JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 6, Number 3, Summer 1994

SINGULARITY PRESERVING GALERKIN METHODS FOR WEAKLY SINGULAR FREDHOLM INTEGRAL EQUATIONS

YANZHAO CAO AND YUESHENG XU

ABSTRACT. Singularity preserving projection methods are developed in this paper for Fredholm integral equations of the second kind with weakly singular kernels. These methods give an optimal order of convergence for the approximate solutions. As an application, the singularity preserving Galerkin approximation for equations with logarithmic or algebraic singular kernels is discussed in detail. This is done by deriving singularity expansions for the solutions of these equations. A numerical example is given to illustrate the error estimates.

1. Introduction. In the last decade there has been considerable interest in the numerical analysis of solutions of integral equations with weakly singular kernels. Most of the existing numerical methods for these equations concentrate on approximating the solutions by functions without singularities, e.g., by polynomials or splines. In this work we establish Galerkin approximations that preserve the singularities of the solutions and possess an optimal order of convergence. This will be done by allowing the projection subspaces to contain some known singular functions that carry the singularities of the exact solutions. The singularities of the approximate solutions will cancel with those of the exact solutions, and consequently, the order of convergence will achieve the optimal rate. The regularity properties and singularity expansions of the solutions play a central role in this work.

Let $L_p = L_p[0,1]$ be the Banach space of pth power integrable functions with norm defined by $||f||_p = (\int_0^1 |f(t)|^p\, dt)^{1/p}$ for $1 \leq p < \infty$. In this paper we study singularity preserving projection methods for solutions of Fredholm integral equations of the second kind that take the form

$$(1.1) \qquad y(s) - \int_0^1 k(s,t)y(t)\, dt = f(s), \qquad 0 \leq s \leq 1,$$

Received by the editors on April 15, 1993, and in revised form on May 5, 1994.
Mathematics Subject Classification (1991). 65R22, 45L10, 45E99.
This work is partially supported by NASA under grant NAG 3-1312.

where k is an L_1 kernel defined on $[0,1]\times[0,1]$ and usually having weak singularities, and f is a given function in a function class to be specified later. It is well known (e.g., see [**1, 2**]) that the integral operator defined by

$$(Kx)(s) = \int_0^1 k(s,t)x(t)\,dt, \qquad 0 \le s \le 1, \tag{1.2}$$

is a compact operator in L_1. Assume that 1 is not an eigenvalue of K. Then equation (1.1) has a unique solution in L_1 (see [**1, 2**]). We rewrite equation (1.1) in operator notation as

$$y = Ky + f. \tag{1.3}$$

The regularity properties of the solution of equation (1.1) have been studied extensively in the literature. If $k(s,t)$ is of the form $|s-t|^\alpha m(s,t)$ for $-1 < \alpha < 0$, or $\log|s-t|m(s,t)$, where m is a smooth kernel, the regularity properties of the solution of equation (1.1) were established in [**12**]. A similar result in a slightly different setting was proved in [**15**]. If $k(s,t) = k(|s-t|)$ for some $k \in L_p$, $1 \le p \le \infty$, a singularity expansion of the solutions of (1.1) was given in [**4, 11**]. From these results, we see that the solutions of the weakly singular Fredholm integral equations usually have singularities in their derivatives, reflecting the singularity of the kernel. Several numerical methods have been designed based on this fact. A product-integration method, a collocation method and a Galerkin method were presented in [**13, 16**] and [**5**], respectively. In these methods, the solutions are approximated by piecewise polynomials with a partition defined corresponding to the singularity of the solution. This nice idea of nonlinear approximation was first introduced by Rice [**10**]. A modified collocation method was introduced in [**7**], where the integral equation was recast as an integro-differential equation with a mild singularity. All the work cited above is dependent on an appropriate choice of the knots of the piecewise polynomials used to approximate the solution. The approximate solutions provided by these methods are piecewise polynomials and have no singularities at the endpoints. As the singularities often describe certain important physical features, a reasonable approximate solution should preserve singularities that the exact solution possesses.

The main purpose of this paper is to present a singularity preserving projection approximation for the solution of equation (1.1) having an optimal order of convergence. In Section 2 we develop some singularity preserving projection approximation methods and establish a theorem about their order of convergence. In Section 3 we apply the general result obtained in Section 2 to develop the singularity preserving Galerkin approximation. In Sections 4 and 5, we extend the singularity expansions of [**4, 11**] for the simple cases where $k(s,t) = \log|s-t|$ and $k(s,t) = |s-t|^\alpha$, with $-1 < \alpha < 0$ to the more general settings where $k(s,t) = \log(|s-t|)m(s,t)$ and $k(s,t) = |s-t|^\alpha m(s,t)$, respectively. The inclusion of the smooth factor $m(s,t)$ into the kernel greatly increases the relevance of the theory to the type of weakly singular equations that arise in practice (see [**5, 12**]). Mathematically, this extension is interesting and technically nontrivial. By making use of this expansion, we define projection subspaces which contain singular functions. This method gives the optimal order of convergence. In Section 6 we discuss the computational implementation of the singularity preserving Galerkin method. A numerical example is given in Section 6 to illustrate the numerical accuracy of the current method in comparison with the conventional Galerkin method.

2. Singularity preserving projection approximation. In this section we develop a singularity preserving projection approximation method for the solution of equation (1.1) and establish a general result about the order of convergence of the approximation.

Let n be a positive integer. Let H^n denote the Sobolev space $H^n[0,1] = \{w : w^{(n)} \in L_2[0,1]\}$, and $(\cdot,\cdot)$ denote the inner product in L_2. It is well known that H^n equipped with the inner product $(u,v)_n = \sum_{i=0}^n (u^{(i)}, v^{(i)})$ is a Hilbert space. Then

$$||u||_{H^n} = \left(\sum_{i=0}^{n} ||u^{(i)}||_2^2 \right)^{1/2}$$

is the norm induced by this inner product. Let W be a finite dimensional subspace of $C[0,1]$ containing mildly singular functions that reflect the singularities of the exact solution of equation (1.1). For this reason W is called a singular subspace for equation (1.1). The choice of this singular subspace depends on the singularity decomposition of the exact solution of the equation under consideration. Throughout this section, we assume that the solution y of equation (1.1) has the decomposition $y = w + v$, where $w \in W$ and $v \in H^n$. We remark that w preserves the singularity of y and v is a smooth function.

An operator P mapping from L_2 into a subspace is said to be a projection if $P^2 = P$. Let $h > 0$ be a parameter to be specified later. Let S_h^n be a finite dimensional linear subspace of $C[0,1]$, depending on n and h. In most applications, this space will be a space of polynomial splines of degree $n-1$ with a certain degree of continuity at given knots and with h being the maximal distance between two successive knots. This is the conventional projection subspace. Let P_h' be a linear projection from L_2 into S_h^n satisfying

1. $||P_h' u - u||_2 \to 0$ for all $u \in L_2$, and
2. $||P_h' u - u||_2 \le Ch^n ||u||_{H^n}$, for all $u \in H^n$.

We define a singularity preserving projection subspace V_h^n to be the direct sum of the singular subspace W and the conventional projection subspace S_h^n, i.e.,

$$V_h^n = W \oplus S_h^n.$$

Let P_h be a linear projection mapping from L_2 into V_h^n. Since this linear projection maps a function in L_2 into the singularity preserving subspace V_h^n, it is called a singularity preserving projection. A function $y_h \in V_h^n$ is called a singularity preserving projection approximation for the solution of (1.1) if it satisfies

$$y_h = P_h K y_h + P_h f. \tag{2.1}$$

Clearly, since V_h^n contains W, y_h will preserve the singularity of the exact solution of equation (1.1). The main theorem of this section will present the order of convergence of y_h in terms of the parameter h. Even though the exact solution of equation (1.1) is not continuously differentiable on $[0,1]$, y_h has an order of convergence as high as conventional projection approximation for an equation with a sufficiently smooth kernel whose solution is in $C^n[0,1]$. To prove this theorem we need the following preliminary results.

Lemma 2.1. *Let X be a Banach space. Suppose that U_1 and U_2 are two subspaces of X with $U_1 \subseteq U_2$. Assume that $P_1 : X \to U_1$ and*

$P_2 : X \to U_2$ are linear operators. If P_2 is a projection, then

$$||x - P_2x||_X \leq (1 + ||P_2||_X)||x - P_1x||_X \quad \textit{for all } x \in X.$$

Proof. Let $x \in X$. We write

$$x - P_2x = (x - P_1x) + (P_1x - P_2x).$$

Since $P_1x \in U_1$ and $U_1 \subseteq U_2$, we have $P_2P_1x = P_1x$. Hence,

$$x - P_2x = x - P_1x + P_2P_1x - P_2x = (I - P_2)(x - P_1x).$$

It follows that

$$||x - P_2x||_X \leq (1 + ||P_2||_X)||x - P_1x||_X \quad \text{for all } x \in X.$$

The proof is complete. □

Next we state a result from [**1**] for the readers' convenience.

Proposition 2.2. *Let X be a Banach space with a norm $|| \cdot ||$. Assume $T, T_n : X \to X$ are bounded linear operators with $T_n \to T$ pointwise. Then*

$$||(T_n - T)K|| \to 0$$

for each compact operator $K : X \to X$.

We are now ready to prove the main theorem of this section.

Theorem 2.3. *Let P_h be a set of linear projections uniformly bounded in L_2 mapping from L_2 into $V_h^n = W \oplus S_h^n$. Assume that for each $h > 0$ there is a linear projection from L_2 into S_h^n that satisfies the assumptions* 1 *and* 2. *Let y be the solution of equation* (1.1) *with a decomposition $y = w + v$, $w \in W$ and $v \in H^n$. Then there exists an $h_0 > 0$ for which equation* (2.1) *has a unique solution $y_h \in V_h^n$ with an estimate*

$$||y - y_h||_2 \leq Ch^n||v||_{H^n}, \quad \textit{whenever } 0 < h < h_0,$$

where $C > 0$ is a constant independent of h.

Proof. Subtract (1.1) from (2.1) and obtain

$$y_h - y = P_h K y_h - Ky + P_h f - f. \tag{2.2}$$

By applying the operator P_h to both sides of equation (1.3), we have

$$P_h y = P_h K y + P_h f.$$

Hence,

$$P_h f - f = P_h y - P_h K y - y + Ky. \tag{2.3}$$

Substituting (2.3) into (2.2) gives

$$y_h - y = P_h K(y_h - y) + P_h y - y. \tag{2.4}$$

Since K is a compact operator and $P_h u \to u$ for all $u \in L_2$, it follows from Proposition 2.2 that $||P_h K - K||_2 \to 0$, as $h \to 0$. Since $(I-K)^{-1}$ exists, by a standard functional analysis argument (cf. [**1**]), we conclude that there exists an $h_0 > 0$ such that for all $0 < h < h_0$ the inverse operator $(I - P_h K)^{-1}$ exists and

$$||(I-K)^{-1}(P_h K - K)||_2 < \frac{1}{2}.$$

Hence,

$$||(I - P_h K)^{-1}||_2 \le \frac{||(I-K)^{-1}||_2}{1 - ||(I-K)^{-1}(P_h K - K)||_2} < C_1, \quad 0 < h < h_0, \tag{2.5}$$

where $C_1 = 2||(I-K)^{-1}||_2$. It follows from (2.4) and the argument above that

$$y - y_h = (I - P_h K)^{-1}(y - P_h y). \tag{2.6}$$

Notice that y has the decomposition $y = w + v$ and that $P_h w = w$ since $w \in W \subset V_h^n$. We conclude that

$$P_h y = P_h(w + v) = P_h w + P_h v = w + P_h v.$$

This implies that

$$P_h y - y = w + P_h v - (w + v) = P_h v - v. \tag{2.7}$$

Therefore, by (2.5), (2.6) and (2.7), we find

$$||y - y_h||_2 \leq C_1 ||y - P_h y||_2 = C_1 ||P_h v - v||_2.$$

Let P'_h be a linear projection from L_2 into S^n_h satisfying the assumptions 1 and 2. Now we apply Lemma 2.1 to obtain

$$||P_h v - v||_2 \leq (1 + ||P_h||_2) ||P'_h v - v||_2.$$

Since P_h is uniformly bounded, there exists a constant $C_2 > 0$ such that

$$||P_h v - v||_2 \leq C_2 ||P'_h v - v||_2, \qquad 0 < h < h_0.$$

By assumption 2 on the projection P'_h, we have

$$||y - y_h||_2 \leq C h^n ||v||_{H^n},$$

where $C = C_1 C_2$. The proof is complete. □

Notice that, in the estimate of Theorem 2.3, the upper bound is given in terms of the H^n norm of v only and v is the smooth part of y. It should be pointed out that, although Theorem 2.3 is stated in terms of L_2 and H^n norms, a similar result holds for L_p and the related Sobolev norms.

3. Singularity preserving Galerkin methods. In this section we apply the general result obtained in the last section to the orthogonal projection (Galerkin) approximation.

We now define S^n_h specifically. Assume we are given the partition

$$\Delta : 0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$$

of $[0, 1]$. Let

$$h = \max_{1 \leq i \leq k+1} (t_i - t_{i-1}),$$

and assume that $h \to 0$ as $k \to \infty$. Let $I_i = (t_{i-1}, t_i)$ for $i = 1, 2, \dots, k+1$. Denote by Π_n the set of polynomials of degree $n-1$. Let

$$S_h^n = S_h^{n,\nu}(\Delta) = \{s \in C^\nu[0,1] : s|_{I_1} \in \Pi_n, \}$$

where $0 \le \nu \le n-1$. This space is called the space of spline functions of degree $n-1$ with knots at $t_1, t_2, \dots, t_k$ of multiplicity $n-1-\nu$; its dimension is $d = n(k+1) - k(1+\nu)$. The smoothest space of nondegenerate splines is the one with $\nu = n-2$, which is of dimension $n+k$. It is well known that the space $S_h^{n,\nu}$ has a basis consisting of B-splines $\{B_i\}_{i=1}^d$.

With this specific S_h^n we define V_h^n as in the last section, i.e., $V_h^n = W \oplus S_h^n$. Let P_h^G be the orthogonal projection from L_2 into V_h^n defined by

$$(P_h^G u, v_h) = (u, v_h) \quad \text{for all } v_h \in V_h^n,$$

where $u \in L_2$ is fixed. We call a solution y_h of the equation

$$(y_h, v_h) = (Ky_h, v_h) + (f, v_h), \quad \text{for all } v_h \in V_h^n \tag{3.1}$$

a Galerkin approximation in V_h^n of the exact solution y of equation (1.1). y_h preserves the singularity of y.

Note that W is a finite dimensional subspace. Assume $w_1, \dots, w_\mu$ is a basis for W. Then, we let

$$y_h(s) = \sum_{i=1}^{\mu} a_i w_i(s) + \sum_{i=1}^{d} b_i B_i(s),$$

where coefficients a_i, $i = 1, 2, \dots, \mu$ and b_i, $i = 1, 2, \dots, d$ are determined by the linear system of equations:

$$\begin{aligned}\sum_{i=1}^{\mu} a_i(w_i, w_j) + \sum_{i=1}^{d} b_i(B_i, w_j) &= \sum_{i=1}^{\mu} a_i(Kw_i, w_j) \\ &\quad + \sum_{i=1}^{d}(KB_i, w_j) + (f, w_j), \\ j = 1, 2, \dots, \mu,& \\ \sum_{i=1}^{\mu} a_i(w_i, B_j) + \sum_{i=1}^{d} b_i(B_i, B_j) &= \sum_{i=1}^{\mu} a_i(Kw_i, B_j) \\ &\quad + \sum_{i=1}^{d}(KB_i, B_j) + (f, B_j), \\ j = 1, 2, \dots, d.&\end{aligned}$$

Before stating and proving Theorem 3.2, the main result of this section, for ready reference we state a known result (cf., [**5, 14**]) concerned with the order of approximation by splines in L_2.

Theorem 3.1. *Let $0 \le \nu \le n-1$. If $g \in H^n$, $n \ge 0$, then for each $h > 0$, there exists $\phi_h \in S_h^n$ such that*

$$||g - \phi_h||_2 \le Ch^n ||g||_{H^n},$$

where $C > 0$ is a constant independent of h.

Theorem 3.2. *Assume the solution y of equation* (1.1) *has a decomposition $y = w + v$ with $w \in W$ and $v \in H^n$. Let $V_h^n = W \oplus S_h^n$. Then there exists an $h_0 > 0$ such that for $0 < h \le h_0$, a unique Galerkin approximation $y_h \in V_h^n$ exists with*

$$||y - y_h||_2 = O(h^n).$$

Proof. Let $P_h^{G'}$ be the conventional Galerkin projection from L_2 into S_h^n, i.e., $P_h^{G'}u$ satisfies the equation

$$(P_h^{G'}u, s) = (u, s), \quad \text{for all } s \in S_h^n.$$

Then we have $P_h^{G'}u \to u$ for all $u \in L_2$. By using Theorem 3.1, we conclude that there is a constant C independent of h and a $\phi_h \in S_h^n$ such that

$$||u - \phi_h||_2 \leq Ch^n ||u||_{H^n}.$$

Noting that $P_h^{G'}u$ is the best L_2-approximation to u from S_h^n, we then have

$$||P_h^{G'}u - u||_2 \leq ||u - \phi_n||_2 \leq Ch^n||u||_{H^n}, \qquad \text{for all } u \in H^n.$$

In addition, since P_h^G is the orthogonal projection from L_2 into V_h^n, for any $x \in L_2$,

$$x = P_h^G x + (x - P_h^G x)$$

with

$$(P_h^G x, x - P_h^G x) = 0.$$

Hence,

$$||x||_2^2 = ||P_h^G x||_2^2 + ||x - P_h^G x||_2^2.$$

This implies that $||P_h^G x||_2 \leq ||x||_2$. Thus, $||P_h^G||_2 \leq 1$. On the other hand, we also have $x = P_h^G x$ for $x \in V_h^n$. This equation yields $||P_h^G||_2 \geq 1$. Therefore, we find $||P_h^G||_2 = 1$ for all $h > 0$. Hence, all assumptions of Theorem 2.3 are satisfied and the validity of this theorem follows directly from Theorem 2.3. The proof is complete. □

4. Applications to equations with logarithmic kernels. In this section we apply the singularity preserving Galerkin method established in the last section to Fredholm integral equations of the second kind with logarithmic kernels.

Consider the equation

$$y(s) - \lambda \int_0^1 \log|s-t| m(s,t) y(t)\, dt = f(s), \qquad 0 \leq s \leq 1, \tag{4.1}$$

where $m \in C^{n+1}([0,1] \times [0,1])$ and $f \in H^n[0,1]$. Let

$$(Ky)(s) = \int_0^1 \log|s-t| m(s,t) y(t)\, dt, \qquad 0 \leq s \leq 1. \tag{4.2}$$

Equation (4.1) may be rewritten in operator form as $y - \lambda K y = f$. To develop the singularity preserving Galerkin method for this equation, we need a singularity expansion for a solution y of equation (4.1). We state the result in a general setting, without assuming that λ is not an eigenvalue of K. This will allow us to include eigenfunctions of K in our theorem.

Theorem 4.1. *Let $m \in C^{n+1}([0,1] \times [0,1])$ and $f \in H^n$. Suppose that y is a solution of equation* (4.1). *Then there exist constants a_{ij} and b_{ij} for $i, j = 1, \dots, n-1$ and a function $v_n \in H^n$ such that*

(4.3)
$$y(s) = \sum_{i=1}^{n-1} \sum_{j=0}^{n-1-i} [a_{ij} s^j (s \log s)^i + b_{ij} (1-s)^j ((1-s)\log(1-s))^i] + v_n(s).$$

Remark. The special case of Theorem 4.1 when $m \equiv 1$ was given in [**4, 11**]. We shall call the expansion of y in this theorem the singularity expansion for y. It can be seen that the proof of this expansion does not depend on the uniqueness of y. Therefore, the conclusion of Theorem 4.1 also holds for eigenfunctions of K if λ is an eigenvalue of K. For further reference, we state this as a corollary of Theorem 4.1.

Corollary 4.2. *The expansion* (4.3) *holds for eigenfunctions of K.*

To prove Theorem 4.1, we need several lemmas. Since the proof of Theorem 4.1 does not depend on the value of λ, to simplify our notation, we let $\lambda = 1$ in the rest of this section.

Lemma 4.3. *Assume $m \in C^1([0,1] \times [0,1])$. Then the operator K defined by* (4.2) *maps L_2 into H^1.*

Proof. If $m(s,t) \equiv 1$, this result is valid by Lemma 2 of [**11**]. We now prove the general result. Since $m \in C^1([0,1] \times [0,1])$, it can be approximated by its bivariate Bernstein polynomial expansion

$$B_n(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} C_n^i C_n^j m\left(\frac{i}{n}, \frac{j}{n}\right) s^i (1-s)^{n-i} t^j (1-t)^{n-j},$$

with

$$\max_{0\le s,t\le 1}|m(s,t)-B_n(s,t)|+\max_{0\le s,t\le 1}|D_s(m(s,t)-B_n(s,t))| \\ +\max_{0\le s,t\le 1}|D_t(m(s,t)-B_n(s,t))|\to 0$$

as $n\to\infty$ (see [**9**]), where C_n^i is the binomial coefficient and D_s and D_t denote the partial derivatives with respect to the variables s and t, respectively. Define the operator associated with the kernel B_n by

$$(G_n y)(s)=\int_0^1 \log(|s-t|)B_n(s,t)y(t)\,dt, \tag{4.4}$$

and

$$(Gy)(s)=\int_0^1 \log(|s-t|)y(t)\,dt.$$

Then,

$$(G_n y)(s)=\sum_{i=0}^n\sum_{j=0}^n C_n^i C_n^j m\left(\frac{i}{n},\frac{j}{n}\right)s^i(1-s)^{n-i}(Gb_j y)(s),$$

where $b_j(t)=t^j(1-t)^{n-j}$. Since G maps L_2 into H^1, $Gb_j y\in H^1$. Thus, G_n also maps L_2 into H^1 for each n and hence $G_n y\in H^1$.

Next we show that $G_n y\to Ky$ in H^1. Clearly, we have $||Ky-G_n y||_2\to 0$ as $n\to\infty$, for any $y\in L_2$. We need only show that $\{DG_n y\}$ is a Cauchy sequence in L_2, where $D=d/ds$. Notice that (see the proof of Lemma 2 in [**11**])

$$D\int_0^1 \log|s-t|y(t)\,dt=\text{P.V.}\int_0^1 \frac{y(t)}{s-t}\,dt,$$

where 'P.V.' denotes the Cauchy principal value of the integral and is defined by

$$\text{P.V.}\int_0^1\frac{y(t)}{s-t}\,dt=\lim_{\varepsilon\to 0}\left[\int_0^{s-\varepsilon}\frac{y(t)}{s-t}\,dt+\int_{s+\varepsilon}^1\frac{y(t)}{s-t}\,dt\right].$$

The prefix P.V. will not be used, it being understood that the principal value is to be taken when appropriate. If $m > n$, then we have

$$DG_m y(s) - DG_n y(s) = \int_0^1 \log(|s-t|) D_s[B_m(s,t) - B_n(s,t)] y(t)\, dt \\ + \int_0^1 \frac{B_m(s,t) - B_n(s,t)}{s-t} y(t)\, dt.$$

We now estimate the two terms in the right-hand side of this equation. Let $\varepsilon > 0$. Since $\{B_n(s,t)\}$, $\{D_s B_n(s,t)\}$ and $\{D_t B_n(s,t)\}$ are all Cauchy sequences in $C[0,1]$, there exists a constant $N > 0$ such that

$$\max_{0 \le s,t \le 1} |B_m(s,t) - B_n(s,t)| < \varepsilon, \quad \text{whenever } m > n > N,$$

$$\max_{0 \le s,t \le 1} |D_s[B_m(s,t) - B_n(s,t)] < \varepsilon, \quad \text{whenever } m > n > N,$$

and

$$\max_{0 \le s,t \le 1} |D_t[B_m(s,t) - B_n(s,t)]| < \varepsilon, \quad \text{whenever } m > n > N.$$

Let

$$Y(s) = \int_0^1 |\log(|s-t|)||y(t)|\, dt,$$

and

$$X(s) = \int_0^1 \frac{y(t)}{s-t}\, dt.$$

Then $Y, X \in L_2$. Clearly,

$$\left| \int_0^1 \log(|s-t|) D_s[B_m(s,t) - B_n(s,t)] y(t)\, dt \right| < \varepsilon Y(s).$$

Moreover,

$$\left| \int_0^1 \frac{B_m(s,t) - B_n(s,t)}{s-t} y(t)\, dt \right| \\ \le \left| \int_0^1 \frac{[B_m(s,t) - B_n(s,t)] - [B_m(s,s) - B_n(s,s)]}{s-t} y(t)\, dt \right| \\ + \left| \int_0^1 \frac{B_m(s,s) - B_n(s,s)}{s-t} y(t)\, dt \right|$$

By the Mean-Value theorem, there exists a number ξ dependent on both s and t such that

$$\left|\frac{[B_m(s,t)-B_n(s,t)]-[B_m(s,s)-B_n(s,s)]}{s-t}\right| \\ = |D_t[B_m(s,\xi)-B_n(s,\xi)]| < \varepsilon.$$

Hence,

$$\left|\int_0^1 \frac{B_m(s,t)-B_n(s,t)}{s-t} y(t)\,dt\right| \le \varepsilon \int_0^1 |y(t)|\,dt + \varepsilon |X(s)|.$$

It follows that for every $s \in [0,1]$

$$|DG_m y(s) - DG_n y(s)| < \varepsilon(M + Y(s) + |X(s)|),$$

where $M = \int_0^1 |y(s)|\,ds$. This implies that

$$||DG_m y - DG_n y||_2 \le \varepsilon(M + ||Y||_2 + ||X||_2),$$

whenever $m > n > N$. We conclude that $\{DG_n y\}$ is a Cauchy sequence in L_2. Thus, $\{G_n y\}$ is a Cauchy sequence in H^1. Since $G_n y \to Ky$ in L_2, $Ky \in H^1$ and $G_n y \to Ky$ in H^1. The proof is complete. □

We remark that for $m \in C([0,1]\times[0,1])$, in general, K does not map L_2 into H^1. The following example illustrates this remark.

Example. Let $m(s,t) = s^{1/2}$ and $y(t) = \log t$. Then $m \in C([0,1]\times[0,1])$ and $y \in L_2[0,1]$. Consider

$$(Ky)(s) = s^{1/2} \int_0^1 \log|s-t| \log t\,dt, \qquad 0 \le s \le 1.$$

The derivative of Ky is given by

$$\frac{d}{ds}(Ky)(s) = \frac{1}{2s^{1/2}} \int_0^1 \log|s-t| \log t\,dt \\ + s^{1/2} \frac{d}{ds} \int_0^1 \log|s-t| \log t\,dt. \tag{4.5}$$

Since $u(s) = \int_0^1 \log|s-t| \log t\, dt$, as a function of s, is in H^1, the second term in the right hand side of (4.5) is in L_2. Notice that $u(0) = \int_0^1 \log^2 t\, dt > 0$ and u is continuous on $[0,1]$. There exist a constant $C > 0$ and $\delta > 0$ for which

$$\left[\frac{1}{2s^{1/2}} \int_0^1 \log|s-t| \log t\, dt\right]^2 \geq \frac{C}{4s}, \qquad 0 < s < \delta.$$

Since $1/s$ is not an integrable function on $[0,1]$, the first term of (4.5) is not in L_2. It follows that $Ky \notin H^1$.

In the following lemma and its proof, we use c_j, d_j, c_{ij} and d_{ij} to denote generic constants whose values may change from time to time, and v_n to denote a function in H^n (possibly being different in different places). Also, we let $u_1(t) = t^p(\log t)^q$ and $u_2(t) = (1-t)^p(\log(1-t))^q$, where $0 \leq t \leq 1$, and p, q are positive integers.

Lemma 4.4. *Let $f \in H^{n-1}$.*

(1) *For the operator G defined by* (4.4)*, we have*

$$\begin{aligned}
(Gf)(s) &= \sum_{j=1}^{n-1} [c_j s^j \log s + d_j (1-s)^j \log(1-s)] + v_n(s), \\
(Gu_1)(s) &= \sum_{j=1}^{q+1} c_j s^{p+1} (\log s)^j \\
&\quad + \sum_{j=q+1}^{n-1} d_j (1-s)^j \log(1-s) + v_n(s),
\end{aligned}$$

and

$$(Gu_2)(s) = \sum_{j=1}^{q+1} c_j (1-s)^{p+1} (\log 1-s)^j + \sum_{j=q+1}^{n-1} d_j s^j \log s + v_n(s).$$

(2) *In addition, assume* $m \in C^{n+1}([0,1]\times[0,1])$. *Then*

$$(Kf)(s) = \sum_{j=1}^{n-1}[c_j s^j \log s + d_j(1-s)^j \log(1-s)] + v_n(s),$$

$$(Ku_1)(s) = \sum_{j=p+1}^{n-1}\sum_{i=1}^{q+1} c_{ij} s^j (\log s)^i + \sum_{j=q+1}^{n-1} d_j(1-s)^j \log(1-s) + v_n(s),$$

and

$$(Ku_2)(s) = \sum_{j=p+1}^{n-1}\sum_{i=1}^{q+1} c_{ij}(1-s)^j(\log(1-s))^i + \sum_{j=q+1}^{n-1} d_j s^j \log s + v_n(s).$$

Proof. (1) The proof of this part may be found in [**11**].

(2) In this proof we denote

$$m^{(i,j)}(s,t) := \frac{\partial^{i+j}}{\partial s^i + \partial t^j} m(s,t).$$

We expand $m(s,t)$ at $t=s$ by using Taylor's expansion and obtain

(4.6)
$$m(s,t) = \sum_{k=0}^{l} \frac{1}{k!} m^{(0,k)}(s,s)(t-s)^k + \frac{1}{l!}\int_s^t m^{(0,l+1)}(s,\sigma)(t-\sigma)^l \, d\sigma.$$

By using (4.6) with $l = n-1$ and the binomial expansion for $(s-t)^k$, we have

$$(Kf)(s)=\sum_{k=0}^{n-1}\frac{1}{k!}m^{(0,k)}(s,s)\sum_{i=0}^{k}(-1)^i C_k^i s^{k-i}\int_0^1 t^i f(t)\log(|s-t|)\,dt+v_n(s),$$

where

$$v_n(s) = \frac{1}{(n-1)!} \int_0^1 \log(|s-t|) \int_s^t m^{(0,n)}(s,\sigma)(t-\sigma)^{n-1}\, d\sigma f(t)\, dt.$$

Next we prove that $v_n \in H^n$. Let

$$g(s,t) = (-1)^{n-1} \log(|s-t|) \int_s^t m^{(0,n)}(s,\sigma)(\sigma-t)^{n-1}\, d\sigma.$$

Differentiating g n times with respect to s gives

$$g^{(n,0)}(s,t) = (-1)^{n-1} \sum_{k=0}^{n} C_n^k \frac{d^k}{ds^k}(\log(|s-t|)) \cdot \frac{d^{n-k}}{ds^{n-k}} \int_s^t m^{(0,n)}(s,\sigma)(\sigma-t)^{n-1}\, d\sigma.$$

Since $m \in C^{n+1}([0,1]\times[0,1])$, we find

$$\left| \frac{d^{n-k}}{ds^{n-k}} \int_s^t m^{(0,n)}(s,\sigma)(\sigma-t)^{n-1}\, d\sigma \right| \le C|s-t|^k, \qquad \text{for } k = 0,1,\dots,n.$$

Hence, the first term of the summation above is bounded by $-\log(|s-t|)$ and the remaining terms are bounded by a constant. Thus, we have $g(\cdot,t) \in H^n$. It follows that $v_n \in H^n$ (see [**3**, p. 88]). Applying (1) of this lemma with $f(t)$ replaced by $t^i f(t)$ and noting that for $w \in C^n[0,1]$

$$w(s)(\log s)^i = \sum_{j=0}^{n-1} s^j (\log s)^i + v_n(s), \tag{4.7}$$

where $v_n \in H^n$, we have

$$(Kf)(s) = \sum_{j=1}^{n-1} [c_j s^j \log s + d_j (1-s)^j \log(1-s)] + v_n(s),$$

for some $v_n \in H^n$. Thus, we obtain the first equation of (2). Using (4.6) with $l = n-1$ and the binomial formula, we conclude that

$$
\begin{aligned}
(Ku_1)(s) &= \sum_{i=0}^{n-1} \frac{m^{(0,i)}(s,s)}{i!} \int_0^1 \log(|s-t|)(s-t)^i t^p (\log t)^q \, dt \\
&+ \frac{1}{(n-1)!} \int_0^1 \log(|s-t|) \int_s^t m^{(0,n)}(s,\sigma)(t-\sigma)^{n-1} \, d\sigma t^p (\log t)^q \, dt \\
&= \sum_{i=0}^{n-1} \frac{m^{(0,i)}(s,s)}{i!} \int_0^1 \log(|s-t|) \sum_{l=0}^{i} C_i^l s^{i-l} t^{l+p} (\log t)^q \, dt \\
&+ v_n(s).
\end{aligned}
$$

By using part (1) of this lemma and (4.7), we obtain the second equation of (2). The proof for the last equation is similar and we omit it. The proof is complete. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We use a proof by induction. For $n = 1$, the result is a direct consequence of Lemma 4.3. Assume that the result holds for $n = k$, that is, if $f \in H^k$, then expansion (4.3) holds with $n = k$. Let

$$u_{ij}(s) = a_{ij} s^j (\log s)^i + b_{ij} (1-s)^j (\log(1-s))^i.$$

Then

$$y(s) = \sum_{i=1}^{k-1} \sum_{j=0}^{k-1-i} u_{ij}(s) + v_k. \tag{4.8}$$

Now we consider the case when $n = k+1$ and suppose that $f \in H^{k+1}$. Substituting (4.8) into the integral of (4.1), we have

$$y(s) = \sum_{i=1}^{k-1} \sum_{j=0}^{k-1-i} (Ku_{ij})(s) + (Kv_k)(s) + f.$$

By using (2) of Lemma 4.4, we obtain the desired result for $n = k+1$. The proof is complete. $\square$

Motivated by the singularity expansion (4.3), we define the singular subspace W of $C[0,1]$ for equation (4.1) by

$$W = \operatorname{span}\{s^j(s\log s)^i, (1-s)^j((1-s)\log(1-s))^i,\\ j = 0,1,\ldots,n-1-i,\ i = 1,2,\ldots,n-1\}.$$

The dimension of the singular subspace W is $n(n-1)^2$. Assume S_h^n is the space of spline functions defined in the beginning of this section. Then $V_h^n = W \oplus S_h^n$. Clearly the dimension of V_h^n is $n(n-1)+d$, where d is the dimension of the space of splines. The Galerkin approximation y_h in V_h^n of equation (4.1) can be written as

$$y_h(s) = \sum_{i=1}^{n-1}\sum_{j=0}^{n-1-i}[a_{ij}s^j(s\log s)^i + b_{ij}(1-s)^j((1-s)\log(1-s))^i] \\ + \sum_{i=1}^{d} c_i B_i(s),$$

where B_i, $i = 1,2,\ldots,d$ is the B-spline basis for the space S_h^n. The coefficients a_{ij}, b_{ij}, $j = 0,1,\ldots,n-1-i$, $i = 1,2,\ldots,n-1$, and c_i, $i = 1,2,\ldots,d$ are determined by the system of linear equations:

$$(y_h, v_h) = (Ky_h, v_h) + (f, v_h), \qquad \text{for all } v_h \in V_h^n.$$

In terms of the basis for V_h^n, we have the following form of linear equations

$$(y_h(s), s^j(s\log s)^i) = ((Ky_h)(s), s^j(s\log s)^i) + (f(s), s^j(s\log s)^i),$$
$$(y_h(s), (1-s)^j((1-s)\log(1-s))^i) \\ = ((Ky_h)(s), (1-s)^j((1-s)\log(1-s))^i) \\ + (f(s), (1-s)^j((1-s)\log(1-s))^i),$$

$$j = 0,1,\ldots,n-1-i,\ i = 1,2,\ldots,n-1,$$

and

$$(y_h(s), B_i(s)) = ((Ky_h)(s), B_i(s)) + (f(s), B_i(s)), \qquad i = 1,2,\ldots,d.$$

The next theorem guarantees the unique existence of the Galerkin approximation y_h in V_h^n of the solution y of equation (4.1) for a sufficiently small $h > 0$ and gives the order of the convergence for this approximation.

Theorem 4.5. *Let* $m \in C^{n+1}([0,1] \times [0,1])$ *and* $f \in H^n$. *Assume that* λ *is not an eigenvalue of* K. *Then there exists an* $h_0 > 0$ *such that a unique Galerkin approximation* y_h *to the solution* y *of equation* (4.1) *exists with*

$$||y - y_h||_2 = O(h^n).$$

Proof. By Theorem 4.1, the solution y of equation (4.1) has the singular form

$$y(s) = w(s) + v(s)$$

where $w \in W$ and $v \in H^n$. Hence, the hypothesis of Theorem 3.2 is satisfied. The result then follows immediately from Theorem 3.1. □

The order of convergence given by Theorem 4.5 is optimal in the sense that it is the order of spline functions used in approximation. Note that the conventional Galerkin method applied to the current problem only gives a convergence order $O(h)$. The price we pay to obtain this optimal convergence rate $O(h^n)$ is that the dimension of the Galerkin subspace increases by $n(n-1)$. It is illustrated by a numerical example presented in Section 6 that, in practice, the choice $n = 2$ or $n = 3$ will give satisfactory numerical results. Even in general, since $d \gg n(n-1)$, the additional cost to achieve the optimal convergence rate is insignificant in comparison with the acceleration convergence that we obtain. This will be demonstrated by a numerical example in Section 6.

5. Applications to equations with algebraic singularities. In this section we establish results analogous to those presented in Section 4 for Fredholm integral equations with algebraic singular kernels. More specifically, we consider the integral equations

$$(5.1) \qquad y(s) - \lambda \int_0^1 |s-t|^\alpha m(s,t) y(t)\, dt = f(s), \qquad 0 \le s \le 1,$$

where $-1 < \alpha < 0$, $m \in C^{n+1}[0,1]$ and $f \in H^n$. Let

$$(Ky)(s) = \int_0^1 |s-t|^\alpha m(s,t)y(t)\,dt, \qquad 0 \le s \le 1. \tag{5.2}$$

We first obtain a singularity expansion for a solution of (5.1) with α being an irrational number in the interval $(-1/2, 0)$ and then describe briefly a way with which it can be extended to other cases. In this section, we use the notation $\{\sum_{i=1}^n a_i(s)\}^*$ to denote a linear combination of functions $a_1(s), a_2(s), \dots, a_n(s)$.

Theorem 5.1. *Let $m \in C^{n+1}[0,1]$ and $f \in H^n$. Assume that α is an irrational number in $(-1/2, 0)$. Then a solution y of equation* (5.1) *has the decomposition*

$$\begin{aligned} y(s) = &\left\{ \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-1} \sum_{i=1}^{2} [s^{(2l+i)(1+\alpha)+j} + (1-s)^{(2l+i)(1+\alpha)+j}] \right\}^* \\ &+ v_n(s), \qquad 0 \le s \le 1, \end{aligned} \tag{5.3}$$

where $v_n \in H^n$.

We remark that the special case of Theorem 5.1 when $m \equiv 1$ was given in [**4, 11**]. To prove this theorem we need two lemmas which are analogous to Lemmas 4.3 and 4.4.

Lemma 5.2. *Let $m \in C^2[0,1]$ and $\alpha \in (-1/2, 0)$. Then K defined by* (5.2) *maps H^δ into $H^{1+\delta+\alpha}$ for $\delta < 1/2$, where H^δ denotes the Sobolev space with real index δ.*

Proof. Let $\psi \in H^\delta$ where $\delta < 1/2$. If $m(s,t) \equiv 1$, the lemma was proved in [**11**]. In general we expand $m(s,t)$ in Taylor's expansion at $t = s$ and obtain

$$\begin{aligned} (K\psi)(s) = \int_0^1 |s-t|^\alpha \bigg[&m(s,s) + (s-t)m^{(0,1)}(s,s) \\ &+ \frac{1}{2}\int_s^t (s-\sigma)m^{(0,2)}(s,\sigma)\,d\sigma \bigg] \psi(t)dt \end{aligned}$$

$$
\begin{aligned}
&= m(s,s)\int_0^1 |s-t|^\alpha \psi(t)\,dt \\
&\quad + m^{(0,1)}(s,s)\int_0^1 |s-t|^\alpha (s-t)\psi(t)\,dt \\
&\quad + \frac{1}{2}\int_0^1 |s-t|^\alpha \int_s^t (s-\sigma)m^{(0,2)}(s,\sigma)d\sigma\psi(t)\,dt \\
&\equiv \psi_1(s)+\psi_2(s)+\psi_3(s).
\end{aligned}
$$

By Lemma 4 of [**11**], $\psi_1, \psi_2 \in H^{1+\alpha+\delta}$. Moreover, similar to the proof of Lemma 4.4, we have

$$\frac{1}{2}\int_s^t (s-\sigma)m^{(0,2)}(s,\sigma)\,d\sigma |s-t|^\alpha \in H^2.$$

Hence, $\psi_3 \in H^2 \subset H^{1+\alpha+\delta}$. The proof is complete. $\square$

Let

$$(Gy)(s) = \int_0^1 |s-t|^\alpha y(t)\,dt, \qquad 0 \le s \le 1.$$

Let $u(s) = s^p + (1-s)^p$ for $p > -1$. In the following lemma v_n always denotes a function in H^n.

Lemma 5.3. *Let $f \in H^{n-1}$.*

(1) *The expansions*

(5.4)
$$
\begin{aligned}
(Gu)(s) = &\left\{ s^{1+\alpha+p} + (1-s)^{1+\alpha+p} + \sum_{j=0}^{n-2}[s^{1+\alpha+j} + (1-s)^{1+\alpha+j}] \right\}^* \\
&+ v_n,
\end{aligned}
$$

and

$$
\begin{aligned}
(G^2u)(s) = &\left\{ s^{2(1+\alpha)+p} + (1-s)^{2(1+\alpha)+p} \right. \\
&\left. + \sum_{i=1}^{2}\sum_{j=0}^{n-1}[s^{i\alpha+j+1} + (1-s)^{i\alpha+j+1}] \right\}^* + v_n
\end{aligned}
\tag{5.5}
$$

hold. If $f \in H^n$*, then*

$$(Gf)(s) = \left\{ \sum_{j=0}^{n-2} [s^{\alpha+j+1} + (1-s)^{\alpha+j+1}] \right\}^* + v_n. \tag{5.6}$$

If $f \in H^{n-1}$*, then*

$$(G^2 f)(s) = \left\{ \sum_{i=1}^{2} \sum_{j=0}^{n-2} [s^{i\alpha+j+1} + (1-s)^{i\alpha+j+1}] \right\}^* + v_n. \tag{5.7}$$

(2) *If* $m \in C^{n+1}([0,1] \times [0,1])$*, then the results of* (1) *hold if the operator* G *is replaced by* K.

Proof. (1) Let $u_1(s) = s^p$ and $u_2(s) = (1-s)^p$.

$$(Gu_1)(s) = \int_0^s (s-t)^\alpha t^p \, dt + \int_s^1 (t-s)^\alpha t^p \, dt \equiv I_1(s) + I_2(s).$$

Let $x = t/s$ in I_1 to obtain

$$I_1(s) = s^{1+\alpha+p} \int_0^1 (1-x)^\alpha x^p \, dx.$$

By the binomial series, we have

$$\begin{aligned} I_2(s) &= \int_s^1 \left(1 - \frac{s}{t}\right)^\alpha t^{\alpha+p} \, dt \\ &= \int_s^1 \sum_{n=0}^\infty \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!} \left(\frac{s}{t}\right)^n t^{\alpha+p} \, dt \\ &= \sum_{n=0}^\infty \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!(\alpha+p-n+1)} (s^n - s^{\alpha+p+1}). \end{aligned}$$

Notice that, for some $\xi_i \in ((-\alpha-1)/i, 0)$

$$\begin{aligned} \log \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!} &= \sum_{i=1}^n \log\left(1 + \frac{-\alpha-1}{i}\right) \\ &= \sum_{i=1}^n \left(\frac{-\alpha-1}{i} + \frac{(1+\alpha)^2}{2i^2(1+\xi_i)^2} \right) \\ &\le (-\alpha-1) \sum_{i=1}^n \frac{1}{i} + C \\ &\le (-\alpha-1)\log n + C, \end{aligned}$$

where

$$C = \frac{1}{2}\left(\frac{1+\alpha}{\alpha}\right)^2 \sum_{i=1}^{\infty} \frac{1}{i^2} < +\infty$$

is a constant independent of n. It follows that

$$\frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!} \le \frac{e^C}{n^{1+\alpha}}.$$

Thus, the series

$$\sum_{n=0}^{\infty} \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!(p+\alpha-n+1)}(s^n - s^{\alpha+p+1})$$

is uniformly convergent for $|s| < 1-\delta$ where $0 < \delta < 1/2$. Hence, $I_2(s) - as^{p+\alpha+1} \in H^\infty[0, 1-\delta]$. On the other hand, for $s \in [\delta, 1]$, again by the binomial series,

$$\begin{aligned} I_2(s) &= \int_s^1 (t-s)^\alpha (1-(1-t))^p \, dt \\ &= \sum_{n=0}^{\infty} \frac{-p(-p+1)\cdots(-p+n-1)}{n!} \int_s^1 (t-s)^\alpha (1-t)^n \, dt \\ &= (1-s)^{1+\alpha} \sum_{n=0}^{\infty} \frac{-p(-p+1)\cdots(-p+n-1)}{n!} \\ &\qquad\qquad \cdot \int_0^1 (1-t)^\alpha t^n \, dt (1-s)^n. \end{aligned}$$

Since

$$\int_0^1 (1-t)^\alpha t^n \, dt \le \frac{1}{\alpha+1},$$

it follows from the proof for the last case that there exists a positive constant C for which

$$\frac{-p(-p+1)\cdots(-p+n-1)}{n!} \int_0^1 (1-t)^\alpha t^n \, dt \le \frac{C}{n^{1+p}}.$$

Since $\lim_{n\to\infty} n^{1/n} = 1$, we conclude that the radius of convergence for the power series above is 1. Therefore, it is uniformly convergent for $s \in [\delta, 1]$. Thus,

$$I_2(s) = (1-s)^{\alpha+1}\{1 + (1-s) + \cdots + (1-s)^{n-1}\}^* + v_n(s),$$

where $v_n \in H^n[\delta, 1]$. Therefore,

$$(5.8) \qquad (Gu_1)(s) = \left\{ s^{1+\alpha+p} + \sum_{j=0}^{n-2} (1-s)^{1+\alpha+j} \right\}^* + v_n(s).$$

By symmetry, we have

$$(5.9) \qquad (Gu_2)(s) = \left\{ (1-s)^{1+\alpha+p} + \sum_{j=0}^{n-2} s^{1+\alpha+j} \right\}^* + v_n(s).$$

Adding these two equations together, we obtain the first expansion.

For $f \in H^n$, integration by parts gives

$$(5.10) \quad (D^{n-1}Gf)(s) = (GD^{n-1}f)(s) + \left\{ \sum_{j=0}^{n-2} [D^j s^\alpha + D^j (1-s)^\alpha] \right\}^*.$$

Integrating both sides of (5.10) $n-1$ times, we obtain

$$(5.11) \qquad (Gf)(s) = \left\{ \sum_{j=0}^{n-2} [s^{\alpha+j+1} + (1-s)^{\alpha+j+1}] \right\}^* + w_n$$

where w_n is a function obtained by integrating $GD^{n-1}f$ $n-1$ times. Since $D^{n-1}f \in H^1 \subset H^{-\alpha}$, by Lemma 5.2 $GD^{n-1}f \in H^1$. Hence, $w_n \in H^n$. It can be shown that if $f \in H^{n-1}$, then

$$(D^{n-1}G^2 f)(s) = (G^2 D^{n-1} f)(s) + (I+G) \left\{ \sum_{j=0}^{n-2} [D^j s^\alpha + D^j (1-s)^\alpha] \right\}^*.$$

Again integrating this equation $n-1$ times gives

$$(G^2 f)(s) = \left\{ \sum_{j=0}^{n-2} [s^{\alpha+j+1} + (1-s)^{\alpha+j+1}] \right\}^* + w_n,$$

where w_n is obtained from $G^2 D^{n-1} f$ by integrating $n-1$ times. Since $f \in H^{n-1}$, $G^2 D^{n-1} f \in H^1$ and thus $w_n \in H^n$.

(2) The proof for this part is similar to that for part (2) of Lemma 4.4, and we omit it. □

Proof of Theorem 5.1. The proof is by induction. Assume that $f \in H^1$ and y is a solution of (5.1). Then $y \in L_2$. Using Lemma 5.2, we conclude that $y \in H^1$. Hence, (5.3) holds for $n = 1$. Assume that (5.3) holds for $n = k$. Suppose that $f \in H^{k+1}$. Equation (5.1) can be written as

$$y = K^2 y + (I + K)f.$$

Substituting (5.3) with $n = k$ into the right hand side of this equation, we have

$$y = K^2 \sum_{l=0}^{k-1} \sum_{j=0}^{n-l-1} \sum_{i=1}^{2} u_{ijl} + K^2 v_k + (I + K)f, \tag{5.12}$$

where u_{ijl} denotes $\{s^{(2l+i)(1+\alpha)+j} + (1-s)^{(2l+i)(1+\alpha)+j}\}^*$. By Lemma 5.3, we have

$$(Kf)(s) = \left\{ \sum_{j=0}^{k-1} [s^{\alpha+j+1} + (1-s)^{\alpha+j+1}] \right\}^* + v_{k+1},$$

where $v_{k+1} \in H^{k+1}$. Applying Lemma 5.3 to the first two terms of the right hand side of (5.12) and rearranging terms, we conclude that (5.3) holds for $n = k + 1$. □

For rational α the exponents $(2l + i)(1 + \alpha)$ in the statement of Theorem 5.1 may become integers, and the foregoing result requires modification. We state the modified result in the next theorem without proof.

Let $1 + \alpha = p/q$, where p, q are coprime and $p < q$, let $n = [q/p] + 1$ and let ρ be the smallest integer such that $q \leq n\rho$, i.e., where

$$q = n(\rho - 1) + \sigma, \quad 0 < \sigma \leq n, \quad \text{and} \quad \rho \in N.$$

Theorem 5.4. *Let $m \in C^{m+1}[0, 1]$ and $f \in H^{m\rho+1}$. Assume that α is a rational number in $(-1/2, 0)$. Then the solution y of equation*

(5.1) *has the decomposition*

$$\begin{aligned} y(t) = \Bigg\{ &\sum_{i=0}^{m-1} (t^p \log t)^i ((t^{1+\alpha} + \cdots + t^{n(1+\alpha)})(1 + \cdots + t^{(m-i)\rho-1}) + \cdots \\ &\qquad + (t^{((\rho-1)n+1)(1+\alpha)} + \cdots + t^p \log t)(1 + \cdots + t^{(m-i-1)\rho})) \\ &+ \sum_{i=0}^{m-1} ((1-t)^p \log(1-t))^i \\ &\times (((1-t)^{1+\alpha} + \cdots + (1-t)^{n(1+\alpha)})(1 + \cdots + (1-t)^{(m-i)\rho-1}) \\ &\quad + \cdots + ((1-t)^{((\rho-1)n+1)(1+\alpha)} + \cdots + (1-t)^p \log(1-t)) \\ &\times (1 + \cdots + (1-t)^{(m-i-1)\rho})) \Bigg\}^* + v_{m\rho+1}, \end{aligned}$$

where $t \in (0,1)$ *and* $v_{m\rho+1} \in H^{m\rho+1}[0,1]$.

For the case $-1 < \alpha \leq -1/2$, we can use the same technique to obtain an analogous decomposition for the solution of equation (5.1) with a modification in proof where we use K^l for $l = 1 + [1/(1+\alpha)]$ instead of using K^2. We leave the straightforward details to the interested readers.

We now apply the method proposed in Section 3 to the algebraic singular case. We only present the method for an irrational $\alpha \in (-1/2, 0)$ since other cases are similar. Similarly to the logarithmic case, we define the singular subspace W of $C[0,1]$ for equation (5.1) by using the singularity expansion (5.3) as follows:

$$\begin{aligned} W = \operatorname{span}\{&s^{(2l+i)(1+\alpha)+j}, (1-s)^{(2l+i)(1+\alpha)+j}, \\ &i = 1, 2,\ j = 0, 1, \ldots, n-l-1,\ l = 0, 1, \ldots, n-1\}. \end{aligned}$$

Clearly, W is of dimension $n(n+1)$. As in Section 4, assume that S_h^n is the space of spline functions defined in Section 3. Let $V_h^n = W \oplus S_h^n$, and let y_h be the Galerkin approximation from V_h^n to the solution y of equation (5.1).

Theorem 5.5. *Let* $m \in C^{n+1}([0,1] \times [0,1])$ *and* $f \in H^n$. *Assume that* λ *is not an eigenvalue of* K. *Then there exists an* $h_0 > 0$ *such that* *whenever $0 < h < h_0$, a unique Galerkin approximation y_h from V_h^n to the solution y of equation* (5.1) *exists with*

$$||y - y_h||_2 = O(h^n).$$

6. Computational implementation. In this section we discuss the computational implementation of the singularity preserving Galerkin method proposed in the previous sections. We assume $f \in C^n[0,1]$ unless stated otherwise. To find the solution for (3.1), we need to evaluate integrals of types

(1) (B_i, B_j), (f, B_j),

(2) (w_i, w_j), (B_i, w_j), (f, w_j),

(3) (KB_i, B_j), (KB_i, w_j), (Kw_i, B_j), (Kw_i, w_j),

where $w_i \in W$ and B_i is a B-spline. The integrals of type (1) can be evaluated by a common quadrature rule which has a convergence of order n.

To evaluate the integrals of types (2) and (3), we construct quadrature formulas using an idea from [**8**]. Let S be a subset of $[0,1]$ consisting of a finite number of points and define a function associated with S by $\omega_S(x) = \inf\{|x-t| : t \in S\}$. For $\alpha > -1$ and a nonnegative integer k, a real-valued function g is said to be of Type$\,(\alpha, k, S)$ if

$$|g^{(k)}(x)| \le C[\omega_S(x)]^{(\alpha-k)}, \qquad x \notin S \quad \text{and} \quad g \in C^k([0,1]\backslash S).$$

The functions in W defined in Sections 4 and 5 are of Type $(\alpha, k, \{0,1\})$ for some $\alpha > -1$ and for all positive integers k. In particular,

$$\begin{aligned} s\log s &\in \text{Type}\,(1, k, \{0\}), \\ (1-s)\log(1-s)) &\in \text{Type}\,(1, k, \{1\}), \\ s^{2(1+\alpha)} &\in \text{Type}\,(2(1+\alpha), k, \{0\}) \end{aligned}$$

and

$$(1-s)^{2(1+\alpha)} \in \text{Type}\,(2(1+\alpha), k, \{1\})$$

for any integer $k \ge 0$. Let $q = (k+1)/(\alpha+1)$ and define a partition Π_α of $[0,1]$ associated with α by

$$\Pi_\alpha : t_0 = 0, \quad t_1 = m^{-q}, \quad t_j = j^q t_1, \quad j = 2, \ldots, m.$$

In addition, assume that

$$t_j \le u_{j1} < \cdots < u_{jn} \le t_{j+1} \qquad \text{for } j = 0, 1, \dots, m-1.$$

Let

$$u_i^{(j)} = \frac{1}{2}(t_{j+1} - t_j)u_{ji} + \frac{1}{2}(t_{j+1} + t_j).$$

It can be proved that if $g \in \mathrm{Type}\,(\alpha, n, \{0\})$ then

(6.1)

$$\int_0^1 g(x)\,dx = \sum_{j=1}^{m-1} \frac{1}{2}(t_{j+1} - t_j) \sum_{i=1}^{n} g(u_i^{(j)}) \int_{-1}^{1} l_{ji}(x)\,dx + O(m^{-n}),$$

where

$$l_{ji}(x) = \prod_{p=1, p\ne i}^{n} \frac{x - u_{jp}}{u_{ji} - u_{jp}}.$$

In particular, if u_{ji}, $i = 1, 2, \dots, n$ are chosen to be the zeros of the Legendre polynomial of degree k and $q = (2k+1)/(\alpha+1)$, then the error term in (6.1) becomes $O(m^{-2n})$ [**8**]. Integrals of type (2) can be evaluated by using this method.

The integrals of type (3) are double integrals that require additional efforts for evaluation. Since the integral (Kw_i, w_j) has the strongest singularity among the four integrals, we take it as an example to demonstrate our treatment. Write

$$(Kw_i, w_j) = \int_0^1 w_j(s) z_i(s)\,ds,$$

where

$$z_i(s) = \int_0^1 k(s,t) w_i(t)\,dt.$$

From the theory established in Sections 4 and 5, $z_i \in \mathrm{Type}\,(\alpha, n, \{0, 1\})$ for some $\alpha \ge 0$. Hence, $w_j z_i \in \mathrm{Type}\,(\alpha', n, \{0, 1\})$ for some $\alpha' \ge 0$. We split it into two integrals so that each has only one singular point and then use (6.1) to evaluate them. Clearly, we need the values of $z_p(u_i^{(j)})$ for computation of the integrals. Since

$$K(u_i^{(j)}, \cdot) w_p(\cdot) \in \mathrm{Type}\,(\alpha, n, \{0, u_i^{(j)}, 1\}),$$

$z_p(u_i^{(j)})$ can be obtained by evaluating four singular integrals using formula (6.1).

Next we present a numerical example to demonstrate the efficiency and accuracy of the method established in this paper and compare our method wth the conventional Galerkin method. The numerical results show that our method improves the convergence rate significantly, which is consistent with our theoretical estimate.

In this estimate, we consider equation

$$y(s) - \int_0^1 \log(|s-t|)\exp(2st)y(t)\,dt = f(s), \qquad 0 \le s \le 1, \tag{6.2}$$

where $f(s)$ is chosen so that $y(s) = s\log s + (1-s)\log(1-s) + s^{3.2}$ is the solution of the equation. That is,

$$\begin{aligned} f(s) &= s\log s + (1-s)\log(1-s) + s^{3.2} \\ &\quad - \int_0^1 \log(|s-t|)\exp(2st)[t\log t + (1-t)\log(1-t) + t^{3.2}]\,dt. \end{aligned}$$

The solution of equation (6.2) has singularities at $s=0$ and $s=1$ and $y \in H^1 \backslash H^2$. We define a uniform partition by letting $h = 1/(k+1)$, and $x_i = ih$ for $i = 0, 1, \ldots, k+1$. Let S_h^2 be the space of piecewise linear functions defined on $[0,1]$ with knots at $x_1, \ldots, x_k$. We denote the B-spline basis of the space S_h^2 by

$$B_i^2(s) = \begin{cases} (s-x_{i-1})/(x_i - x_{i-1}), & x_{i-1} \le s < x_i \\ (s-x_{i+1})/(x_i - x_{i+1}), & x_i \le s < x_{i+1} \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, k$,

$$B_0^2(s) = \begin{cases} (s-x_1)/(x_0 - x_1), & x_0 \le s < x_1 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B_{k+1}^2(s) = \begin{cases} (s-x_k)/(x_{k+1} - x_k), & x_k \le s < x_{k+1} \\ 0, & \text{otherwise.} \end{cases}$$

In addition, we define the singular subspace W by

$$W = \operatorname{span}\{s\log s, (1-s)\log(1-s)\}.$$

Then the Galerkin subspace is $V_h^2 = W \oplus S_h^2$. We use both the conventional Galerkin method and the singularity preserving Galerkin method to solve the equation and obtain two approximate solutions, y_h and $\hat{y}_h$, respectively. Numerical resuls are contained in the following table, where $e_i = y(s_i) - y_h(s_i)$ and $\hat{e}_i = y(s_i) - \hat{y}_h(s_i)$.

	$h = 0.1$	$h = 0.1$	$h = 0.025$	$h = 0.025$
x_i	$\lvert\hat{e}_i\rvert$	$\lvert e_i\rvert$	$\lvert\hat{e}_i\rvert$	$\lvert e_i\rvert$
0.0	2.78E-2	4.09E-2	3.44E-2	1.28E-2
0.1	5.77E-3	1.69E-1	9.83E-4	9.37E-2
0.2	3.61E-3	2.25E-1	7.55E-5	1.72E-1
0.3	3.89E-3	2.51E-1	2.31E-3	4.27E-2
0.4	3.77E-3	2.36E-1	1.65E-3	1.68E-1
0.5	4.84E-3	1.69E-1	2.07E-3	2.38E-1
0.6	5.08E-3	4.07E-2	1.10E-3	1.72E-1
0.7	6.52E-3	1.69E-1	7.46E-4	4.27E-2
0.8	6.95E-3	4.84E-1	3.08E-4	1.68E-1
0.9	1.28E-2	8.44E-2	2.22E-4	4.83E-1
1.0	2.96E-2	1.00E-0	8.65E-3	1.00E-0
L_2 error	1.12E-2	3.87E-1	3.00E-3	3.77E-1

The L_2 norm of the errors of the approximate solutions in this table is evaluated by the formula

$$||y - y_h||_2 \approx \left\{ \sum_{i=1}^{10} \frac{1}{10} [y(x_i) - y_h(x_i)]^2 \right\}^{1/2}.$$

Notice that the theoretical error estimate for the current method is $||y - \hat{y}_h||_2 = O(h^2)$. The numerical results are consistent with our theoretical estimate.

Acknowledgment. The authors would like to thank the referees for their constructive suggestions that led to this revised version. The authors are grateful to Professors Ronald Mathsen and Joseph Brennan for their careful and thorough examination of the presentation of this paper.

REFERENCES

1. P.M. Anselone, *Collectively compact operator approximation theory and applications to integral equations*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

2. K.E. Atkinson, *A survey of numerical mehtods for the solution of Fredholm integral equations of the second kind*, SIAM, Philadelphia, PA, 1976.

3. A. Friedman, *Foundations of modern analysis*, Dover, New York, 1982.

4. I. Graham, *Singularity expansions for the solution of second kind Fredholm integral equations with weakly singular convolution kernels*, J. Integral Equations **4** (1982), 1–30.

5. ———, *Galerkin methods for second kind interal equations with singularities*, Math. Comp. **39** (1982), 519–533.

6. H. Kaneko, R. Noren and Y. Xu, *A characterization of the solution of a Fredholm integral equation with L_∞ forcing term*, J. Integral Equations Appl. **2** (1990), 581–594.

7. H. Kaneko and Y. Xu, *Numerical solutions for weakly singular Fredholm equations of the second kind*, Appl. Numer. Math. **7** (1991), 167–177.

8. ———, *Gaussian-type quadratures for weakly singular integrals and their applications to the Fredholm integral equation of the second kind*, Math. Comp. **62** (1994), 739–753.

9. G.G. Lorentz, *Bernstein polynomials*, University of Toronto Press, Toronto, 1953.

10. J. Rice, *On the degree of convergence of nonlinear spline approximations*, in *Approximation with special emphasis on spline functions* (I.J. Schoenberg, ed.), Academic Press, NY, 349–365, 1969.

11. G.R. Richter, *On weakly singular Fredholm integral equations with displacement kernels*, J. Math. Anal. Appl. **55** (1976), 32–42.

12. C. Schneider, *Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind*, Integral Equations Operator Theory **2** (1979), 62–68.

13. ———, *Product integration for weakly singular integral equations*, Math. Comp. **36** (1981), 207–213.

14. I.H. Sloan, *Superconvergence*, in *Numerical solution of integral equations*, M.A. Goldberg, ed., Plenum Press, NY, 35–70, 1990.

15. G. Vainikko and A. Pedas, *The properties of solutions of weakly singular integral equations*, J. Austral. Math. Soc. **22** (1981), 419–430.

16. G. Vainikko and P. Ubas, *A piecewise polynomial approixmation to the solution of an integral equation with weakly singular kernel*, J. Austral. Math. Soc. **22** (1981), 431–438.

Department of Mathematics, North Dakota State University, Fargo, ND 58105