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A FREDHOLM EQUATION FOR THE HANKEL SINGULAR VALUES OF SYSTEMS WITH DISTRIBUTED INPUT DELAYS

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ABSTRACT. We study a system with continuous input delays (no state delay). We show that its Hankel operator is compact and that the singular values are the square roots of the positive eigenvalues of a Fredholm integral operator.

1. Introduction. Hankel norm and Hankel singular values of distributed systems have been studied by several authors in recent times [**2**]. Some of these papers deal specifically with delay systems ([**3, 1, 10, 11**] for example) usually assuming *discrete* delays. Very few authors considered specifically systems with distributed delays (see [**11**] where a transfer function of a system with distributed delays—and no pole—was studied). Actually, a system with only distributed delays admits a state space representation which is quite simple, and this suggests that we consider this abstract state space representation—described below—as a starting point for formulas which can lead to an alternative characterization of singular values. A consequence of this fact is that the operator $\mathcal{P}\mathcal{Q}$ introduced below is compact so that $\sigma(\mathcal{P}\mathcal{Q})/\{0\} = \sigma_p(\mathcal{P}\mathcal{Q})/\{0\}$ and the singular values can only accumulate at zero.

In this paper we study the control system

$$(1)\qquad \dot{x} = Ax + \int_{-\tau}^{0} B(s)u(t+s)\,ds + B_0u(t) \qquad y = Cx(t).$$

The letters x, y, u denote respectively n-, p- and m-vectors and the matrices have suitable dimensions. The entries of the matrix $B(\cdot)$ are square integrable functions while the remaining matrices are constant.

We assume that A is a stable matrix, i.e., that $\sigma(A) \subseteq \{z, \operatorname{Re} z < 0\}$ so that it is possible to define the Hankel operator $\Gamma : L^2(0, +\infty) \to$

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$L^2(0, +\infty)$:

$$(2) \qquad (\Gamma u)(t) = C \int_0^{+\infty} e^{A(t+s)} \left\{ B_0 u(s) + \int_{-\tau}^{0} B(r) u(s+r)\, dr \right\} ds.$$

Here we intend that

$$u(t) = 0 \quad \text{if } t < 0.$$

The *singular values* of the operator Γ, i.e., the square roots of the *positive* eigenvalues σ_i^2 of $\Gamma^*\Gamma$, have a great importance in the theory of linear system [**5**], in particular, in model reduction, and the largest of them, σ_0, is related to the model matching problem: $\inf\{||T(z) - Q(z)||_\infty, Q \text{ antistable matrix}\} = \sigma_0$. See [**4**] for references and applications.

In this paper we show that the singular values are the square roots of the eigenvalues of a certain Fredholm operator which can easily be constructed from the coefficients of the equation (1). This reduces the computation of the singular values to a standard problem in numerical analysis.

2. Abstract representation of systems with distributed delays. An abstract state space representation of system (1) was first proposed in [**13**]. It is based on the following simple observation: the solution to the problem

$$\frac{d}{dt} V(t, \vartheta) = -\frac{d}{d\vartheta} V(t, \vartheta) + B(\vartheta) u(t), \quad -\tau < \vartheta < 0,$$
$$V(t, -\tau) = 0, \qquad V(0, \vartheta) = \eta(\vartheta)$$

is:

$$(3) \qquad V(t, \vartheta) = \begin{cases} \eta(\vartheta - t) & \text{for } -\tau < \vartheta - t < 0 \\ 0 & \text{otherwise} \end{cases} + \int_{\max(\vartheta - t, -\tau)}^{\vartheta} B(r) u(r + t - \vartheta)\, dr.$$

If $\eta(\cdot) = 0$ and $t > \tau$, then we get $V(t, 0) = \int_{-\tau}^{0} B(r) u(t + r)\, dr$. This observation suggests the following framework: the state space is $M^2 = \mathbf{R}^n \times L^2(-1, 0; \mathbf{R}^n)$ and $\mathcal{A}$ is the operator:

$$\operatorname{dom}(\mathcal{A}) = \{\operatorname{col}(h_0, h(\cdot)) \in M^2, h(\cdot) \in W^{1,2},\ h(-\tau) = 0\},$$

$$\mathcal{A}\begin{pmatrix} h_0 \\ h(\cdot) \end{pmatrix} = \begin{pmatrix} Ah_0 + h(0) \\ -\frac{d}{d\vartheta}h(\cdot) \end{pmatrix}. \tag{4}$$

If $\mathcal{B} : \mathbf{R}^m \to M^2$ is given by: $\mathcal{B}u = \operatorname{col}(B_0u, B(\cdot)u)$, an abstract model for system (1) is:

$$\frac{d}{dt}X(t) = \mathcal{A}X(t) + \mathcal{B}u(t), \qquad y(t) = \mathcal{C}X(t). \tag{5}$$

Here, X denotes a vector of M^2, $X(t) = \operatorname{col}(x(t), x(t,\cdot))$ and $\mathcal{C}$ is the operator defined by $\mathcal{C}X = Cx$.

The precise relation between the solutions of (1) and (5) is described as follows (see [**13**]). Let us define the operator $\mathcal{M} : M^2 \to M^2$:

$$\mathcal{M}\begin{pmatrix} x_0 \\ v(\cdot) \end{pmatrix} = \begin{pmatrix} x_0 \\ \int_{-\tau}^{\vartheta} B(s)v(s-\vartheta)\,ds \end{pmatrix}.$$

Moreover, let $u_t(\cdot)$ be the segment of control $u_t(\vartheta) = u(t+\vartheta)$, $-\tau < \vartheta < 0$. Then,

Theorem 1. *If $x(\cdot)$ solves* (1) *with $x(0) = x_0$, $u(\vartheta) = \eta(\vartheta)$ for $-\tau < \vartheta < 0$ and $X(\cdot)$ is the mild solution to* (5) *with initial datum $\mathcal{M}(x_0, \eta(\cdot)) = (x_0, \int_{-\tau}^{\vartheta} B(s)\eta(s-\vartheta)\,ds)$, then $X(t) = \mathcal{M}(x(t), u_t(\cdot))$.*

Let $x_0 = 0$, $\eta(\cdot) = 0$. Then, from (3) and (4), the first line of Equation (5) takes the form $\dot{x}(t) = Ax(t) + B_0u(t) + \int_{\max(-t,-\tau)}^{0} B(r)u(t+r)\,dr$. Consequently,

Theorem 2. *The Hankel operator for system* (5):

$$(\Gamma u)(t) = \mathcal{C}\int_0^{+\infty} \mathcal{E}(t+s)\mathcal{B}u(s)\,ds \tag{6}$$

(*where $\mathcal{E}(t)$ is the semigroup—indeed a group—generated by $\mathcal{A}$*) *is the same operator as the Hankel operator of system* (1).

Remark 1. i) A more general output operator can be associated with system (5) since a linear bounded operator on M^2 has the general form:

$$\hat{\mathcal{C}}\begin{pmatrix} x^0 \\ v(\cdot) \end{pmatrix} = Cx^0 + \int_{-\tau}^{0} C(s)v(s)\,ds.$$

But, the corresponding output operator for system (1) looks quite artificial; the integrated term gives:

$$\int_{-\tau}^{0} C(\vartheta)\left\{ \int_{\max(\vartheta-t,-h)}^{\vartheta} B(r)u(r+t-\vartheta)\,dr \right\} d\vartheta.$$

For this reason we consider the simpler output described by the operator $\mathcal{C}$.

ii) It is proved in [**8**] that a system with both input and output delays can always be represented as a system with only input delays.

The operator $v(\cdot) \to \phi(\cdot)$, $\phi(t) = \mathcal{C}\int_0^{+\infty} \mathcal{E}(t+s)v(s)\,ds$ is continuous as an operator from $L^2(0,+\infty)$ to $L^2(0,+\infty)$ since $\mathcal{E}(\cdot)$ decays exponentially. Moreover, $\operatorname{Im}\mathcal{B}$ is finite dimensional. Hence,

Theorem 3. *The operator Γ is compact.*

An important fact, proved in [**5**], also holds for distributed systems: a number σ is a singular value of the Hankel operator Γ of system (5) if and only if $\sigma^2 \in \sigma_p(\mathcal{P}\mathcal{Q})$ where $\mathcal{P}, \mathcal{Q}$ are the symmetric positive solutions to the Lyapunov equations

$$\mathcal{A}^*\mathcal{P}x + \mathcal{P}\mathcal{A}x = -\mathcal{C}^*\mathcal{C}x \quad \forall x \in \operatorname{dom}\mathcal{A} \tag{7}$$

$$\mathcal{A}\mathcal{Q}x + \mathcal{Q}\mathcal{A}^*x = -\mathcal{B}\mathcal{B}^*x \quad \forall x \in \operatorname{dom}\mathcal{A}^* \tag{8}$$

(it is implicitly required that $\mathcal{P}x \in \operatorname{dom}\mathcal{A}^*$ for all $x \in \operatorname{dom}\mathcal{A}$ and $\mathcal{Q}x \in \operatorname{dom}\mathcal{A}$ for all $x \in \operatorname{dom}\mathcal{A}^*$). See [**5**] for the proof which goes exactly as for finite dimensional systems (the notations $\mathcal{P}, \mathcal{Q}$ are interchanged with respect to those in [**5**]. But, of course, the nonzero eigenvalues of $\mathcal{P}\mathcal{Q}$ and of $\mathcal{Q}\mathcal{P}$ coincide).

The semigroup $\mathcal{E}$ of the free evolution of system (1) is exponentially stable since A is a stable matrix. Hence, both the previous Lyapunov equations *admit unique solutions* and $\mathcal{P} = \mathcal{P}^* \geq 0$, $\mathcal{Q} = \mathcal{Q}^* \geq 0$. We shall see in the next section that these solutions are easily represented.

The number σ^2 is an eigenvalue of $\mathcal{P}\mathcal{Q}$ if for some nonzero vector $V \in M^2$ we have: $\mathcal{P}\mathcal{Q}V = \sigma^2 V$ so that we must find nonzero solutions to the equation:

$$\mathcal{P}W = \sigma V \qquad \mathcal{Q}V = \sigma W. \tag{9}$$

We shall see that the system of equations (9) can be reduced to the solution of a system of Fredholm integral equations of the second kind.

Remark 2. If $\sigma \in \sigma_p(\mathcal{P}\mathcal{Q})$, then σ is nonnegative.

3. Solution of the Lyapunov equations. The elements of M^2 will be represented both by upper case letters and by column vectors as: $H = \operatorname{col}(h_0, h(\cdot))$. The notation $h(\cdot)$ is used for a function on $[-\tau, 0]$ whose regularity, if not explicitly declared, will be clear from the context. It is easily seen that $\mathcal{C}^*\mathcal{C}H = \operatorname{col}(C^*Ch_0, 0)$. Let us represent $\mathcal{P}$ in block form

$$\mathcal{P} = \begin{pmatrix} P_0 & P_1 \\ P_1^* & \Pi \end{pmatrix}$$

(as an operator from M^2 to itself, M^2 being a product space). The entry P_0 is an $n \times n$ matrix while P_1 takes the form $P_1(h(\cdot)) = \int_{-\tau}^0 P_1(s)h(s)\,ds$ for each $h(\cdot) \in L^2(0, +\infty)$ so that $(P_1^* h_0)(\vartheta) = P_1^*(\vartheta)h_0$.

We recall that if $\operatorname{col}(h_0, h(\cdot)) \in \operatorname{dom}\mathcal{A}$, then $\mathcal{P}(h_0, h(\cdot)) \in \operatorname{dom}\mathcal{A}^*$. In particular, the second component is in $W^{1,2}$ if $h(\cdot) \in W^{1,2}$ with $h(-\tau) = 0$ since:

$$\operatorname{dom}\mathcal{A}^* = \left\{ \begin{pmatrix} h_0 \\ h(\cdot) \end{pmatrix}, h(\cdot) \in W^{1,2}, h_0 = h(0) \right\} \tag{10}$$

and

$$\mathcal{A}^* \begin{pmatrix} h_0 \\ h(\cdot) \end{pmatrix} = \begin{pmatrix} A^* h_0 \\ (d/d\vartheta)h(\cdot) \end{pmatrix}. \tag{11}$$

In particular, $\vartheta \to [\Pi h(\cdot)](\vartheta)$ is a continuous function. This suggest the following form for the operator $\Pi : [\Pi h(\cdot)](\vartheta) = \int_{-\tau}^0 \Pi(\vartheta, s)h(s)\,ds$ (in fact, the operator Π must be continuously extendable to L^2 so that it cannot depend on $\dot{h}(\cdot)$. Moreover, the distributional derivative of $\vartheta \to \int_{-\tau}^0 \Pi(\vartheta, s)h(s)\,ds$ should exist in L^2). If the functions $(\vartheta, s) \to \Pi(\vartheta, s)$, $\vartheta \to P(\vartheta)$ and the matrix P_0 can be determined, then they identify the unique solution $\mathcal{P}$ to the Lyapunov equation (7).

As $\mathcal{P}$ is self-adjoint, the operator Π must be self-adjoint and this means that:

$$\Pi^*(s,\vartheta) = \Pi(\vartheta, s). \tag{12}$$

Moreover, the condition that $\mathcal{P}\mathrm{dom}\,\mathcal{A} \subseteq \mathrm{dom}\,\mathcal{A}^*$ implies that:

$$P_0 h_0 + \int_{-\tau}^0 P_1(s)h(s)\,ds = P_1^*(0)h_0 + \int_{-\tau}^0 \Pi(0,s)h(s)\,ds$$

so that

$$P_1^*(0) = P_0, \qquad \Pi(0,s) = P(s). \tag{13}$$

It is now easy to decouple equation (7) into the three equations:

$$A^*P_0 + P_0 A = -C^*C \tag{14}$$

$$\frac{d}{d\vartheta}P^*(\vartheta) + P^*(\vartheta)A = 0 \tag{15}$$

$$\frac{d}{d\vartheta}\Pi(\vartheta,s) + \frac{d}{ds}\Pi(\vartheta,s) = 0. \tag{16}$$

Equation (14) is a finite dimensional Lyapunov equation. It can be solved with algebraic methods and we assume that its solution, analytically given by $P_0 = \int_0^{+\infty} e^{A^*t}C^*Ce^{At}\,dt$, is known. In term of this matrix, we have:

$$P(\vartheta) = e^{-A^*\vartheta}P_0, \qquad \Pi(\vartheta,s) = \begin{cases} P(s-\vartheta) & -\tau \le s \le \vartheta \le 0 \\ P^*(\vartheta - s) & -\tau \le \vartheta \le s \le 0. \end{cases} \tag{17}$$

We note that $(\vartheta,s) \to \Pi(\vartheta,s)$ is a continuous function, $\Pi(\vartheta,\vartheta) = P_0$.

The second Lyapunov equation is a bit more involved since the operator on the right hand side is:

$$\mathcal{B}\mathcal{B}^*\begin{pmatrix} h_0 \\ h(\cdot)\end{pmatrix} = \begin{pmatrix} B_0B_0^*h_0 + \int_{-\tau}^0 B_0B^*(s)h(s)\,ds \\ B(\vartheta)B_0^*h_0 + \int_{-\tau}^0 B(\vartheta)B^*(s)h(s)\,ds \end{pmatrix}. \tag{18}$$

The operator $\mathcal{Q} : \mathrm{dom}\,\mathcal{A}^* \to \mathrm{dom}\,\mathcal{A}$ takes the form:

$$\mathcal{Q}\begin{pmatrix} h_0 \\ h(\cdot)\end{pmatrix} = \begin{pmatrix} Q_0h_0 + \int_{-\tau}^0 Q(s)\,ds \\ Q^*(\vartheta)h_0 + (\Omega h(\cdot))(\vartheta)\end{pmatrix}, \quad h_0 = h(0). \tag{19}$$

Arguments very much like the previous ones suggest

$$(20) \qquad (\Omega h(\cdot))(\vartheta) = \int_{-\tau}^{0} \Omega(\vartheta, s) h(s)\, ds, \qquad \Omega^*(\vartheta, s) = \Omega(s, \vartheta).$$

As (8) admits a *unique* solution, this is found if we can calculate $Q_0, Q(\cdot), \Omega(\cdot,\cdot)$. Now it is easily seen that the second Lyapunov equation (8) is solved if we can solve

$$(21) \qquad \frac{d}{d\vartheta}\Omega(\vartheta, s) + \frac{d}{ds}\Omega(\vartheta, s) = B(\vartheta)B^*(s) \quad \Omega(\vartheta, -\tau) = 0$$

$$(22) \qquad \frac{d}{ds}Q(s) = AQ(s) + \Omega(0, s) + B_0 B^*(s) \quad Q(-\tau) = 0$$

$$(23) \qquad AQ_0 + Q_0 A^* = -\{B_0 B_0^* - Q(0) - Q^*(0)\}.$$

This system has the unique solution

$$(24) \qquad \Omega(\vartheta, s) = \begin{cases} \int_{-\tau}^{s} B(\vartheta - s + r) B^*(r)\, dr & -\tau \le s \le \vartheta \le 0 \\ \int_{-\tau}^{\vartheta} B(r) B^*(s - \vartheta + r)\, dr & -\tau \le \vartheta \le s \le 0 \end{cases}$$

and

$$(25) \quad Q(\vartheta) = \int_{-\tau}^{\vartheta} e^{A(\vartheta - s)} \left\{ B_0 B^*(s) - \int_{-\tau}^{s} B(r - s) B^*(r)\, dr \right\} ds$$

$$(26) \qquad Q_0 = \int_{0}^{+\infty} e^{As} \{B_0 B_0^* + Q(0) + Q(0)^*\} e^{A^* s}\, ds$$

We note that the function $\Omega(\cdot,\cdot)$ given by (24) satisfies equation (21) only in the weak sense: it is the mean square limit of sequences $\Omega_n(\cdot,\cdot)$ which solves (21) with a differentiable function $B_n(\cdot)$ instead then $B(\cdot), B_n(\cdot) \to B(\cdot)$ in the L^2 sense. In spite of this,

Theorem 4. *The function $(\vartheta, s) \to \Omega(\vartheta, s)$ is continuous.*

Proof. In fact, let $h(\vartheta, s) = \int_{-\tau}^{s} B(\vartheta - s + r) B^*(r)\, dr$ for $-\tau \le s \le \vartheta \le 0$ (i.e., $h(\vartheta, s) = \Omega(\vartheta, s)$ on the specified set). We note that

$$(27) \quad h(\vartheta, s) - h(t, \xi)$$
$$= \int_{\xi}^{s} B(t - \xi + r) B^*(r)\, dr + \int_{-\tau}^{s} [B(\vartheta - s + r) - B(t - \xi + r)] B^*(r)\, dr.$$

The first integral is minor than the square root of $(\int_{-\tau}^{0} ||B(r)||^2\, dr)$ $(\int_{\xi}^{s} ||B(r)||^2\, dr)$.

This quantity tends to zero for $s - \xi \to 0$ uniformly with respect to t.

The second integral gives:

$$\left\| \int_{-\tau}^{s} [B(\vartheta - s + r) - B(t - \xi + r)] B^*(r)\, dr \right\|^2 \\ \leq \int_{-\tau}^{0} ||B(\vartheta - s + r) - B(t - \xi + r)||^2\, dr \int_{-\tau}^{0} ||B(r)||^2\, dr. \tag{28}$$

Continuity follows since the shift operator is continuous on L^2.

Consequently, $(\vartheta, s) \to \Omega(\vartheta, s)$ is continuous on $-\tau \leq s \leq \vartheta \leq 0$. An analogous argument proves continuity also on $-\tau \leq \vartheta \leq s \leq 0$. □

At this point we must characterize those nonzero numbers σ such that there exist vectors V, W in M^2 (not both zero) such that (9) is satisfied.

4. Fredholm integral equation for the singular values. Equation (9) is an equation on M^2: a number σ such that (9) admits a nonzero solution is an eigenvalue of the operator

$$\begin{pmatrix} 0 & \mathcal{Q} \\ \mathcal{P} & 0 \end{pmatrix}.$$

It is easy to see that this is a compact operator so that its spectrum is finite, or it is the image of a sequence of eigenvalues which converges to zero plus the point zero.

Actually, (9) can be decoupled as follows:

$$\sigma v_0 = P_0 w_0 + \int_{-\tau}^{0} P(s) w(s)\, ds \tag{29}$$

$$\sigma w_0 = Q_0 v_0 + \int_{-\tau}^{0} Q(s) v(s)\, ds \tag{30}$$

$$\sigma v(\vartheta) = P^*(\vartheta) w_0 + \int_{-\tau}^{0} \Pi(\vartheta, s) w(s)\, ds \tag{31}$$

$$\sigma w(\vartheta) = Q^*(\vartheta) v_0 + \int_{-\tau}^{0} \Omega(\vartheta, s) v(s)\, ds. \tag{32}$$

The first equation states that $v(0) = v_0$, $v(\cdot)$ given by (31). This is already known since the image of $\mathcal{P}$ must be contained in $\operatorname{dom}\mathcal{A}^*$.

The previous equations must be solved for σ *different from zero*. Let us forget equation (29) for the moment. Substitution of equations (30) and (32) into (31) gives that:

Proposition 5. *The number $\sigma > 0$ is a singular value if and only if there exists a nonzero function $v(\cdot)$ which solves*:

$$\begin{aligned}\sigma^2 v(\vartheta) = &\left\{P^*(\vartheta)Q_0 + \int_{-\tau}^0 \Pi(\vartheta,s)Q^*(s)\,ds\right\}v_0 \\ &+ \int_{-\tau}^0 \left\{P^*(\vartheta)Q(s) + \int_{-\tau}^0 \Pi(\vartheta,\rho)\Omega(\rho,s)\,d\rho\right\}v(s)\,ds\end{aligned} \tag{33}$$

and: $v(0) = v_0$.

Equation (33) is a Fredholm equation of the second kind of a quite special form:

$$\sigma^2 v(\vartheta) = J(\vartheta)v_0 + \int_{-\tau}^0 K(\vartheta,s)v(s)\,ds \quad v(0) = v_0. \tag{34}$$

It can be considered as a Fredholm equation whose kernel contains a Dirac's delta function. The usual theory of Fredholm equations extends to this case ([**7**, pg. 286–287]). Moreover,

Theorem 6. *The functions $J(\cdot)$ and $K(\cdot,\cdot)$ which appear in* (34) (*as defined by comparison with* (33)) *are continuous functions.*

Proof. Continuity of the functions $P(\cdot), Q(\cdot), \Pi(\cdot,\cdot)$ is seen directly from the definitions. Continuity of the function $\Omega(\cdot,\cdot)$ is proved in Theorem 4. $\square$

A very special case is when the system has no lag: $B(\cdot) = 0$. In this case $\Omega(\cdot,\cdot) = 0$ and also $Q(\cdot)$ is zero so that $\sigma^2 v(\vartheta) = P^*(\vartheta)Q_0 v_0 = e^{-A^*\vartheta}P_0 Q_0 v_0$. We see that σ^2 is an eigenvalue of P_0Q_0 by taking $\vartheta = 0$. This is known from finite dimensional theory [**5**].

In the special case that the kernel is degenerate, it is easy to find an algebraic characterization of the singular values, whose number is now finite. This characterization can also be used to approximate the singular values in general, since any continuous kernel can be approximated with degenerate kernels (see [**7**]). By definition, the kernel $K(\vartheta, s) = \{P^*(\vartheta)Q(s) + \int_{-\tau}^0 \Pi(\vartheta,\rho)\Omega(\rho,s)\,d\rho\}$ is degenerate when

$$\begin{aligned} \int_{-\tau}^0 \Pi(\vartheta,\rho)\Omega(\rho,s)\,d\rho &= \sum_{i=1}^r L_i(\vartheta)\Lambda_i(s) \\ \text{i.e., } K(\vartheta,s) &= \sum_{i=1}^N H_i(\vartheta)K_i(s). \end{aligned} \tag{35}$$

Let $x_i = \int_{-\tau}^0 K_i(s)v(s)\,ds$. We have:

$$\sigma^2 v(\vartheta) = J(\vartheta)v_0 + \sum_{i=1}^N H_i(\vartheta)x_i \tag{36}$$

and x_r satisfies:

$$\begin{aligned} \sigma^2 x_r &= \int_{-\tau}^0 K_r(\vartheta)J(\vartheta)\,d\vartheta v_0 + \sum_{i=1}^N k_{r,i}x_i, \\ k_{r,i} &= \int_{-\tau}^0 K_r(\vartheta)H_i(\vartheta)\,d\vartheta. \end{aligned} \tag{37}$$

Let $\mathcal{K}$ be the block matrix $\mathcal{K} = (k_{r,i})$ and $\mathcal{J}$ be the block matrix $\mathcal{J} = \mathrm{col}\,(\int_{-\tau}^0 K_r(\vartheta)J(\vartheta)\,d\vartheta)$ so that

$$(\sigma^2 I - \mathcal{K})\mathrm{col}\,(x_r) = \mathcal{J}v_0. \tag{38}$$

Theorem 7. *Let us assume that the functions $H_i(\cdot)$ are continuous. Then,*

i) *If $\det(\sigma^2 I - \mathcal{K})) \neq 0$, then $\sigma \neq 0$ is a Hankel singular value if and only if*

$$\det(\sigma^2 I - J(0) - [H_1(0), \dots, H_N(0)](\sigma^2 I - \mathcal{K})^{-1}\mathcal{J}) = 0; \tag{39}$$

ii) *If* $\det(\sigma^2 I - \mathcal{K}) = 0$*, then* $\sigma \neq 0$ *is a Hankel singular value if and only if there exist vectors* $v_0 \neq 0$ *and* ξ_0 *such that*

$$(\sigma^2 I - J(0) - [H_1(0), \ldots, H_N(0)]\mathcal{R}\mathcal{J})v_0 = 0 \qquad Jv_0 = (\sigma^2 I - \mathcal{K})\xi_0. \tag{40}$$

Here, $\mathcal{R}$ *is the inverse of* $(\sigma^2 I - \mathcal{K})$ *from* $\{\mathrm{Ker}\,(\sigma^2 I - \mathcal{K})\}^{\perp}$*.*

Proof. If $\det(\sigma^2 I - \mathcal{K}) \neq 0$, then (39) is obtained from (40) with $\xi_0 = (\sigma^2 I - \mathcal{K})^{-1} J$. So, we prove the second statement. We note that $\sigma^2 \neq 0$ is a singular value if and only if the function $v(\vartheta)$ defined by (36) and (38) satisfies $v(0) = v_0$. Equation (38) can be solved if $\mathrm{Im}\,\mathcal{J} \cap \mathrm{Im}\,(\sigma^2 I - \mathcal{K}) \neq \{0\}$, i.e., if and only if $Jv_0 = (\sigma^2 I - \mathcal{K})\xi_0$ for some ξ_0. In this case, $\mathrm{col}\,(x_r) = \mathcal{R}(\mathcal{J}v_0)$ so that

$$v(\vartheta) = (1/\sigma^2)\{J(\vartheta)v_0 + [H_1(\vartheta), \ldots, H_N(\vartheta)]\mathcal{R}\mathcal{J}v_0\}. \tag{41}$$

The condition $v(0) = v_0$ is equivalent to the first equality in (40) since the functions $J(\cdot), H_i(\cdot)$ are continuous. Of course, the function $v(\vartheta)$ given by (41) is not zero since in this case the right hand side would be zero in particular for $\vartheta = 0$ (because the functions $J(\cdot), H_i(\cdot)$ are continuous). This fact, (41) and the first equality in (40) would imply that $v_0 = 0$. $\square$

The assumption that the functions $H_i(\vartheta)$ are continuous is not restrictive.

Equality (41) in the previous proposition gives an explicit expression for $v(\vartheta)$ (and for $v_0 = v(0)$). This can be replaced in (30–32) to get both w_0 and $w(\cdot)$. A Smith pair for the singular value σ can be constructed from this.

The previous proposition takes in account a very special, and unusual, case. But it is well known that any continuous kernel can be approximated with degenerate kernels. Such approximation is the first step of a method due to Smith ([**7**] or [**12**]) for the solution of the original Fredholm equation for an unspecified value of v_0. The condition $v(0) = v_0$ must then be imposed. Alternatively, other methods, like Galerkin type approximations, can be invoked [**9**].

We recall now Theorem 6 according to which both the functions $J(\cdot)$ and the kernel $K(\cdot, \cdot)$ of the integral are continuous functions so that the search for the eigenvalues expressed by (34) can be seen as an eigenvalue problem over the space $C(-\tau,0)$ of the continuous functions on the interval $[-\tau,0]$. In this setting, both the integral operator and the operator $v(\cdot)\to J(\vartheta)v(0)$ which appear in (34) are continuous operators. Let I_n be a sequence of operators, $I_n\in\mathcal{L}(C(-\tau,0))$, which converges to the integral operator on the right hand side of (33). It may be a sequence of operators which are obtained by an approximation with degenerate kernels. Then, if σ_n^2 is an eigenvalue with the operator I_n replacing the integral operator in (34), and *if the sequence* $\{\sigma_n^2\}$ *is convergent to* α^2, then α^2 belongs to the spectrum of the original equation [**6**]. If α is not zero, it is a singular value.

We note that, the operators being bounded, it is not restrictive to assume that the sequence $\{\sigma_n^2\}$ is bounded.

A system which is, in principle, without delays, may in fact contains "small delays" due to small "transportation delays" between its different parts. These delays can be considered "small" or because τ is a small time interval (almost instantaneous transmission) or because the matrix function $B(\cdot)$ has a very small norm (weak coupling). The previous observation implies that in both these cases the "real system," which is affected by delays, has singular values which are close to those of the "nominal system" (the one without delays) plus, possibly, a sequence of singular values which are in a neighborhood of zero.

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