - \left(\frac{h'(t)}{h(t)} - \frac{\mu}{t}\right) g(t). \tag{4.9}$$

It would be a mess to write down an analytic expression of a general solution of the above ordinary differential equation, and hence it is doubtful if such a complicated solution formula for the most general case will be useful. For practical applications, however, we can expect to have a less general function $h(t)$, and we can obtain a simple-minded formula for solutions of the equation (4.9). Then we can proceed similarly as above to draw conclusions on the existence, uniqueness and smoothness properties of a solution of the equation (1.2). Similar work can be done on the integral equation (1.1). In our future work, we will employ the idea presented here to study the integral equations (1.1) and (1.2) in various other cases. Finally, let us notice that, in order to obtain results on the integral equations (1.1) and (1.2), by using the technique presented here, it is not necessary to have an analytic expression for solutions of the equation (4.9). It will be sufficient if we can prove the existence, uniqueness, smoothness and regularity estimate for solutions of the equations (4.7) and (4.8) directly. This way may be proved to be successful to study the integral equations (1.1) and (1.2) for the most general case.

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