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A DECREASING REARRANGEMENT APPROACH FOR A CLASS OF ILL-POSED NONLINEAR INTEGRAL EQUATIONS

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ABSTRACT. A special class of nonlinear Fredholm integral equations of the first kind is considered where the kernel depends on t only via the unknown function $x(t)$. Due to this structure, one can achieve at best uniqueness up to rearrangement. To overcome this ambiguity, the decreasing rearrangement is used as a canonical solution exploiting the fact that continuous, L^p- and $W^{1,p}$-solutions have decreasing rearrangements which are in the same space, respectively. Conditions for existence and uniqueness of monotone solutions are presented. The equation can be reformulated as a linear integral equation of the first kind for the distribution function (which is, essentially, the inverse of the decreasing rearrangement). As an alternative, the general theory of Tikhonov regularization for nonlinear ill-posed problems can be applied and provides convergence and rates under certain conditions. For a specific example arising in optics, all conditions needed for these results are fulfilled.

1. Introduction. This paper is devoted to a study of the class

$$\int_0^1 k(s,x(t))\,dt = y(s), \quad 0 \le s \le 1, \tag{1.1}$$

of nonlinear Fredholm integral equations of the first kind. Inverse problems, e.g., in material sciences, require the solution of such an equation when the local distribution profile $x(t)$, $0 \le t \le 1$, of a physical quantity in a layer has to be recovered from an observable function $y(s)$, $0 \le s \le 1$, and the transmissibility properties of the layer forming the kernel k depend on the location t only via the profile value $x(t)$. An example arising in optics, where $k(s,x) = \sqrt{x^2 - s}$, is discussed in [**13**] (see also Example 3.7). Solutions of equation (1.1) are

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characterized by a strong ambiguity, because, e.g., the reflected profile $x(1-t)$, $0 \le t \le 1$, is always a solution to (1.1) when $x(t)$, $0 \le t \le 1$, is.

We are going to formulate an approach to overcome these difficulties by using decreasing rearrangements of solution profiles. Under weak assumptions it will be shown that decreasing rearrangements of solutions of (1.1) also solve (1.1). These decreasing rearrangements are by definition monotone; a sufficient condition for the uniqueness of monotone solutions is given, so that we can enforce "uniqueness up to rearrangements." We then apply descriptive regularization and Tikhonov regularization in order to solve (1.1) in a stable manner and prove convergence (rates) results.

Throughout this paper, (1.1) is considered as a nonlinear operator equation

$$F(x) = y, \tag{1.2}$$

where the operator $F : D(F) \subseteq B_1 \to B_2$ acts between Banach spaces B_1 and B_2 of real-valued functions $x = x(t)$ on the interval $[0,1]$, and $D(F)$ consists of functions bounded below by a positive number c.

In Section 2 we review the definition and main properties of decreasing rearrangements of Lebesgue measurable real functions. It will turn out to be important for our approach that the decreasing rearrangements x^* of functions x in $C[0,1]$, $L^p[0,1]$, and $W^{1,p}[0,1]$, respectively, are again in the same spaces.

Based on the concept of equimeasurable functions, assertions on the existence and uniqueness of decreasing solutions to equation (1.1) are made in Section 3. Restricted to decreasing solutions, the injectivity of the nonlinear integral operator F of (1.2) corresponds to the injectivity of an associated linear integral operator.

Section 4 gives some main ideas of descriptive regularization using monotonicity. This approach can be used to the determination of monotone representatives of equation (1.1) when in addition an upper bound C for the profile function values is known.

In Section 5 the general theory of Tikhonov regularization for ill-posed nonlinear operator equations is applied to equation (1.2) on the Hilbert spaces $L^2[0,1]$ and $W^{1,2}[0,1]$, respectively. A basic assumption needed for proving convergence of Tikhonov regularization is the weak closedness of operator F ([**8, 4**]). If F maps from $W^{1,2}[0,1]$ into $L^2[0,1]$, then this property is ensured by the compactness of associated embedding operators. However, in the case that $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$, weak closedness requires additional restrictions on $D(F)$ and on the kernel k of the integral operator, as will finally be described in Section 5.

2. Decreasing rearrangements. In this section we review some results on the decreasing rearrangement. General references about this topic are, e.g., [**2, 3, 6** and **14**]. We denote by λ the Lebesgue measure and by Λ^+ the set of nonnegative Lebesgue measurable real-valued functions defined almost everywhere on the interval $[0,1]$. Moreover, let, for a given $0 < c < \infty$, $D_\lambda^c \subset \Lambda^+$ be the set of Lebesgue measurable functions $x : [0,1] \to [c,\infty[$. For any function x of Λ^+ the *distribution function*

$$d_x(s) := \lambda(\{t \in [0,1] : x(t) > s\}), \quad 0 \le s < \infty \tag{2.1}$$

is well-defined. Then, the *decreasing rearrangement* (also called Schwarz symmetrization) $x^* : [0,1] \to [0,\infty[$ of x is defined by

$$x^*(t) := \begin{cases} \operatorname{ess\,sup}_{0\le\tau\le 1} x(\tau) & \text{if } t = 0 \\ \inf\{s > 0 : d_x(s) \le t\} & \text{if } 0 < t < 1 \\ \operatorname{ess\,inf}_{0\le\tau\le 1} x(\tau) & \text{if } t = 1. \end{cases} \tag{2.2}$$

Evidently, $x \in D_\lambda^c$ implies that $x^* \in D_\lambda^c$. Note that both d_x and x^* are for all $x \in D_\lambda^c$ nonincreasing and right-continuous functions.

Definition 2.1. Let $x_1 \in \Lambda^+$ and $x_2 \in \Lambda^+$. Then we say x_1 and x_2 are *equimeasurable* whenever $d_{x_1} = d_{x_2}$, i.e., the distribution functions coincide pointwise.

The following lemmata characterize the main properties of equimeasurability and decreasing rearrangements.

Lemma 2.2 ([**15**, p. 48], [**3**, p. 43]). *Let $x \in \Lambda^+$. Then x and x^* are equimeasurable. Moreover, for $x \in L^p[0,1]$, $1 \le p < \infty$, we have that*

$x^* \in L^p[0,1]$ *and*

$$(2.3) \qquad ||x||_{L^p[0,1]} = ||x^*||_{L^p[0,1]} = \left(\int_0^\infty p s^{p-1} d_x(s)\, ds \right)^{1/p}.$$

Lemma 2.3 ([**6**, p. 24–26]). *Let $x \in D_\lambda^c$ be a nonincreasing right-continuous function. Then $x = x^*$, i.e., the function and its decreasing rearrangement coincide.*

Lemma 2.4 ([**11**, p. 218]). *Let $x_1 \in \Lambda^+$ and $x_2 \in \Lambda^+$ be equimeasurable, $\Phi : \mathbf{R} \to \mathbf{R}_0^+$ a nonnegative Borel measurable function. Then $\Phi(x_1)$ and $\Phi(x_2)$ are equimeasurable functions in Λ^+, and*

$$(2.4) \qquad \int_0^1 \Phi(x_1(t))\, dt = \int_0^1 \Phi(x_2(t))\, dt.$$

Lemma 2.5. *Let $f : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ be a differentiable function and $x(t) \in \Lambda^+$ a simple function, i.e., $x(t) = \sum_{n=1}^N \alpha_n \chi_{A_n}(t)$ with measurable $A_n \subseteq [0,1]$, $\alpha_n \geq 0$. Then*

(a) $\int_0^t f(x^*(\tau))\, d\tau = f(x^*(t)) \cdot t + \int_{x^*(t)}^\infty f'(s)\, d_x(s)\, ds$ *for every* $0 \leq t \leq 1$,

(b) $\int_0^1 f(x(\tau))\, d\tau = f(c) + \int_c^\infty f'(s)\, d_x(s)\, ds$ *for every* $0 \leq c \leq \operatorname{ess\,inf}_{0 \leq \tau \leq 1} x(\tau)$.

Proof. Formally, Lemma 2.5(a) can be obtained by substituting $\tau = d_x(s)$ and integrating by parts. However, since $d_x(s)$ is not necessarily invertible and differentiable, the proof requires some care in detail. For simple functions, both sides of (a) can be computed directly; (b) follows from $\int_0^1 f(x(\tau))\, d\tau = \int_0^1 f(x^*(\tau))\, d\tau$ and part (a). □

If x is an arbitrary Borel measurable function in Λ^+, it is a pointwise limit of an increasing sequence of nonnegative simple functions, and the formulas given in Lemma 2.5 are still valid as soon as the limit can be interchanged with all integrals appearing in Lemma 2.5. This fact is used in Section 3 to transform the nonlinear integral equation (1.1) into a linear integral equation for the distribution function $d_x(s)$.

As Lemma 2.2 indicates, the membership in L^p of a function $x \in \Lambda^+$ carries over to its decreasing rearrangement x^*. This also holds for $C[0,1]$ and $W^{1,p}$:

Lemma 2.6 ([**2**, p. 48]). *Let* $x \in D_\lambda^c \cap C[0,1]$. *Moreover, let* $0 < c \leq m := \min_{0 \leq t \leq 1} x(t)$ *and* $M := ||x||_{C[0,1]} = \max_{0 \leq t \leq 1} x(t)$. *Then* $x^*(0) = M$, $x^*(1) = m$, *and* $x^* \in D_\lambda^c \cap C[0,1]$.

Lemma 2.7 ([**11**, p. 218, **5**]). *Let* x *be an element of the Sobolev space* $W^{1,p}[0,1]$, $p \in [1, +\infty]$. *Then the decreasing rearrangement* x^* *is again in* $W^{1,p}[0,1]$.

In [**7**] it is shown that, for a continuously differentiable function x, the derivative of x^* satisfies

$$\frac{1}{(x^*)'(u)} = -\sum_{\{t_j : x(t_j) = x^*(u)\}} \frac{1}{|x'(t_j)|} \tag{2.5}$$

at all points u where $x'(t_j) \neq 0$ for every point t_j in the set $\{t_j : x(t_j) = x^*(u)\}$. Using this formula, one finds that the rearrangement $x^*(u)$ of $x(t) = -54t^3 + 81t^2 - 36t + 94$ has a jump discontinuity in its first derivative at $u_0 = 1/6$; $x^*(u_0) = 5$ equals the value of the local maximum of $x(t)$. Thus, Lemma 2.7 cannot be generalized to Sobolev spaces of higher order: this is an example of an analytic function with a decreasing rearrangement that is not continuously differentiable and hence not in $W^{2,p}$ for any $p \geq 1$. However, the decreasing rearrangement of a continuously differentiable function is, by Lemma 2.7, still in $W^{1,\infty}$ (and hence in $W^{1,2}$). This motivates our use of $W^{1,2}$ (instead of C^1) in Section 5.

3. Existence and uniqueness of monotone solutions.

In this section we will show that under weak assumptions on the kernel, the integral equation (1.1) has nonincreasing solutions whenever it is solvable. Moreover, all $\hat{x} \in D_\lambda^c$ equimeasurable to a solution $x \in D_\lambda^c$ are also solutions to (1.1).

The next result immediately follows from Lemma 2.4:

Proposition 3.1. *Let, for all* $0 \le s \le 1$, *the functions* $\Phi_s(x) := k(s,x)$, $0 < c \le x < \infty$, *be Borel measurable and nonnegative. Moreover, let* $x \in D_\lambda^c$ *be such that* $y(s) := \int_0^1 k(s, x(t))\, dt < \infty$ *for all* $s \in [0,1]$. *Then, for every* $\hat{x} \in D_\lambda^c$ *with* $d_x = d_{\hat{x}}$, $\int_0^1 k(s, \hat{x}(t))\, dt = y(s)$ *for all* $s \in [0,1]$.

As a consequence of Proposition 3.1, and Lemmata 2.2, 2.6 and 2.7, we obtain

Corollary 3.2. *If* $x \in D_\lambda^c$ *solves* (1.1) *for given* $y(s)$, $0 \le s \le 1$, *then, under the assumptions of Proposition* 3.1 *about* k, *the decreasing rearrangement* x^* *is also a solution to the integral equation* (1.1). *In particular,* x^* *is in* $L^p[0,1]$, $C[0,1]$ *and* $W^{1,p}[0,1]$, *respectively, whenever at least one solution* $x \in D_\lambda^c$ *of* (1.1) *belongs to that space.*

Since monotone, continuous and simple functions are Borel measurable, we moreover have:

Corollary 3.3. *If* $x \in D_\lambda^c$ *solves* (1.1) (*with* y *such that* $y(s) < \infty$ *for all* $s \in [0,1]$) *and the nonnegative kernel* $k(s,\cdot)$ *is, for all* $0 \le s \le 1$, *a monotone, a continuous, or a simple function of* x *on* $[c, \infty[$, *then* x^* *is a nonincreasing right-continuous solution of* (1.1).

Although (1.1) is in general not uniquely solvable, the injectivity of the linearized integral operator will in fact imply the uniqueness of nonincreasing solutions of (1.1), as we will see in Theorem 3.6.

Assumption 3.4. *Let*

(i) $B_1 = B_2 = L^2[0,1]$, $D(F) := \{x \in L^2[0,1] : x(t) \ge c > 0 \text{ a.e.}\}$,

(ii) $k(s,x)$, $(s,x) \in [0,1] \times [c, \infty[$ *be nonnegative and continuous,*

(iii) $|k_x'(s,x)| \le H < \infty$, $k_x'(s,x)$, $(s,x) \in [0,1] \times [c,\infty[$, *be continuous.*

Lemma 3.5. *Under Assumption* 3.4, *the nonlinear integral operator* $F : D(F) \subseteq B_1 \to B_2$ *defined by*

$$[F(x)](s) := \int_0^1 k(s, x(t))\, dt, \quad 0 \le s \le 1 \tag{3.1}$$

is continuous and compact. At any $\bar{x} \in D(F)$, *the* (*formal*) *derivative* $F'(\bar{x}) : B_1 \to B_2$, *given by*

$$|F'(\bar{x})h](s) = \int_0^1 k'_x(s, \bar{x}(t))h(t)\, dt, \quad h \in B_1, \ 0 \le s \le 1, \tag{3.2}$$

is continuous and compact. If, in addition, there is a $K > 0$ *such that*

$$|k'_x(s, x_1) - k'_x(s, x_2)| \le K|x_1 - x_2|, \quad s \in [0, 1], \ x_1, x_2 \in [c, \infty[\,, \tag{3.3}$$

then the (*formal*) *derivative* $F'(\bar{x})$ *satisfies*

$$F(x) = F(\bar{x}) + F'(\bar{x})(x - \bar{x}) + r(x) \tag{3.4}$$

with $||r(x)|| \le (K/2)||x - \bar{x}||^2$ *for every* $x \in D(F)$.

Proof. From $|k'_x(s, x)| \le H$ we have

$$\begin{aligned} |k(s, x)| &\le |k(s, c)| + |(x - c)H| \\ &\le (k(s, c) + cH) + Hx \le \alpha x + \beta, \end{aligned}$$

where $\alpha := H \ge 0$ and $\beta := \max_{0 \le s \le 1} k(s, c) + cH \ge 0$. If, however, the kernel $k(s, x)$ is bounded in such a way, the general theory of Urysohn integral equations (see, e.g., [**16**, Chapter 7]) asserts that F is a continuous and compact operator in $L^2[0, 1]$. It is well-known that $F'(\bar{x})$ is continuous and compact. (3.3) implies

$$|k(s, x) - k(s, \bar{x}) - k_x(s, \bar{x})(x - \bar{x})| \le \frac{K}{2}|x - \bar{x}|^2. \tag{3.5}$$

This inequality and the definition of F immediately yield (3.4). □

Observe that the derivative F' looks formally like a Fréchet derivative. The Fréchet derivative, however, is usually defined only at interior points, while the interior of our set $D(F)$ is empty in L^2. Lemma 3.5 will be used in Section 5 to prove convergence rates for Tikhonov regularization.

The decreasing rearrangement approach gives a chance for overcoming the intrinsic ambiguity of solutions. We introduce the domain

$$D^*(F) := \{x \in D(F) : x \text{ nonincreasing}\}. \tag{3.6}$$

This definition is a bit sloppy since the set $D(F) \subseteq L^2[0,1]$ consists of equivalence classes of functions that are equal a.e. More precisely, we require that the a.e. equivalence class of x contains a nonincreasing function. In the following we assume that $x \in D^*(F)$ is always chosen as such a (not necessarily unique) nonincreasing representative.

Theorem 3.6. *Let $x \in D(F)$ solve* (1.1). *Then, under Assumption* 3.4, *the decreasing rearrangement $x^* \in D^*(F)$ also solves* (1.1). *Moreover, this solution is unique in $D^*(F)$ for all $y \in B_2$ (i.e., F is injective on $D^*(F)$) whenever the linear operator $G : L^1\,[c,\infty[\, \to L^\infty[0,1]$ defined by*

$$[G(f)](s) := \int_c^\infty k'_x(s,\tau) f(\tau)\, d\tau \tag{3.7}$$

is injective.

Proof. Obviously, $x \in D(F)$ implies that $x^* \in D^*(F)$. Moreover, Proposition 3.1 applies, because k is continuous and nonnegative, i.e., x^* solves (1.1) when x is a solution.

Let x_n be an increasing sequence of simple functions convergent to $x \in D(F)$. Since $|k(s,x_n)| \le \alpha x_n + \beta \le \alpha x + \beta =: g_1(x)$, $\int_0^1 g_1(x)\,dx \le \alpha ||x||_{L^1} + \beta$, $|k'_x(s,x_n(\tau))\,d_x(\tau)| \le H d_x(\tau) =: g_2(\tau)$, and $\int_c^\infty g_2(\tau)\,d\tau \le H \int_0^\infty d_x(\tau)\,d\tau = H||x||_{L^1}$ (cf. Lemma 2.2), the dominated convergence theorem applies, and the formula given in Lemma 2.5(b) can be used for x, as remarked after Lemma 2.5:

$$y(s) = \int_0^1 k(s,x(t))\,dt = k(s,c) + \int_c^\infty k'_x(s,\tau)\,d_x(\tau)\,d\tau. \tag{3.8}$$

Thus, the nonlinear integral equation (1.1) may be reformulated as a linear Fredholm integral equation of the first kind for the distribution function d_x:

$$\int_c^\infty k_x'(s,\tau)\,d_x(\tau)\,d\tau = y(s) - k(s,c), \quad 0 \le s \le 1. \tag{3.9}$$

Since $||d_x||_{L^1[c,\infty[} = \int_c^\infty |d_x(\tau)|\,d\tau \le \int_0^\infty d_x(\tau)\,d\tau = ||x||_{L^1[0,1]}$ by Lemma 2.2, distribution functions of functions in $D(F)$ are in $L^1\,[c,\infty[$. If the integral operator G is injective, then the distribution function is uniquely determined by (3.9), and thus F is injective on $D^*(F)$.

It remains to be shown that G maps $L^1\,[c,\infty[$ into $L^\infty[0,1]$, which follows from

$$\begin{aligned} |G(f)(s)| &\le \int_c^\infty |k_x'(s,\tau)f(\tau)|\,d\tau \\ &\le H\int_c^\infty |f(\tau)|\,d\tau = H\cdot ||f||_{L^1[c,\infty[}. \end{aligned} \tag{3.10}$$

□

Note that, because of Proposition A.3 in [**8**] and Lemma 3.5, (1.1) (considered on $D^*(F)$) is ill-posed in the sense that the unique non-increasing solution does not depend continuously on the data y in the L^2-sense. As shown in the proof, the nonlinear integral equation (1.1) can be reformulated as the linear integral equation (3.9) for the distribution function (which is quite straightforward). Note that, as an integral equation of the first kind, (3.9) is again ill-posed. In [**10**], this linear equation has been studied. It has turned out that plain Tikhonov regularization does not yield very good results, especially if no good upper and lower bounds for the solution are known, and that one can improve the results by imposing monotonicity. However, by doing this, one loses the linearity of the problem anyway, which, in our opinion, justifies to consider direct regularization of the nonlinear equation (1.1) (without transforming it to (3.9) first). Also, (3.9) is an equation for the distribution function d_x. If one wants to recover the decreasing rearrangement solution x^* from d_x, one has to note that while the L^1-error in d_x remains the same for x^*, a small error in the L^2-norm or in the uniform norm may correspond to a larger error in x^*. Thus, if one wants to find x^*, a direct regularization of (1.1) is at least an alternative worth being discussed, which will be done in Sections 4 and 5.

We now present a class of kernels, for which the injectivity of the operator G can be shown:

Let k be such that

$$k'_x(s,x) = \varphi\left(\frac{s}{g(x)}\right) = \sum_{i=0}^{\infty} \frac{\varphi^{(i)}(0)}{i!} \cdot \frac{s^i}{(g(x))^i} \tag{3.11}$$

with a smooth function φ ($\varphi^{(i)}(0) \neq 0$ for $i \geq i_0$) and a differentiable strictly increasing function g with $0 < g(c) < \infty$, $g'(x) > 0$ ($x \in [c,\infty[$) and $\lim_{x\to\infty} g(x) = \infty$, where we assume that the power series expansion (3.11) converges for all $0 \leq s \leq 1$ and $c \leq x < \infty$.

We check the injectivity of the integral operator G defined in (3.7):

If $G\psi = 0$, then

$$\int_c^{\infty} \frac{1}{(g(x))^i}\psi(x)\,dx = 0 \quad \text{for all } i \geq i_0$$

and, with

$$h = \frac{1}{g(x)}, \qquad f(h) = \frac{\psi(g^{-1}(1/h))}{g'(g^{-1}(1/h))},$$

$$\int_0^{1/g(c)} h^{i-2} f(h)\,dh = 0 \quad \text{for all } i \geq i_0.$$

The uniqueness of the solution of the Hausdorff moment problem implies that $f(h) = 0$ a.e. for $h \in [0, 1/g(c)]$. Since g is differentiable and, thus, the function $x(h) = g^{-1}(1/h)$ is absolutely continuous, this implies that $\psi(x) = 0$ a.e. on $[c,\infty[$. Thus, G is injective, so that the conclusion of Theorem 3.6 holds for such kernels, of which we now give a specific example:

Example 3.7. Consider the kernel

$$k(s,x) = \sqrt{x^2 - s}, \quad 1 < c \leq x < \infty,\ 0 \leq s \leq 1; \tag{3.12}$$

this kernel arises in optics (cf. [**13**]).

We have

$$k'_x(s,x) = \frac{x}{\sqrt{x^2 - s}} = 1 + \sum_{i=1}^{\infty} \frac{v_i}{i!} \cdot \frac{s^i}{x^{2i}}, \tag{3.13}$$

where $v_1 = 1/2$, $v_{i+1} = ((2i+1)/2)v_i$, $i \in \mathbf{N}$, and $1 < k'_x(s,x) \leq c/\sqrt{c^2-1}$, $0 < k(s,x) \leq x$, so that Assumption 3.4 is fulfilled. Note that the convergence radius of the power series expansion $\sum_{i=1}^{\infty}(v_i/i!)s^i y^i$ with respect to y is $\rho = (\limsup_{i\to\infty}(v_{i+1}s)/(v_i(i+1)))^{-1} = 1/s$. Consequently, (3.13) converges uniformly for all $1 < c \leq x < \infty$ and $0 \leq s \leq 1$.

Since k_x has the form (3.11) with $\varphi(\tau) = 1/\sqrt{1-\tau}$, $\varphi^{(i)}(0) = v_i$, $g(x) = x^2$, G is injective, so that, by Theorem 3.6, the equation

$$\int_0^1 \sqrt{x(t)^2 - s}\, dt = y(s) \tag{3.14}$$

has at most one nonincreasing solution.

4. A descriptive regularization approach. As motivated in the remarks following Theorem 3.6, we also consider a direct regularization of (1.1) as an alternative to regularizing the linear equation (3.9). We present two approaches: In this section, we regularize the problem by assuming that an upper bound $C < \infty$ for the solution is known. In Section 5, we will apply Tikhonov regularization.

Assumption 4.1. *Let Assumption* 3.4 *hold where, however, condition* (i) *is replaced by*

(i*) $B_1 = B_2 = L^2[0,1]$, $D(F) := D^*_C(F) := \{x \in L^\infty[0,1]$: x *nonincreasing and* $0 < c \leq x(t) \leq C < \infty$, $t \in [0,1]\}$, *and the conditions*

(iv) (1.2) *has a solution* $x^* \in D^*_C(F)$,

(v) F *is injective on* $D^*_C(F)$,

are added.

Note that, while (iv) is an existence assumption to be made, (v) can be deduced from Theorem 3.6.

Lemma 4.2. *$D^*_C(F)$ is a compact subset of $L^p[0,1]$ for $1 \le p < \infty$.*

Proof. The monotone functions $x \in D^*_C(F)$ are uniformly bounded in the form $|x(t)| \le C$ and $\vee_{t=0}^{1} x(t) \le C$. Then, it follows from Helly's theorem (see, e.g., [**17**, p. 250]) that every infinite subset of $D^*_C(F)$ contains a sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n(t) = \bar{x}(t)$, $0 \le t \le 1$, and $\bar{x} \in D^*_C(F)$. Since $|x_n(t) - \bar{x}(t)|^p \le |2C|^p$ is bounded by an integrable function and converges a.e. to 0, Lebesgue's dominated convergence theorem provides strong convergence:

(4.1)
$$\lim_{n\to\infty} ||x_n - \bar{x}||_{L^p[0,1]} = \left(\int_0^1 \lim_{n\to\infty} |x_n(t) - \bar{x}(t)|^p \, dt \right)^{1/p} = 0. \qquad \square$$

Lemma 4.3 ([**19**, p. 63]). *A sequence of* (*not necessarily continuous*) *monotone functions converging pointwise on a compact interval* $[a,b]$ *to a continuous function converges uniformly to that function.*

Now we are going to exploit the compactness of $D^*_C(F)$ in $L^p[0,1]$ to show the convergence of least-squares solutions of (1.2) when the data errors tend to zero. This approach is sometimes called the "method of quasi-solutions." If the compactness is forced by additional assumptions describing the expected shape of solutions, as monotonicity in our considerations, the approach is also called "descriptive regularization." For details, see [**12, 21**].

Let $\{y_n\}_{n=1}^{\infty}$ denote a sequence of right-hand sides of (1.2) with

$$||y_n - y||_{L^2[0,1]} \le \delta_n \tag{4.2}$$

and $\{x_n\}_{n=1}^{\infty}$ be an associated sequence in $D^*_C(F)$ such that

$$||F(x_n) - y_n||_{L^2[0,1]} \le \inf_{x \in D^*_C(F)} ||F(x) - y_n||_{L^2[0,1]} + \eta_n. \tag{4.3}$$

For such a sequence $\{x_n\}_{n=1}^{\infty}$, we have the following convergence result:

Theorem 4.4. *Let $\delta_n \to 0$ and $\eta_n \to 0$ as $n \to \infty$; then, under Assumption* 4.1,

$$\lim_{n\to\infty} ||x_n - x^*||_{L^p[0,1]} = 0, \quad 1 \le p < \infty, \tag{4.4}$$

where x^* is the uniquely determined solution of (1.2) *in* $D_C^*(F)$*. If, moreover,* $x^* \in C[0,1]$*, then the sequence* $\{x_n\}_{n=1}^\infty$ *converges uniformly to* x^* *on every interval* $[\alpha, \beta] \subset]0,1[$.

Proof. From Assumption 4.1 ((iv), (v)) it follows that there is a unique solution $x^* \in D_C^*(F)$ of (1.2). We will show that, for any subsequence of $\{x_n\}_{n=1}^\infty$ (which we again denote by $\{x_n\}_{n=1}^\infty$), there is a further subsequence $\{x_{n_k}\}_{k=1}^\infty$ that converges to x^* in the L^p-norm. This then implies (4.4).

Since $D_C^*(F)$ is compact by Lemma 4.2, we have, for any $1 \le p < \infty$, a subsequence $x_{n_k} \stackrel{L^p}{\to} \bar{x} \in D_C^*(F)$ with

$$(4.5) \quad ||F(x_{n_k}) - y_{n_k}||_{L^2[0,1]} \le ||F(x^*) - y_{n_k}||_{L^2[0,1]} + \eta_{n_k} \le \delta_{n_k} + \eta_{n_k}.$$

Now, $F : D_C^*(F) \subset L^2[0,1] \to L^2[0,1]$ is a continuous operator by Lemma 3.5, and $x_{n_k} \stackrel{L^2}{\to} \bar{x}$. Consequently, (4.5) implies that $F(\bar{x}) = y$ and, hence, by the injectivity of F on $D_C^*(F)$, $\bar{x}(t) = x^*(t)$ a.e. on $[0,1]$. Thus, (4.4) holds.

In order to prove uniform convergence on compact subintervals of $]0,1[$, we use Lemma 4.3 and show that, for sequences of monotone functions x_n, L^p-converges implies pointwise convergence on every interior point where the limit function x^* is continuous:

Choose a point $t \in]0,1[$ where x^* is continuous, and assume that there is an $\varepsilon > 0$ such that $|x_n(t) - x^*(t)| > \varepsilon$ for infinitely many n. Without loss of generality, assume $x_n(t) > x^*(t) + \varepsilon$ infinitely often. Since x_n is monotonically nonincreasing, $x_n(\tau) \ge x_n(t) > x^*(t) + \varepsilon$ for every $\tau < t$. Furthermore, since x^* is continuous at t, there is a $\delta(\varepsilon, t) > 0$ such that $x^*(\tau) < x^*(t) + \varepsilon/2$ for every $\tau \in]t - \delta, t[$. For those values of τ, $x_n(\tau) - x^*(\tau) > \varepsilon/2$. Now

$$||x_n - x^*||_{L^p}^p \ge \int_{t-\delta}^t |x_n(\tau) - x^*(\tau)|^p \, d\tau \ge \delta(\varepsilon, t)(\varepsilon/2)^p$$

infinitely often, which contradicts $x_n \stackrel{L^p}{\to} x^*$. ◻

Note that there is sort of a boundary layer effect; the convergence need not be uniform on all of $[0,1]$.

For the special kernel of Example 3.7, L^p-convergence and this boundary layer effect have been illustrated in [**13**].

In principle, this "regularization by compactness" cannot provide convergence rates. In the next section, we will also be able to provide convergence rates for a different regularization method, namely, for Tikhonov regularization.

5. A Tikhonov regularization approach. We apply the general framework for Tikhonov regularization developed in [**8**] (cf. also ([**4**]) to our problem. We first outline that approach for an abstract equation of the form (1.2) and then give conditions for (1.1) to fulfill the assumptions needed in that framework.

Assumption 5.1. (a) B_1 *and* B_2 *are Hilbert spaces,* $D(F)$ *is a convex subset of* B_1.

(b) $F : D(F) \subset B_1 \to B_2$ *is continuous and weakly* (*sequentially*) *closed.*

(c) (1.2) *is solvable in* $D(F)$.

Let $y_\delta \in B_2$ symbolize perturbed data with

$$||y_\delta - y||_{B_2} \le \delta \tag{5.1}$$

and denote by $x_\alpha^{\delta,\eta} \in D(F)$ any element satisfying

$$\begin{aligned} ||F(x_\alpha^{\delta,\eta}) - y_\delta||_{B_2}^2 &+ \alpha||x_\alpha^{\delta,\eta} - \bar{x}||_{B_1}^2 \\ &\le \inf_{x\in D(F)} \{||F(x) - y_\delta||_{B_2}^2 + \alpha||x - \bar{x}||_{B_1}^2\} + \eta \end{aligned} \tag{5.2}$$

for given $\alpha > 0$ and $\delta, \eta \ge 0$, $\bar{x} \in B_1$. This is Tikhonov regularization with a possible tolerance in the minimization represented by η; for the role of $\bar{x}$, see [**8**].

Theorem 5.2. *Let Assumption* 5.1 *be fulfilled, and let* $\alpha = \alpha(\delta, \eta)$ *be chosen such that* $\alpha \to 0$, $\delta^2/\alpha \to 0$ *and* $\eta/\alpha \to 0$ *as* $\delta \to 0$ *and* $\eta \to 0$. *Then, for* $\delta_n \to 0$, $\eta_n \to 0$, *and* $\alpha_n = \alpha_n(\delta_n, \eta_n)$, *any sequence* $\{x_{\alpha_n}^{\delta_n,\eta_n}\}_{n=1}^\infty$ (*as defined by* (5.2)) *has a convergent subsequence.*

The limit x_0 of every convergent subsequence is an $\bar{x}$-minimum norm solution of (1.2)*, i.e., $F(x_0) = y$, and*

$$||x_0 - \bar{x}||_{B_1} = \min\{||x - \bar{x}||_{B_1} : F(x) = y, x \in D(F)\}. \tag{5.3}$$

Proof. [**8**]. □

Assumption 5.3. (a) *F is Fréchet-differentiable*;

(b) *Let x_0 be an $\bar{x}$-minimum-norm solution of* (1.2) *such that $||F'(x_0) - F'(z)|| \leq L||x_0 - z||_{B_1}$ holds for all $z \in D(F)$ with an $L > 0$*;

(c) *there exists $\omega \in B_2$ satisfying $x_0 - \bar{x} = (F'(x_0))^*\omega$*;

(d) $L||\omega||_{B_2} < 1$.

Theorem 5.4. *Let Assumptions* 5.1 *and* 5.3 *be satisfied. Then, for the choices $\alpha \sim \delta$ and $\eta = O(\delta^2)$, we obtain the convergence rate*

$$||x_\alpha^{\delta,\eta} - x_0||_{B_1} = O(\sqrt{\delta}). \tag{5.4}$$

Proof. [**8**]. □

For generalizations of these results to the case where (1.2) is not solvable but has only least-squares solutions, see [**4**].

This general theory of regularization for nonlinear operator equations immediately applies to the determination of decreasing rearrangement solutions $x^* \in D^*(F)$ to Urysohn integral equations (1.1) when we set $B_1 = W^{1,2}[0,1]$, $B_2 = L^2[0,1]$ and $D(F) := \{x \in W^{1,2}[0,1] : x \text{ nonincreasing}, x(t) \geq c > 0\}$. In this case, F as defined in (1.1) is continuous, if the kernel k satisfies conditions (ii) and (iii) of Assumption 3.4. Since the norm of $W^{1,2}$ is stronger than the norm of L^2, this is a consequence of Lemma 3.5. Moreover, the Rellich-Kondrachov theorem (see, e.g., [**1**, p. 144]) asserts that the embedding of $W^{1,2}[0,1]$ in $C[0,1]$ is compact. Consequently, weak convergence in $W^{1,2}[0,1]$ implies strong convergence in $C[0,1]$. This yields the weak continuity of the operator $F : D(F) \subset W^{1,2}[0,1] \to L^2[0,1]$ and also the weak closedness of this operator.

If we set $B_1 = B_2 = L^2[0,1]$ and $D(F) := D_C^*(F)$ (see Assumption 4.1), then Assumption 5.1 is also satisfied. In particular, the weak closedness of $F : D_C^*(F) \subseteq L^2[0,1] \to L^2[0,1]$ is a direct consequence of the compactness of $D_C^*(F)$ (see Lemma 4.2) and of the fact that a continuous operator on a compact set is weakly closed.

In this L^2-setting with domain $D_C^*(F)$, Theorem 5.4 does not immediately apply. As remarked after Lemma 3.5, no Fréchet derivative exists since $D_C^*(F)$ has an empty interior. However, [**20**] points out that the results of [**8**] are still valid when Assumptions 5.3 (a) and (b) are replaced by the following assumption:

There exists a linear operator $F'(x_0)(\cdot) : B_1 \to B_2$ such that, for every $x \in D(F)$,

$$F(x) = F(x_0) + F'(x_0)(x - x_0) + r(x)$$

with

$$||r(x)|| \le \frac{K}{2}||x - x_0||^2$$

for some positive K and x sufficiently close to the "a priori guess" $\bar{x}$. Hence, by Lemma 3.5, Theorem 5.4 applies with $x_0 = x^*$ if there is a $K > 0$ such that

$$|k_x'(s,x_1) - k_x'(s,x_2)| \le K|x_1 - x_2|, \quad s \in [0,1],\ x_1, x_2 \in [c, \infty[\,. \tag{5.5}$$

Assumptions 5.3 (c) and (d) are fulfilled if an element $\omega \in L^2[0,1]$ with $||\omega||_{L^2[0,1]} < 1/K$ and

$$x^*(t) - \bar{x}(t) = \int_0^1 k_x'(s, x^*(t))\omega(s)\, ds \quad \text{a.e. on } [0,1] \tag{5.6}$$

exists. The latter condition can only hold if

$$||x^* - \bar{x}||_{L^2[0,1]} < H/K \tag{5.7}$$

holds, where H is as in Assumption 3.4 (iii), i.e., if the "a priori guess" $\bar{x}$ is sufficiently close to x^*; (5.6) is a smoothness assumption for the initial error $x^* - \bar{x}$.

This theory does not seem to carry over to the case of not necessarily bounded decreasing rearrangements x^*, i.e., $D(F) := D^*(F)$ (see

(3.6)). In that case, the required weak closedness of F may be missing. The following Proposition shows that weak closedness of $F : D^*(F) \subset L^2[0,1] \to L^2[0,1]$ can be achieved for a specific class of Urysohn kernels in equation (1.1):

Proposition 5.5. *Under Assumption* 3.4, *the operator* $F : D^*(F) \subset L^2[0,1] \to L^2[0,1]$ *defined by the Urysohn integral equation* (1.1) *satisfies Assumption* 5.1 *when the kernel* $k(s,x)$ *has a Laurent series expansion*

$$k(s,x) = f_{-1}(s)x + f_0(s) + \sum_{i=1}^{\infty} f_i(s)\frac{1}{x^i} \tag{5.8}$$

converging uniformly on $x \in [c,\infty[$ *for all* $0 \le s \le 1$.

Proof. From Lemma 3.5, we obtain the continuity of F. Now, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $D^*(F)$ with $x_n \overset{L^2}{\rightharpoonup} \bar{x}$. Then $||x_n||_{L^2[0,1]} \le K < \infty$ and $\int_0^1 x_n(t)\,dt \to \int_0^1 \bar{x}(t)\,dt$ as $n \to \infty$. We now show that $\int_0^1 1/(x_n(t))^i\,dt \to \int_0^1 1/(\bar{x}(t))^i\,dt$ for $i = 1, 2, \dots$ as $n \to \infty$ by proving that every subsequence of x_n (which we denote again by x_n) contains a subsequence which has this property:

If $x_n \in D^*(F)$, then by Assumption 3.4, $0 < c \le x_n(t) < \infty$ for all $t > 0$. We consider the reciprocal function

$$\frac{1}{x_n}(t) := \begin{cases} \frac{1}{x_n(t)} & \text{if } x_n(t) < \infty \\ 0 & \text{if } x_n(t) = \infty; \end{cases}$$

$1/x_n$ is a nondecreasing function in $L^\infty[0,1]$ with $0 \le (1/x_n)(t) \le 1/c < \infty$. By Helly's theorem (see, e.g., [**17**, p. 250]), there is a subsequence $1/x_{n_k}$ converging pointwise on $[0,1]$ to a nondecreasing function $\hat{x}$ with $0 \le \hat{x}(t) \le 1/c$. If we can show that $\hat{x}$ vanishes on no interval $[0,\varepsilon]$, then x_{n_k} converges almost everywhere and hence in measure to the almost everywhere finite function $1/\hat{x}$. Then $\bar{x}$ and $1/\hat{x}$ are the same elements of $L^2[0,1]$ since convergence of x_{n_k} to $1/\hat{x}$ in measure and uniform boundedness of $\{||x_{n_k}||_{L^2[0,1]}\}$ implies that x_{n_k} converges to $1/\hat{x}$ weakly in $L^2[0,1]$ ([**17**, p. 224]). Hence, by Lebesgue's dominated convergence theorem, $||1/x_{n_k}||_{L^p[0,1]} \to ||1/\bar{x}||_{L^p[0,1]}$ as $k \to \infty$ holds for all $p = 1, 2, \dots$.

To make this argument valid, it remains to be shown that there is no interval $[0,\varepsilon]$ where $\hat{x}$ vanishes. Assume that such an interval would exist. Then $\lambda\{t \in [0,\varepsilon] : 1/x_{n_k}(t) \geq \alpha\} \to 0$ as $k \to \infty$ for all $\alpha > 0$, since $(1/x_{n_k})|_{[0,\varepsilon]}$ converges pointwise and hence in measure to 0. This implies that $\lambda\{t \in [0,\varepsilon] : x_{n_k}(t) > Z\} \to \varepsilon$ as $k \to \infty$ for all $Z > 0$. Now choose $Z > \sqrt{2K^2/\varepsilon}$. Then, for $k > k_0(Z)$, the distribution function $d_{x_{n_k}}$ satisfies the inequality $d_{x_{n_k}}(z) \geq \varepsilon/2$ for $0 \leq z \leq Z$, since $d_{x_{n_k}}$ is nonincreasing. Hence, by (2.3),

$$\begin{aligned} ||x_{n_k}||^2_{L^2[0,1]} &= 2\int_0^\infty z\, d_{x_{n_k}}(z)\, dz \geq 2\int_0^Z z\, d_{x_{n_k}}(z)\, dz \\ &\geq 2\int_2^Z z\frac{\varepsilon}{2}\, dz \\ &= \frac{Z^2\varepsilon}{2} > K^2. \end{aligned}$$

This contradicts the definition of K as an upper bound for $||x_{n_k}||_{L^2[0,1]} \leq K$.

We have thus proven that, with $\bar{\mu}_{-1} := ||\bar{x}||_{L^1[0,1]}$, $\bar{\mu}_i := ||1/\bar{x}||^i_{L^i[0,1]}$ and $\mu_{-1,n} := ||x_n||_{L^1[0,1]}$, $\mu_{i,n} := ||1/x_n||^i_{L^i[0,1]}$, we have

$$\lim_{n\to\infty} \mu_{i,n} = \bar{\mu}_i, \quad i = -1, 1, 2, \dots . \tag{5.9}$$

From the uniform convergence of (5.8) we derive (for $0 \leq s \leq 1$) that, for $x \in D^*(F)$,

$$[F(x)](s) = \int_0^1 k(s, x(t))\, dt = \mu_{-1} f_{-1}(s) + f_0(s) + \sum_{i=1}^\infty \mu_i f_i(s) \tag{5.10}$$

holds with

$$\mu_{-1} = ||x||_{L^1[0,1]}, \quad \mu_i = ||1/x||^i_{L^i[0,1]}, \quad x \in D^*(F)$$

(see [**9**, p. 452]). The majorant criterion for uniform convergence and $\mu_{i,n} \leq 1/c^i$ for all n guarantee, with the fact that $k(s,c) = f_{-1}(s)c + f_0(s) + \sum_{i=1}^\infty f_i(s)(1/c^i) < \infty$, that the integration in (5.10) can be interchanged with the limits in (5.9), which yields that, for all $0 \le s \le 1$,

$$\lim_{n\to\infty}[F(x_n)](s) = [F(\bar{x})](s). \tag{5.11}$$

The uniform bounds $0 \le [F(x_n)](s) \le \alpha + \beta K$ coming from the assumption that $k(s,x) \le \alpha+\beta x$ now imply that $F(x_n) \overset{L^2}{\to} F(\bar{x})$. Thus, F is weakly continuous (and even compact). Since $D^*(F)$ is closed and convex, hence weakly closed, this proves the weak closedness of F. □

Finally, note that the assumptions of Proposition 5.5 are satisfied for the kernel of Example 3.7. Integration of (3.13) yields that

$$k(s,x) = x + \sum_{i=1}^{\infty}\left(\frac{\chi_i s^i}{i!}\right)\frac{1}{x^{2i-1}} \tag{5.12}$$

with $\chi_1 = -1/2$, $\chi_{i+1} = ((2i-1)/2)\chi_i$, $i = 1,2,\ldots$, i.e., $f_{-1}(s) = 1$, $f_{2i}(s) = 0$, $i = 0,1,2,\ldots$, and $f_{2i-1}(s) = \chi_i s^i/i!$, $i = 1,2,\ldots$.

Consequently, for this specific kernel, Proposition 5.5 and hence Theorem 5.2 apply.

To summarize: The general convergence theory for Tikhonov regularization can be applied to (1.1) in one of the following situations:

(i) (1.1) is considered on $D(F) := \{x \in W^{1,2}[0,1] : x \text{ nonincreasing}, x(t) \ge c > 0\}$.

(ii) (1.1) is considered on $D^*_C(F) \subseteq L^2[0,1]$.

(iii) (1.1) is considered on $D^*(F)$ for a kernel having the properties required in Proposition 5.5.

In all these cases, we obtain convergence results for Tikhonov regularization (where the minimization in (5.2) takes place over nonincreasing functions only!) to the decreasing rearrangement solution of (1.1). The concrete formulation of these results is now quite obvious and hence omitted. Finally, note that the theory developed in [**18**] for obtaining convergence (with rates) for (5.2) combined with projection into finite-dimensional spaces can also be applied.

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