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# RECONSTRUCTION OF SOME POTENTIALS USED IN THE BOUNDARY ELEMENT METHOD

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ABSTRACT. We present a fast method for evaluation of some potentials arising when the boundary element method (BEM) is applied to the investigation of simply supported plates resting on an elastic foundation of Winkler type. The main idea is to reformulate this problem as a differential one and to approximate it by a difference scheme. The solution of the difference scheme approximates the potential with second order of accuracy in the $W_2^2$ mesh norm. It can be obtained by fast solvers. Exact formulas for the jumps of some derivatives of the potentials are derived as well.

**Introduction.** Let $D$ be a finite simply connected domain, bounded by a closed smooth contour $\Sigma$, contained in the rectangle $\Omega$. As observed in [**1, 2**], the investigation of simply supported plates, resting on an elastic foundation of Winkler type, is governed by the boundary value problem (BVP) with constant coefficients

$$\Delta^2 u(r) + \beta^2 u(r) = 0, \quad r \in D, \tag{0.1}$$

$$u(r) = u_0(r), \qquad \Delta u(r) = u_1(r), \quad r \in \Sigma. \tag{0.2}$$

In order to reduce the problem (0.1), (0.2) to boundary integral equations, we begin with the definitions of the Green's function $G(r,s)$, $s \in \Sigma$, as the solution to the BVP

$$\Delta^2 G(r,s) + \beta^2 G(r,s) = -\delta(r-s), \quad r \in \Omega, \tag{0.3}$$

$$G(r,s) = \partial^2 G(r,s)/\partial n^2 = 0, \quad r \in \partial\Omega. \tag{0.4}$$

Here $n$ denotes the exterior normal to $\partial\Omega$. Representing the solution $u(r)$ of (0.1), (0.2) in the form

$$u(r) = \int_\Sigma \Delta G(r,s)\Phi_1(s)\,ds + \int_\Sigma G(r,s)\Phi_2(s)\,ds, \tag{0.5}$$

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the boundary conditions (0.2) give two coupled integral equations for the unknown densities $\Phi_1$ and $\Phi_2$.

Suppose now that these integral equations are solved numerically (some efficient methods are proposed and investigated in [**3, 4, 5**]). Having found the approximate densities to $\Phi_1$ and $\Phi_2$, we substitute them into representation (0.5) and arrive at the problem for evaluation of the solution $u(r)$ to the initial BVP (0.1), (0.2) by formula (0.5). A similar problem for evaluation of potentials arises also for the direct version of the BEM.

The direct evaluation of the potentials by quadrature formulas has at least two drawbacks. The first one is the high cost when one is calculating the solution $u(r)$ at $N_1N_2$ equi-spaced points of the rectangle $\Omega$. The second one is that the good accuracy computation of the potentials at points near $\Sigma$ needs special procedures (see [**7, 12**]).

One way to avoid the first problem is to compute the potential values at a small number of carefully chosen mesh points close to $\Sigma$ and then to extend the approximate solution to the rest of the domain using fast solvers. This idea is proposed and numerically implemented in [**7, 13**].

Another way to evaluate the potential is to represent it as a solution of an appropriate BVP. Then this BVP is approximated and the solution of the discrete problem is found effectively by fast solvers. The difficulty which occurs here is the numerical treatment of the differential equation because it always has the Dirac delta function as a singularity. In [**6, 11, 18**] Steklov averaging operators are used for approximation of the delta function. The jumps of the derivatives of the potentials are included in the approximation of the delta function in [**6, 12, 15**]. Theoretical investigation and numerical implementation of the last idea is carried out in [**8, 12, 15**] only for potentials arising from second order elliptic problems.

We follow the last method and apply it to the evaluation of the logarithmic potential and potentials from (0.5) relative to the fourth order elliptic equation (0.1). New difficulties arise from the more complicated behavior of the kernels and the intention to get a scheme with higher accuracy $-O(h^2)$ in $W_2^2$ mesh norm, compared with $O(h^{3/2})$ in [**15**]. Moreover, the method is applied to evaluate the potentials for fourth order elliptic equations.

The assumptions imposed in the paper on the potentials are in terms of the classical $C^k$ smoothness spaces in $\overline{D}$ and $\overline{\Omega\backslash D}$. In a forthcoming paper [**10**] the case of Sobolev $W_2^k$ smoothness in $\overline{D}$ and $\overline{\Omega\backslash D}$ is treated. The $W_2^k$ smoothness is more suitable for getting error estimates which agree very well with the required smoothness of the densities and the boundary curve. But, as one may expect, they are harder to be obtained. Our numerical experiments have shown a good agreement with convergence results presented in this article. For these experiments we refer to the paper mentioned above.

The kernels of the potentials considered in the paper (see (0.5)) contain the Green's function and its Laplacian. As is well known, the evaluation of the Green's function is a difficult problem, especially when more complicated BVP are treated. The investigated method does not require computation of Green's function. This fact is of basic importance when we consider effective numerical methods for solving the integral equation system in the unknown densities $\Phi_1$ and $\Phi_2$ (the first step of the BEM). For example, one can apply the Galerkin method to the system of integral equations and then solve iteratively the obtained system of linear equations using appropriate preconditioners. At each step of the iterative process, one evaluates only potentials with densities known from the previous iteration and, according to the results of this paper, this can be done without evaluation of $G(r,s)$. Hence the numerical approximations to the densities $\Phi_1$ and $\Phi_2$ can be done without numerical evaluation of the Green's function (see, e.g., [**16**] for some iterative methods for solving integral equations).

Let us also mention that the method considered is not confined to a special kind of domain (except the smoothness of the boundary). It allows evaluation of the potential in $O(N_1N_2)$ points with the same accuracy and almost the same computational cost as in the well-studied case of rectangular domains.

The paper consists of two parts. In Part 1 we evaluate all the jumps in the coordinate directions, derivatives of the logarithmic and biharmonic potential. In Part 2 we present fast numerical methods for evaluation of the considered potentials.

**Part 1. Jump relations of some potentials.** Our main goal in this part is to evaluate the jump relations between some derivatives of potentials with kernels $\ln|r-s|$, $|r-s|^2\ln|r-s|$, $\Delta G(r,s)$ and $G(r,s)$.

These potentials appear in the BEM treatment of the Laplace equation, the biharmonic equation and equation (0.1). For the construction of an effective numerical method for evaluation of these potentials we need explicit expressions for their jumps across $\Sigma$.

Our primary tools in the investigations are well known jump relations of the normal derivatives of the logarithmic potential and the continuity of its tangential derivatives across the boundary. The jump relations for the first and second derivatives in the coordinate directions of the logarithmic potential can be found in Mokin [**15**]. We extend our consideration to potentials with more complicated kernels. Similar ideas applied to the jump relations of double layer logarithmic potential are investigated by A. Mayo [**12**].

Section 1 contains preliminary results. In Section 2 we obtain jump relations for the third derivatives of the logarithmic potential. Then in Section 3 we consider the biharmonic potential. Some properties of Green's function defined by (0.3), (0.4) are included in Section 4. Jump relations of potentials with kernels $\Delta G(r,s)$ and $G(r,s)$ are derived at the end of part 1 in Section 5.

**1. Preliminaries.** The following notations will be used in the paper. Let $r(\lambda)=(x(\lambda),y(\lambda))$, $0\le\lambda\le 2\pi$, be a parametrization of the curve $\Sigma$ and let $\kappa(r)$ denote its curvature. Set $D_1=\Omega\backslash\overline{D}$. Without loss of generality suppose $\operatorname{diam}(\Omega)<1$.

Let $C^k(\Omega)$, $k\in\mathbf{N}$, be the space of all $k$-times continuously differentiable functions defined on $\Omega$. Let $H^{k,\alpha}(\Omega)$, $k\in\mathbf{N}$, $0<\alpha\le 1$, be the Hölder spaces $\{u\in C^k(\Omega): |D^k u(r)-D^k u(s)|\le c|r-s|^\alpha, \forall r,s\in\Omega\}$.

Let $f(r)$, $r\in\Omega$, be a continuous function relative to the curve $\Sigma$, i.e., continuous up to the boundary in $D$ and $D_1$. At the points of $\Sigma$ we denote by $f_i$ the interior limit of this function and by $f_e$ the exterior limit. Then $[f](r)$, $r\in\Sigma$ (or $[f](\lambda)$, $0\le\lambda\le 2\pi$) will stand for the jump of this function at the point $r=r(\lambda)\in\Sigma$, i.e., $[f](r)=f_i(r)-f_e(r)$.

Consider the logarithmic potential

$$w(r)=-\frac{1}{2\pi}\int_\Sigma g(s)\ln|r-s|\,ds,\quad r\in\Omega. \tag{1.1}$$

The following lemma summarizes some known results (see, e.g., [**15**]).

**Lemma 1.** *Assume* $g \in H^{1+\varepsilon}(\Sigma)$ *and* $\Sigma \in H^{2+\varepsilon}[0,2\pi] \times H^{2+\varepsilon}[0,2\pi]$, $\varepsilon > 0$. *Then the jumps in the derivatives of* $w$ *at* $r = r(\lambda) \in \Sigma$ *are given by*

$$[w] = 0, \qquad \left[\frac{\partial w}{\partial x}\right] = \frac{y'}{|r'|}g, \qquad \left[\frac{\partial w}{\partial y}\right] = -\frac{x'}{|r'|}g,$$
$$\left[\frac{\partial^2 w}{\partial x^2}\right] = -\left[\frac{\partial^2 w}{\partial y^2}\right] = -\frac{1}{|r'|^2}\left\{(y'^2 - x'^2)\kappa g - \frac{2x'y'g'}{|r'|}\right\}.$$

**2. Jumps of the third derivatives of the logarithmic potential.** In this section the jumps of the third derivatives of the logarithmic potential are found. For this purpose we extend the method used for obtaining the formulas in Lemma 1.

Assume that $g \in H^{2+\varepsilon}(\Sigma)$ and $\Sigma \in H^{3+\varepsilon}[0,2\pi] \times H^{3+\varepsilon}[0,2\pi]$, $\varepsilon > 0$. Then $w$ given by (1.1) is a harmonic function in $D$ and $D_1$ and $(\partial\Delta w/\partial x)(r) = (\partial\Delta w/\partial y)(r) = 0$ for $r \in D$ and $r \in D_1$. These equalities immediately imply

$$\left[\frac{\partial^3 w}{\partial x^3}\right] = -\left[\frac{\partial^3 w}{\partial y^2 \partial x}\right] \quad \text{and} \quad \left[\frac{\partial^3 w}{\partial x^2 \partial y}\right] = -\left[\frac{\partial^3 w}{\partial y^3}\right].$$

We change the variables in a neighborhood of $\Sigma$ introducing new curvilinear coordinates $(\lambda, t)$ relative to $\Sigma : r = r(\lambda) + tn(\lambda)$, $0 \le \lambda \le 2\pi$, $|t| \le t_0$. We differentiate the composite function $w(r(\lambda) + tn(\lambda))$ three times, substitute $t = 0$ and subtract the limits from both sides of $\Sigma$ in the obtained formulas. Then we get

$$\left[\frac{\partial^3 w}{\partial \lambda \partial t^2}\right] = \left[\frac{\partial^3 w}{\partial x^3}\right]\frac{x'(3y'^2 - x'^2)}{|r'|^2} + \left[\frac{\partial^3 w}{\partial x^2 \partial y}\right]\frac{y'(y'^2 - 3x'^2)}{|r'|^2} + 2\kappa g'. \tag{1.2}$$

On the other hand, if we differentiate the well known jump relations of the normal and tangential derivatives of $w$ and use the definition of the distribution, we will arrive at the equality $[\partial^3 w/\partial\lambda\partial t^2] = -\kappa g' - \kappa' g$. Thus (1.2) leads to

$$\left[\frac{\partial^3 w}{\partial x^3}\right]\frac{x'(3y'^2 - x'^2)}{|r'|^2} + \left[\frac{\partial^3 w}{\partial x^2 \partial y}\right]\frac{y'(y'^2 - 3x'^2)}{|r'|^2} = -3\kappa g' - \kappa' g. \tag{1.3}$$

Similarly, we treat the other third mixed derivative of $w$ and obtain

$$\left[\frac{\partial^3 w}{\partial x^3}\right]\frac{y'(3x'^2-y'^2)}{|r'|^3}+\left[\frac{\partial^3 w}{\partial x^2\partial y}\right]\frac{x'(3y'^2-x'^2)}{|r'|^3}=-2\kappa^2 g-g'\frac{|r'|'}{|r'|^3}+\frac{g''}{|r'|^2}. \tag{1.4}$$

From the system (1.3) and (1.4), we find the jumps of the third derivatives of $w$. Thus, we have proved the following.

**Lemma 2.** *Assume $g \in H^{2+\varepsilon}(\Sigma)$ and $\Sigma \in H^{3+\varepsilon}[0,2\pi]\times H^{3+\varepsilon}[0,2\pi]$, $\varepsilon > 0$. Then the jump relations of the third derivatives of the logarithmic potential with density $g$ are*

$$\begin{aligned}\left[\frac{\partial^3 w}{\partial x^3}\right] &= -\left[\frac{\partial^3 w}{\partial y^2\partial x}\right]\\ &= -\frac{g}{|r'|^4}\{x'(3y'^2-x'^2)\kappa'+2\kappa^2|r'|y'(3x'^2-y'^2)\}\\ &\quad + g''\frac{y'(3x'^2-y'^2)}{|r'|^5}\\ &\quad - \frac{g'}{|r'|^6}\{3\kappa|r'|^2x'(3y'^2-x'^2)+|r'|'y'(3x'^2-y'^2)\},\\ -\left[\frac{\partial^3 w}{\partial y^3}\right] &= \left[\frac{\partial^3 w}{\partial y\partial x^2}\right] = \frac{g}{|r'|^4}\{y'(3x'^2-y'^2)\kappa'-2\kappa^2|r'|x'(3y'^2-x'^2)\}\\ &\quad + g''\frac{x'(3y'^2-x'^2)}{|r'|^5}\\ &\quad + \frac{g'}{|r'|^6}\{3\kappa|r'|^2y'(3x'^2-y'^2)-|r'|'x'(3y'^2-x'^2)\}.\end{aligned}$$

**3. Jumps of the biharmonic potential.** Consider the biharmonic potential $\nu$

$$\nu(r)=\int_\Sigma G_0(|r-s|)g(s)\,ds,\quad r\in\Omega, \tag{1.5}$$

with kernel $G_0(|r-s|)$, $G_0(|t|)=-t^2\ln t/(8\pi)$–the fundamental solution of the biharmonic equation.

The first and second derivatives $D^m\nu$, $|m|\le 2$, are continuous functions in $\Omega$. This fact is an immediate consequence of the well known theorems about continuity of integral operators depending on a parameter (see, e.g., [**14**]). We study the jumps of the third and fourth derivatives of $\nu$. From the identity

$$2\pi \frac{\partial^3 G_0(r,s)}{\partial x^3} = \frac{\partial}{\partial x} \ln \frac{1}{|r-s|} + \frac{1}{2}(x - x(\lambda)) \frac{\partial^2}{\partial x^2} \ln \frac{1}{|r-s|},$$

valid for arbitrary $r = (x,y) \in \Omega$ and $s = (x(\lambda), y(\lambda)) \in \Sigma$, the calculation of $[\partial^3 \nu / \partial x^3]$ is reduced to the calculation of the known jumps of the logarithmic potential. Thus, $[\partial^3 \nu/\partial x^3]$ is found to be $(y'^3/|r'|^3)g$. Similarly, we treat the other third and fourth derivatives of $\nu$ and obtain the following

**Lemma 3.** *Assume $g \in H^{2+\varepsilon}(\Sigma)$ and $\Sigma \in H^{3+\varepsilon}[0,2\pi] \times H^{3+\varepsilon}[0,2\pi]$, $\varepsilon > 0$. Then the jumps of the derivatives of the biharmonic potential $\nu$ with density $g$ are*

$$\left[\frac{\partial^m \nu}{\partial x^i \partial y^{m-i}}\right] = 0, \quad 0 \le i \le m \le 2,$$

$$\left[\frac{\partial^3 \nu}{\partial x^3}\right] = \frac{y'^3}{|r'|^3} g, \qquad \left[\frac{\partial^3 \nu}{\partial y^3}\right] = -\frac{x'^3}{|r'|^3} g,$$

$$\left[\frac{\partial^3 \nu}{\partial x^2 \partial y}\right] = -\frac{x' y'^2}{|r'|^3} g, \qquad \left[\frac{\partial^3 \nu}{\partial x \partial y^2}\right] = \frac{x'^2 y'}{|r'|^3} g.$$

$$\left[\frac{\partial^4 \nu}{\partial x^4}\right] = 2\kappa g y'^2 (3x'^2 - y'^2)|r'|^{-4} + 4g' x' y'^3 |r'|^{-5},$$

$$\left[\frac{\partial^4 \nu}{\partial y^4}\right] = 2\kappa g x'^2 (3y'^2 - x'^2)|r'|^{-4} - 4g' x'^3 y' |r'|^{-5},$$

$$\left[\frac{\partial^4 \nu}{\partial x^2 \partial y^2}\right] = \kappa g \{x'^4 + y'^4 - 6x'^2 y'^2\}|r'|^{-4} + 2g' x' y' (x'^2 - y'^2)|r'|^{-5}.$$

**4. Green's function.** Let $p \ge 1$ and let $d$ be the distance between $\Sigma$ and $\partial\Omega$. Fix a $C^\infty[0,\infty)$ function $e(t)$ such that $e(t) = 1$ for $t \le d/2$; $0 < e(t) < 1$ for $d/2 < t < d$ and $e(t) = 0$ for $d \le t$. For $d/2 < t < d$ one can take, for example,

$$e(t) = \int_{2t/d-1}^{1} \exp(-1/(s - s^2))\, ds \Big/ \int_0^1 \exp(-1/(s-s^2))\, ds.$$

With a standard technique it can be shown that Green's function and its Laplacian (with respect to the first variable) $\Delta_1 G$ are symmetric functions in $r$ and $s$:

$$G(r,s) = G(s,r), \qquad \Delta_1 G(r,s) = \Delta_1 G(s,r).$$

Representing Green's function in the form

$$G(r,s) = e(|r-s|)G_0(|r-s|) + G_1(r,s) \tag{1.6}$$

with $G_0$ defined in Section 3, we study the properties of $G_1$. The definitions of $G$, $G_0$ and some simple calculations lead to the BVP for $G_1(\cdot, s)$ for every $s \in \Sigma$

$$\Delta^2 G_1(r,s) + \beta^2 G_1(r,s) = -f(|r-s|), \quad r \in \Omega, \tag{1.7}$$

$$G_1(r,s) = \frac{\partial^2 G_1(r,s)}{\partial n^2} = 0, \quad r \in \partial\Omega, \tag{1.8}$$

where

$$\begin{aligned} f(t) = (\beta^2 e(t) + \Delta^2 e(t))G_0(t) + (4e'''(t) + 6e''(t)/t - 2e'(t)/t^2)G_0'(t) \\ + (6e''(t) + 6e'(t)/t)G_0''(t) + 4e'(t)G_0'''(t). \end{aligned}$$

The singularity of $f$ is of order $|r-s|^2 \ln |r-s|$ because of the smoothness of $e$ and $G_0'$. Hence $f$ belongs to $W_p^2(\Omega)$, $p \geq 1$. Moreover, in view of the compactness of $\Sigma$, $W_p^2(\Omega)$-norm of $f(|r-s|)$ can be bounded from above by a constant independent of $s$. Function $G_1$, as a solution to (1.7) and (1.8), belongs to $W_p^6(\Omega)$. In view of the imbedding theorem $G_1$ belongs to $H^{5+\alpha}(\Omega)$, $\alpha = 1 - 2/p$, $p > 2$, and its norm in this space is estimated from above by a constant independent of $s$.

The evaluation of $\Delta G(r,s)$ from (1.6) gives the equality

$$\Delta G(r,s) = -(1/2\pi) \ln |r-s| + G_2(r,s), \quad r \in \Omega. \tag{1.9}$$

Here $G_2$ depends on $e$, $G_1$ and their first and second derivatives only. Hence, $G_2$ is a smooth function from the Sobolev space $W_p^4(\Omega)$, $p \geq 1$. In this way we have proved the following

**Lemma 4.** *Let* $p \geq 1$, *and let the Green's function defined by* (0.3), (0.4) *be written in the form* (1.6). *Then*:

(i) *Green's function and its Laplacians are symmetric functions of their arguments.*

(ii) *Green's function differs from* $e(|r-s|)G_0(|r-s|)$ *by the* $W_p^6$-*smooth solution* $G_1(r,s)$ *to the problem* (1.7), (1.8).

(iii) *For* $r$ *close to* $s$ *the Laplacian* $\Delta G(r,s)$ *behaves like* $-(1/2\pi) \ln|r-s|$. *These functions differ by a* $W_p^4$-*smooth function* $G_2(r,s)$, *which can be explicitly written in terms of* $e$, $G_1$ *and their first and second derivatives.*

Since Green's function satisfies (1.6), it can be computed by determining $G_1$ as the smooth solution to the standard problem (1.7), (1.8) in a rectangular domain by any fast method.

**5. Jump relations of potentials with kernels $G$ and $\Delta G$.** Consider the relative to formulas (0.5) potential $V$, given by

$$V(r) = \int_\Sigma G(r,s)g(s)\,ds, \quad r \in \Omega. \tag{1.10}$$

Rewrite (1.6) in the form

$$G(r,s) = G_0(|r-s|) + (e(|r-s|)-1)G_0(|r-s|) + G_1(r,s).$$

From the properties of $G$ (see the proof of Lemma 4), $e$ and $G_0$, we immediately see that the potentials with kernels $(e(|r-s|)-1)G_0(|r-s|)$ and $G_1(|r-s|)$ are $H^{5+\alpha}$-continuous functions on $\Omega$. Hence, the smoothness of $D^mV$, $|m| \leq 4$, is determined by the smoothness of the $m$-th derivatives of the biharmonic potential $\nu$ investigated in Section 3. Therefore, we have proved

**Lemma 5.** *Assume* $g \in H^{2+\varepsilon}(\Sigma)$ *and* $\Sigma \in H^{3+\varepsilon}[0,2\pi] \times H^{3+\varepsilon}[0,2\pi]$, $\varepsilon > 0$. *Let* $\nu$ *and* $V$ *be the potentials defined by* (1.5) *and* (1.10), *respectively, with a common density* $g$. *Then the jumps of the derivatives of* $V$ *are equal to the jumps in the derivatives of* $\nu$ (*explicitly given in Lemma* 3):

$$[D^mV] = [D^m\nu], \quad |m| \leq 4.$$

Let us study the potential

$$W(r) = \int_{\Sigma} \Delta G(r,s) g(s)\, ds, \quad r \in \Omega. \tag{1.11}$$

By analogy with the previous arguments using representation (1.9) for the kernel $\Delta G$, we obtain the following

**Lemma 6.** *Assume $g \in H^{2+\varepsilon}(\Sigma)$ and $\Sigma \in H^{3+\varepsilon}[0,2\pi] \times H^{3+\varepsilon}[0,2\pi]$, $\varepsilon > 0$. Let $w$ and $W$ be the potentials defined by* (1.5) *and* (1.11)*, respectively, with a common density $g$. Then the jumps of the derivatives of $W$ are equal to the jumps in the derivatives of $w$* (*explicitly given in Lemmas* 1 *and* 2):

$$[D^m W] = [D^m w], \quad |m| \leq 3.$$

Let us note that all jumps of the derivatives of the potentials, obtained in Lemmas 1, 2, 3, 5 and 6, are expressed in terms of the given functions–the density $g$ and the parametrization of $\Sigma$.

**Part 2. Numerical evaluation of some potentials.** The aim of this part is to propose effective numerical methods for evaluation of the values of the potentials investigated in Part 1 on a uniform mesh in the rectangle $\Omega$. The main idea used here is to reformulate the problem–instead of the direct computation of the potential we use the fact that it is a solution of an appropriate BVP with a Dirac delta function on the right-hand side. Then we approximate the differential problem by a difference scheme, which contains terms with jumps of derivatives of the potentials across $\Sigma$. The solution of the difference scheme can be found by fast solvers using $O((N_1^2 + N_2^2)\log(N_1 N_2))$ arithmetic operations.

Section 1 contains some notations and definitions. In Section 2 we give an $O(h^2)$ accurate numerical method for evaluation of the logarithmic potential. In Sections 3 and 4 we discuss the computation of potentials with kernels $G(r,s)$ and $\Delta G(r,s)$, respectively.

**1. Notations and definitions.** We cover the closed rectangle $\Omega = [a_1, b_1] \times [a_2, b_2]$ with a uniform mesh with parameter $h = (h_1, h_2)$

($h_i = (b_i - a_i)/N_i$ with integer $N_i$). Denote by $\omega$ and $\gamma$ the set of all mesh points belonging to the interior of $\Omega$ and to $\partial\Omega$, respectively. Thus, $\omega \cap \gamma = \varnothing$ and $\omega \cup \gamma$ is a uniform mesh on $\Omega$. Set

$$\gamma_1 = \{(x,y) \in \gamma : a_1 < x < b_1, y = a_2 \text{ or } y = b_2\},$$
$$\gamma_2 = \{(x,y) \in \gamma : a_2 < y < b_2, x = a_1 \text{ or } x = b_1\}.$$

Denote by $\gamma_1^+$ and $\gamma_2^+$ these points of $\gamma$ for which $y = b_2$ and $x = b_1$, respectively.

For any mesh function $z(r)$, $r = (x,y) \in \omega$, define the difference operators

$$z_x(r) = (z(x+h_1,y) - z(x,y))/h_1, \qquad z_{\bar{x}}(r) = (z(x,y) - z(x-h_1,y))/h_1$$

for the first difference quotients in the $x$-direction ($z_y(r)$ and $z_{\bar{y}}(r)$ are defined analogously in the $y$-direction).

The norm in $C(\omega)$ is defined as usual by $|z|_{C(\omega)} = \max_{r\in\omega} |z(r)|$. For defined $\omega$ functions $z_1(r)$, $z_2(r)$, the scalar product and the norm in $L_2(\omega)$ are given by

$$(z_1, z_2) = \sum_{r\in\omega} h_1 h_2 z_1(r) z_2(r), \qquad ||z||_{0,\omega}^2 = (z,z).$$

When the above sum runs over $\omega \cup \gamma_2^+$, $\omega \cup \gamma_1^+$ or $\omega \cup \gamma_1^+ \cup \gamma_2^+$, then the corresponding norms are denoted by $||\cdot\rangle|$, $|\langle\cdot||$ or $|\langle\cdot\rangle|$, respectively. The Sobolev spaces $W_2^m(\omega)$, $m = 1,2,3,4$, consist of mesh functions defined on $\omega \cup \gamma$ and vanishing on $\gamma$. Their discrete semi-norms are given by

$$|z|_{1,\omega}^2 = ||z_{\bar{x}}\rangle|^2 + |\langle z_{\bar{y}}||^2, \qquad |z|_{2,\omega}^2 = ||z_{\bar{x}x}||^2 + ||z_{\bar{y}y}||^2 + 2|\langle z_{\bar{x}\bar{y}}\rangle|^2,$$
$$|z|_{3,\omega}^2 = ||z_{x\bar{x}\bar{x}}\rangle|^2 + 3|\langle z_{x\bar{x}\bar{y}}||^2 + 3||z_{y\bar{y}\bar{x}}\rangle|^2 + |\langle z_{y\bar{y}\bar{y}}||^2,$$
$$|z|_{4,\omega}^2 = ||z_{\bar{x}\bar{x}xx}||^2 + 4\langle z_{x\bar{x}\bar{x}\bar{y}}\rangle|^2 + 6||z_{y\bar{y}\bar{x}x}||^2 + 4|\langle z_{\bar{x}\bar{y}\bar{y}y}\rangle|^2 + ||z_{yy\bar{y}\bar{y}}||^2.$$

At a point $r = (x,y) \in \omega$, define the discrete Laplacian $\Delta_h z(r) = z_{x\bar{x}}(r) + z_{y\bar{y}}(r)$. We relate this Laplacian to the pattern $P(r) = \{(x,\eta) \in \Omega : |y-\eta| \le h_2\} \cup \{(\xi,y) \in \Omega : |x-\xi| \le h_1\}$. Denote by $\omega^*$ the set of all points $r$ of $\omega$ such that $P(r) \cap \Sigma \ne \varnothing$ (in other words, $\omega^*$ is the subset of $\omega$ consisting of those mesh points for which at least one of the

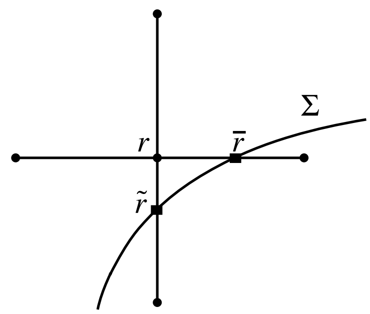


FIGURE 1.

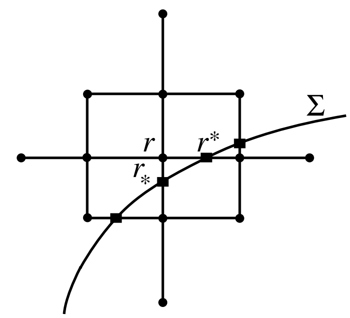


FIGURE 2.

four neighbor mesh points lie on the other side of $\Sigma$, provided $h$ is small enough). For example, the point $r$ on Figure 1 belongs to $\omega^*$ and the intersection points of $\Sigma$ and $P(r)$ in $x$ and $y$ directions are $\bar{r} = (\bar{x}, y)$ and $\tilde{r} = (x, \tilde{y})$. The mesh points are marked by dots.

By analogy to the previous, define the discrete approximation to the biharmonic operator by $\Delta_h^2 z(r) = z_{\bar{x}\bar{x}xx}(r) + 2z_{x\bar{x}y\bar{y}}(r) + z_{yy\bar{y}\bar{y}}(r)$. It uses 13 values of $z$ at the mesh points $\rho \in \overline{P}(r) \cap \omega$, where the pattern $\overline{P}(r)$ is given by $\overline{P}(r) = P(x+h_1, y) \cup P(x-h_1, y) \cup P(x, y+h_2) \cup P(x, y-h_2)$. Denote by $\omega^{**}$ the set of all points $r$ of $\omega$ such that $\overline{P}(r) \cap \Sigma \neq \varnothing$. Figure 2 illustrates a typical point $r \in \omega^{**}$, along with some intersection points of $\overline{P}(r)$ and $\Sigma$ (two of them are $r^* = (x^*, y)$ and $r_* = (x, y_*)$).

We make use of the truncated power notation $(x)_+^n$, given by $(x)_+^n = x^n$ if $x > 0$ and 0 otherwise.

Throughout Part 2 the symbol $M$ with different subscripts denotes positive constants, which may depend only on $\Omega$ and $\Sigma$. The values of $M$ may differ at each occurrence.

**2. Evaluation of the logarithmic potential.** The purpose of this section is to present a numerical method for calculation at the points of $\omega$ of the logarithmic potential $w(r)$, defined by (1.1). The method has a second order of accuracy and requires $O((N_1^2 + N_2^2)\log(N_1 N_2))$ arithmetic operations.

Instead of the direct evaluation of $w(r)$ we shall find it as a solution to the BVP

$$\Delta w(r) = -g(r)\delta_\Sigma(r), \quad r \in \Omega, \tag{2.1}$$

$$w(r) = -\frac{1}{2\pi}\int_\Sigma g(s)\ln|r-s|\,ds, \quad r \in \partial\Omega. \tag{2.2}$$

Let the logarithmic potential $w(r)$, $r \in \Omega$, be a four times continuously differentiable function relative to the curve $\Sigma$ (four times continuous up to the boundary in $D$ and $D_1$). Conditions $g \in H^{3+\varepsilon}(\Sigma)$ and $\Sigma \in H^{4+\varepsilon}[0,2\pi] \times H^{4+\varepsilon}[0,2\pi]$, $\varepsilon > 0$, will guarantee such smoothness (see, e.g., [**14**]).

First, we approximate (2.1) by a difference scheme on the mesh $\omega$ using the discrete Laplacian $\Delta_h w$. At points of $\omega\backslash\omega^*$ (being away from $\Sigma$) $\Delta_h w$ is zero up to terms of second order of $h$. The difficulties arise from the approximation of $\Delta_h w$ at the points of $\omega^*$ (close to $\Sigma$), where the right-hand side of (2.1) includes the delta function as a singularity.

Consider a point $r \in \omega^*$ (see Figure 1). We expand $w(\rho)$, $\rho \in P(r)\cap\omega$, in Taylor series and substitute these expansions into the definition of the discrete Laplacian. Then we obtain

$$\Delta_h w(r) = \Delta w(r) + (2\chi_D(r) - 1)\Phi(r) + O(h_1^2 + h_2^2), \quad r \in \omega^*, \tag{2.3}$$

where $\chi_D$ is the characteristic function of the set $D$ and

$$\begin{aligned}\Phi(r) &= \operatorname{sign}(\bar{x} - x)\frac{(h_1 - |\bar{x} - x|)_+}{h_1^2}\left[\frac{\partial w}{\partial x}\right](\bar{r})\\ &\quad + \frac{(h_1 - |\bar{x} - x|)_+^2}{2h_1^2}\left[\frac{\partial^2 w}{\partial x^2}\right](\bar{r})\\ &\quad + \operatorname{sign}(\bar{x} - x)\frac{(h_1 - |\bar{x} - x|)_+^3}{6h_1^2}\left[\frac{\partial^3 w}{\partial x^3}\right](\bar{r})\\ &\quad + \operatorname{sign}(\tilde{y} - y)\frac{(h_2 - |y - \tilde{y}|)_+}{h_2^2}\left[\frac{\partial w}{\partial y}\right](\tilde{r})\\ &\quad + \frac{(h_2 - |y - \tilde{y}|)_+^2}{2h_2^2}\left[\frac{\partial^2 w}{\partial y^2}\right](\tilde{r}) + \operatorname{sign}(\tilde{y} - y)\\ &\quad \cdot \frac{(h_2 - |y - \tilde{y}|)_+^3}{6h_2^2}\left[\frac{\partial^3 w}{\partial y^3}\right](\tilde{r}).\end{aligned}$$

Let us note that the truncated power notation allows the application of the above definition of $\Phi(r)$ not only for the case demonstrated in Figure 1, but also when $\Sigma$ intersects only one of the segments of $P(r)$. The same definition can be used also for $r \notin \omega^*$, when obviously $\Phi(r) = 0$. The function $\Phi(r)$ can be explicitly written (see Lemmas 1 and 2 of Part 1) in terms of the density $g(r)$, the parametrization of $\Sigma$ and the distances between $r$ and $\Sigma$ in the $x$ and $y$ direction.

Second, let us approximate boundary condition (2.2). Since the integrand

$$F(t,r) = -(1/2\pi) g(r(t)) |r'(t)| \log |r(t) - r|$$

is a $2\pi$-periodic function of $t$, it is convenient to use the rectangular rule, due to its extreme accuracy in this case. For any $r \in \partial\Omega$, set $\varphi_n(r) = n^{-1} \sum_{i=1}^{n} F(in^{-1}, r)$ and $R_n(F,r) = w(r) - \varphi_n(r)$, where $w(r)$ is known from (2.2).

Since the problem (2.1), (2.2) is linear, we separate it to two problems: for finding function $w_1$ as the solution to the problem

$$\Delta w_1(r) = 0, \quad r \in \Omega, \tag{2.4}$$

$$w_1(r) = R_n(F,r), \quad r \in \partial\Omega \tag{2.5}$$

and a function $w_2(r) = w(r) - w_1(r)$, which is a solution to the problem

$$\Delta w_2(r) = -g(r)\delta_\Sigma(r), \quad r \in \Omega,$$
$$w_2(r) = \varphi_n(r), \quad r \in \partial\Omega.$$

Function $w_1$ gives the influence inside the rectangle of the error of quadrature formulas $\varphi_n(r)$ used for evaluation of the integral from (2.2).

**Theorem 1.** *Let $k, m \in \mathbf{N}$, $g \in C^k(\Sigma)$ and $\Sigma \in C^{k+1}[0,2\pi] \times C^{k+1}[0,2\pi]$. Assume $w_1$ is a solution to the* BVP (2.4), (2.5). *Then we have*

$$|w_1|_{W_2^m(\Omega)} \le Mn^{-k} |g|_{C^k(\Sigma)} |r(\cdot)|_{C^{k+1}(\Sigma)}.$$

*Proof.* First we estimate the $W_2^m(\partial\Omega)$ norm of $R_n(F,r)$. Since $\partial^l R_n(F,r)/\partial x^i \partial y^{l-i} = R_n(F_1, r)$ with $F_1(r,t) = g(r(t))|r'(t)|\partial^l$

$\log|r(t)-r|/\partial x^i \partial y^{l-i}$, the problem is reduced to the estimation of the error of the rectangular rule with step $1/n$ applied to the new function $F_1(r,t)$. Following the results of [**17**], the error is bounded from above by the $\tau_k(F_1, 1/n)_1$–the $k$-th averaged modulus of smoothness of $F_1$. The smoothness of $F_1$ is determined by the smoothness of $g(r(t))|r'(t)|$, since $r(t) \in \Sigma$, $r \in \partial\Omega$, $\operatorname{dist}(\Sigma, \partial\Omega) > 0$. Using the properties of the $\tau_k$ modulus, we conclude $\tau_k(F_1, 1/n)_1 = O(n^{-k})$ under the assumptions on $g$ and $\Sigma$. Hence, for any natural $m$, the $W_2^m$ norm of $R_n(F,r)$ on the sides of the rectangle $\Omega$ is of order $O(n^{-k})$ (with a constant in "big $O$" depending on $m$).

We apply the a priori estimates in $W_2^m(\Omega)$ norm for the solution of the BVP (2.4), (2.5), in the case of the convex polygonal domain. Necessary and sufficient conditions [**9**] for the coupling of the boundary conditions at the corner points of the rectangle are satisfied, since $R_n(F,r)$ is a harmonic function at each point $r \in \partial\Omega$. Hence, the a priori estimate $|w_1|_{m,\Omega} \le M|R_n|_{m,\partial\Omega}$ holds. We combine it with the $W_2^m$ norm of $R_n(F,r)$ on the sides of the rectangle $\Omega$ and obtain the desired estimate. □

Due to Theorem 1 the effect of replacing the integral in the boundary condition (2.2) with $\varphi_n(r)$ is negligible from a numerical point of view. Moreover the jumps of the derivatives of $w_2$ at the points of $\Sigma$ coincide with the jumps of the derivatives of the logarithmic potential $w$ because of the harmonicity of $w_1$ inside the rectangle. Thus, to the end of this section we freely use the notation $w$ for $w_2$.

We approximate the problem (2.1), (2.2), by the difference scheme

$$\Delta_h w_h(r) = (2\chi_D(r) - 1)\Phi(r), \quad r \in \omega, \tag{2.6}$$

$$w_h(r) = \varphi_n(r), \quad r \in \gamma. \tag{2.7}$$

The approximation error of the mesh function $w_h(r)$ to $w(r)$ is given by

**Theorem 2.** *Let the logarithmic potential $w(r)$ be a $C^4$ smooth function relative to $\Sigma$, and let $w_h(r)$ be the solution to the finite difference scheme* (2.6) *and* (2.7). *Then one has*

$$|w - w_h|_{2,\omega} \le M_1(h_1^2 + h_2^2)||w||_{C^4(D)\cap C^4(D_1)},$$

$$|w - w_h|_{C(\omega)} \le M_2(h_1^2 + h_2^2)||w||_{C^4(D)\cap C^4(D_1)}.$$

*Proof.* The error $z(r) = w(r) - w_h(r)$ is a solution to the problem $\Delta_h z(r) = \psi(r)$, $r \in \omega$ and $z(r) = 0$, $r \in \gamma$ with $\psi(r) = \Delta_h w(r) - \Phi(r)$. In view of (2.3), function $\psi(r)$ is of order $O(h_1^2 + h_2^2)$ for each $r \in \omega$, hence $|\psi|_{0,\omega} = O(h_1^2 + h_2^2)$. Thus, the a priori estimate $|w - w_h|_{2,\omega} \leq M|\psi|_{0,\omega}$ and the imbedding inequality

$$|z|_{C(\omega)} \leq M|z|_{2,\omega} \tag{2.8}$$

prove the theorem. □

Thus, we have shown that the method has a second order accuracy in $C(\omega)$ in the whole domain, in particular at the mesh points close to $\Sigma$.

Let us compare the accuracy of the presented method and the method proposed by Mokin [**15**] under one and the same smoothness requirements for the solution. The rate of convergence in [**15**] is $O(h^{1.5})$ in $W_2^2(\omega)$ norm. We have obtained a higher accuracy in Theorem 2 because the terms of first order in $h$ were included in $\Phi(r)$.

Let us count the arithmetic operations used for evaluation of the approximation $w_h$ of the potential $w(r)$ at the $N_1N_2$ points of $\omega$. The determination of $\Phi(r)$ at points $r \in \omega^*$ requires $O(N_1+N_2)$ operations. The evaluation of the logarithmic potential at $O(N_1 + N_2)$ points of the boundary $\gamma$ by the $n$ points rectangular rule needs $O((N_1 + N_2)n)$ operations. The solution $w_h$ of (2.6) and (2.7) can be found by any fast solver. If we use FFT, $O((N_1^2 + N_2^2)\log(N_1N_2))$ operations are required. Therefore, the proposed algorithm for evaluation of $w(r)$ at $N_1N_2$ points of $\omega$ can be realized by $O((N_1^2+N_2^2)\log(N_1N_2))$ arithmetic operations.

**3. Evaluation of a potential with kernel $G(r,s)$.** The aim of this section is to present a second order accurate numerical method for calculation of the potential $V(r)$ with kernel $G(r,s)$ at the $N_1N_2$ points of $\omega$ with operation count $O((N_1^2 + N_2^2)\log(N_1N_2))$.

Let the (defined by (1.10)) potential $V(r)$ be a $C^2$ smooth function in $\Omega$ and a $C^6$ smooth function relative to the curve $\Sigma$ (these conditions are fulfilled if we assume $g \in H^{3+\varepsilon}(\Sigma)$ and $\Sigma \in H^{4+\varepsilon}[0, 2\pi]$, $\varepsilon > 0$). Our method is based on the fact that the potential $V(r)$ is a solution to the problem

$$\Delta^2 V(r) + \beta^2 V(r) = -g(r)\delta_\Sigma(r), \quad r \in \Omega, \tag{2.9}$$

$$V(r) = \frac{\partial^2 V(r)}{\partial n^2} = 0, \quad r \in \partial\Omega. \tag{2.10}$$

We approximate (2.9) and (2.10) by a difference scheme on the mesh $\omega$. For the differential operator, we use the well-known 13-point discrete approximation $\Delta_h^2 V(r) + \beta^2 V(r)$. At the points of $\omega \backslash \omega^{**}$ this discrete operator is zero up to terms of the order $O(h_1^2 + h_2^2)$. We shall find with accuracy $O(h_1 + h_2)$ the values of the discrete operator at the points of $\omega^{**}$ explicitly.

Expanding $V(r)$, $V(x + h_1, y)$, $V(x + 2h_1, y)$, $V(x - 2h_1, y)$ and $V(x - h_1, y)$ in Taylor series and replacing into the definition of $V_{\bar{x}\bar{x}xx}(r)$, we obtain

$$V_{xx\bar{x}\bar{x}}(r) = \frac{\partial^4 V(r)}{\partial x^4} + (2\chi_D(r) - 1)\Phi_1(r) + O(h_1 + h_2), \quad r \in \omega^{**},$$

where

$$\Phi_1(r) = \operatorname{sign}(x^* - x)\frac{(2h_1 - |x^* - x|)_+^3 - 4(h_1 - |x^* - x|)_+^3}{6h_1^4}\left[\frac{\partial^3 V}{\partial x^3}\right](r^*) + \frac{(2h_1 - |x^* - x|)_+^4 - 4(h_1 - |x^* - x|)_+^4}{24h_1^4}\left[\frac{\partial^4 V}{\partial x^4}\right](r^*).$$

By analogy with the $x$-direction we obtain in the $y$-direction

$$V_{yy\bar{y}\bar{y}}(r) = \frac{\partial^4 V(r)}{\partial y^4} + (2\chi_D(r) - 1)\Phi_2(r) + O(h_1 + h_2), \quad r \in \omega^{**},$$

where

$$\Phi_2(r) = \operatorname{sign}(y_* - y)\frac{(2h_2 - |y_* - y|)_+^3 - 4(h_2 - |y_* - y|)_+^3}{6h_2^4}\left[\frac{\partial^3 V}{\partial y^3}\right](r_*) + \frac{(2h_2 - |y_* - y|)_+^4 - 4(h_2 - |y_* - y|)_+^4}{24h_2^4}\left[\frac{\partial^4 V}{\partial y^4}\right](r_*).$$

Since the finite difference operator $V_{y\bar{y}x\bar{x}}$ is the tensor product of the univariate operators $V_{x\bar{x}}$ and $V_{y\bar{y}}$, we apply $V_{x\bar{x}}$ for points $(x, y+h_2)$, $(x, y)$ and $(x, y-h_2)$ and evaluate $V_{y\bar{y}x\bar{x}}$. We find

$$V_{x\bar{x}y\bar{y}}(r) = \frac{\partial^4 V(r)}{\partial x^2 \partial y^2} + (2\chi_D(r) - 1)\Phi_3(r) + O(h_1 + h_2), \quad r \in \omega^{**},$$

where

$$\begin{aligned}\Phi_3(r) = &\left\{\operatorname{sign}(x^* - x)\frac{(h_1 - |x^* - x|)_+^3}{6h_1^2}\left[\frac{\partial^3 V}{\partial x^3}\right](r^*)\right\}_{y\bar{y}} \\ &+ \left\{\frac{(h_1 - |x^* - x|)_+^4}{24h_1^2}\left[\frac{\partial^4 V}{\partial x^4}\right](r^*)\right\}_{y\bar{y}} \\ &+ \frac{h_1^2}{12h_2^2}(h_2 - |y - y_*|)_+^0\left[\frac{\partial^4 V}{\partial x^4}\right](r_*) \\ &+ \operatorname{sign}(y_* - y)\frac{(h_2 - |y - y_*|)_+^1}{h_2^2}\left[\frac{\partial^3 V}{\partial x^2 \partial y}\right](r_*) \\ &+ \frac{(h_2 - |y - y_*|)_+^2}{2h_2^2}\left[\frac{\partial^4 V}{\partial x^2 \partial y^2}\right](r_*)\end{aligned}$$

Let us note that the truncated power notation allows us to apply the above definitions of $\Phi_i(r)$, $i = 1, 2, 3$, not only for the case demonstrated in Figure 2, but also when $\Sigma$ intersects $\overline{P}(r)$ in another way. The same definition can be used also for $r \in \omega \backslash \omega^{**}$, when obviously $\Phi_i(r) = 0$. By Lemma 5, the jumps in the third and fourth derivatives of $V(r)$ are explicitly evaluated through the density function $g(r)$ and the parametrization of $\Sigma$. Hence the functions $\Phi_i(r)$, $i = 1, 2, 3$, are determined in terms of $g(r)$, the parametrization of $\Sigma$ and some distances from given points to the curve $\Sigma$.

We approximate problem (2.9) and (2.10) by the following difference problem for the mesh function $V_h$

(2.11)
$$\Delta_h^2 V_h(r) + \beta^2 V_h(r) = (2\chi_D(r) - 1)(\Phi_1(r) + \Phi_2(r) + 2\Phi_3(r)), \quad r \in \omega,$$

(2.12)
$$V_h(r) = 0, \quad r \in \gamma, \qquad V_{h,y\bar{y}}(r) = 0, \quad r \in \gamma_1, \qquad V_{h,x\bar{x}}(r) = 0, \quad r \in \gamma_2.$$

Note that $V_h$ is determined not only at the points of $\omega \cup \gamma$, but also at all mesh points lying at distance $h_1$ or $h_2$ from $\partial\Omega$ outside $\Omega$. The number of unknown values of $V_h(r)$ coincides with the number of equations in (2.11) and (2.12), and the problem (2.11) and (2.12) is well posed.

The convergence analysis of (2.11) and (2.12) is carried out by using the framework of finite difference schemes methods.

**Theorem 3.** *Suppose $V(r)$ is a potential defined by* (1.10) *which is a $C^6$ smooth function relative to $\Sigma$. Assume $V_h(r)$ is the solution to the problem* (2.11) *and* (2.12). *Then there exist positive constants $M_i$, $i = 1, 2, 3$, depending only on $\Sigma$, such that*

$$|V - V_h|_{4,\omega} \le M_1 h^{1.5} |V|_{C^6(D)\cap C^6(D_1)},$$

$$|V - V_h|_{3,\omega} \le M_2 h^2 (\log h^{-1})^{0.5} |V|_{C^6(D)\cap C^6(D_1)},$$

$$|V - V_h|_{2,\omega} \le M_3 h^2 |V|_{C^6(D)\cap C^6(D_1)}.$$

*Proof.* We prove Theorem 3 in three steps.

*Step* 1. The error $z(r) = V(r) - V_h(r)$ satisfies a problem of type (2.11) and (2.12) with a right-hand side given by $\psi(r) = \Delta_h^2 V(r) + \beta^2 V(r) - (2\chi_D(r) - 1)(\Phi_1(r) + \Phi_2(r) + 2\Phi_3(r))$. The local representations of $V_{\bar{x}\bar{x}xx}, V_{yy\bar{y}\bar{y}}$ and $V_{x\bar{x}y\bar{y}}$ obtained above give $\psi(r) = O(h_1 + h_2)$ for $r \in \omega^{**}$. We have $\psi(r) = O(h_1^2 + h_2^2)$ for $r \in \omega \backslash \omega^{**}$.

*Step* 2. Regroup the terms in the scalar products $(\Delta_h^2 z + \beta^2 z, \Delta_h^2 z + \beta^2 z)$, $(\Delta_h^2 z + \beta^2 z, \Delta_h z)$ and $(\Delta_h^2 z + \beta^2 z, z)$. Then we obtain the following a priori estimates

$$|z|_{4,\omega}^2 + 2\beta^2 |z|_{2,\omega}^2 + \beta^4 |z|_{0,\omega}^2 \le |\psi|_{0,\omega}^2, \tag{2.13}$$

$$|z|_{3,\omega}^2 + \beta^2 |z|_{1,\omega}^2 \le |(\Delta_h z, \psi)|, \tag{2.14}$$

$$|z|_{2,\omega}^2 + \beta^2 |z|_{0,\omega}^2 \le |(z, \psi)|. \tag{2.15}$$

Consider (2.14). The estimate

$$\max_{r\in\omega} |u(r)| \le M (\log h^{-1})^{0.5} |u|_{1,\omega}$$

is valid for any mesh function $u(r)$ vanishing on $\gamma$. Applying the above estimate for $u = \Delta_h z$ and combining with (2.14), we get

$$|z|_{3,\omega}^2 \le (1, |\psi|) \max_{r\in\omega} |\Delta_h z| \le M(1, |\psi|)(\log h^{-1})^{0.5} |z|_{3,\omega}.$$

Finally, we obtain

$$|z|_{3,\omega} \le M(1, |\psi|)(\log h^{-1})^{0.5}. \tag{2.16}$$

Using (2.8), we derive in a similar way from (2.15) that

$$|z|_{2,\omega} \le M(1, |\psi|). \tag{2.17}$$

*Step* 3. We substitute the formulas for $\psi(r)$ obtained in Step 1 into $|\psi|_{0,\omega}$ and $(1, |\psi|)$. Since the number of points in $\omega^{**}$ is $O(N_1+N_2)$, we conclude that $|\psi|_{0,\omega} = O(h^{1.5})$ and $(1, |\psi|) = O(h_1^2 + h_2^2)$. Using these estimates in (2.13), (2.16) and (2.17), we complete the proof. □

The presented method for evaluation of potential $V(r)$ leads to the difference problem (2.11) and (2.12). The determination of $\Phi_i(r)$, $i = 1, 2, 3$, at $O(N_1+N_2)$ points of $\omega^{**}$ requires $O(N_1+N_2)$ operations. The solution $V_h(r)$ of (2.11) and (2.12) can be found by using fast algorithms. For example, if we set $u_h = \Delta_h V_h$, problem (2.11) and (2.12) can be treated as a couple of second order finite difference schemes for $V_h(r)$ and $u_h(r)$ with zero boundary conditions in a rectangular domain. Each of the two schemes can be solved using FFT. Therefore, the total number of arithmetic operations is $O((N_1^2 + N_2^2)\log(N_1N_2))$.

**4. Evaluation of a potential with kernel $\Delta G(r, s)$.** Consider the potential $W(r)$ defined by (1.11). Using a procedure similar to the previous cases we can find $W(r)$ as a solution to the BVP

$$\Delta W(r) = -g(r)\delta_\Sigma(r) - \beta^2 \int_\Sigma G(r, s)g(s)\, ds, \quad r \in \Omega, \tag{2.18}$$

$$W(r) = 0, \quad r \in \partial\Omega. \tag{2.19}$$

Let the potential $W(r)$, $r \in \Omega$, be a four times continuously differentiable function relative to the curve $\Sigma$. Conditions $g \in H^{3+\varepsilon}(\Sigma)$ and $\Sigma \in H^{4+\varepsilon}[0, 2\pi] \times H^{4+\varepsilon}[0, 2\pi]$, $\varepsilon > 0$, provides the smoothness required.

Since the potential $V_1(r)$, defined by the kernel $G(r, s)$ and the same density $g(r)$, satisfies the assumptions of Theorem 3, it can be approximated as in Section 3 of Part 2 by the mesh function $V_{1,h}(r)$ with an error bound

$$|V_1 - V_{1,h}|_{2,\omega} \leq M_4(h_1^2 + h_2^2)|V_1|_{C^6(D)\cap C^6(D_1)}. \tag{2.20}$$

At the points $r \in \omega^*$ with the same function $\Phi(r)$ we write, similarly to (2.3),

$$\begin{aligned}\Delta_h W(r) &= \Delta W(r) + (2\chi_D(r) - 1)\Phi(r) + O(h_1^2 + h_2^2) \\ &= -\beta^2 V_1(r) + (2\chi_D(r) - 1)\Phi(r) + O(h_1^2 + h_2^2).\end{aligned}$$

Approximate (2.15) and (2.16) by the finite difference scheme

$$\Delta_h W_h(r) = -\beta^2 V_{1,h}(r) + (2\chi_D(r) - 1)\Phi(r), \quad r \in \omega, \tag{2.21}$$

$$W_h(r) = 0, \quad r \in \gamma. \tag{2.22}$$

**Theorem 4.** *Suppose the potential $W(r)$ defined by* (1.11) *is a four times continuously differentiable function relative to the curve $\Sigma$. Let $W_h(r)$ be the solution to the difference scheme (2.21) and (2.22). Then there exists a positive constant $M_5$ such that*

$$|W - W_h|_{2,\omega} \leq M_5(h_1^2 + h_2^2)|W|_{C^4(D)\cap C^4(D_1)}.$$

The arguments given in the proof of Theorem 1 combined with inequality (2.20) yield the estimate of Theorem 4.

Numerical computation of the solution $W_h(r)$ to (2.21) and (2.22) can be performed in three steps. First we evaluate an approximation $V_{1,h}(r)$ to the potential $V_1(r)$ with kernel $G(r, s)$ and the same density $g(r)$ at all points of the mesh $\omega$. This part was described in Section 3 of Part 2. Second we evaluate $\Phi(r)$ at the $O(N_1 + N_2)$ points of $\omega^*$. In this way we obtain the right-hand side terms of (2.21). Third we solve (2.21) and (2.22) using any fast solver (e.g., FFT). The amount of arithmetic operations is $O((N_1^2 + N_2^2)\log(N_1 N_2))$ at the first and third step and $O(N_1 + N_2)$ at the second step. Hence, the total number of operations required is approximately equal to the number of mesh values (up to a logarithmic factor).

The methods considered in Sections 3 and 4 of Part 2 do not require the evaluation of Green's function.

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