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A NYSTRÖM METHOD FOR BOUNDARY INTEGRAL EQUATIONS ON DOMAINS WITH A PIECEWISE SMOOTH BOUNDARY

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ABSTRACT. We consider the Dirichlet problem for Laplace's equation on planar domains with corners. A Nyström method is used to solve the corresponding double layer boundary integral equation. The Nyström method is based on the trapezoidal rule with a graded mesh method as in [**13**], but we adopt a generalized mesh grading. An improved convergence is observed.

1. Introduction. We investigate a Nyström method for the numerical solution of the double layer boundary integral equation of the second kind for the planar harmonic Dirichlet problem in domains with corners. For a smooth boundary which is at least twice continuously differentiable, a Nyström method with the trapezoidal rule can be applied; and a standard argument using the collectively compact operator theory in [**1**] gives us an easy error analysis. If the boundary curve is analytic, the double layer density solution is of the same class as the boundary function. The numerical solution converges exponentially when the boundary data is analytic.

For a domain with corners, the corresponding integral operator is no longer compact. But the integral operator can be expressed as the sum of a compact operator and a noncompact operator with norm less than 1, in suitable function spaces. Then we can use again an error analysis based on collectively compact operator theory, with some modification. Also, it is observed in Costabel and Stephan [**7**] and Grisvard [**9**] that the double layer density function may have a singularity of the type

$$x^{\alpha^*}, \qquad \alpha^* = \frac{\pi}{\pi + |\pi - \theta|}$$

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around corners with sufficiently smooth boundary data. Here x is the distance from the corner and θ is the interior angle at the corner.

Because of the singularity in the solution, the Galerkin and collocation methods with a uniform mesh show poor convergence. To cope with this poor convergence, Chandler and Graham [**6**] adopt a graded mesh method, both with the Galerkin and collocation methods. In addition, with the collocation methods, a slight modification around each corner may be needed to avoid instability in the approximating system.

Basically the concept of a graded mesh method is to make a solution sufficiently smooth through a change of variable such as $x = t^n$ for some positive integer n. For example, if the original solution is $\psi(x) = x^{\alpha^*}$ for some $0 < \alpha^* < 1$, the new solution with the above substitution becomes $\tilde{\psi}(t) = t^{n\alpha^*}$, which is sufficiently smooth if n is sufficiently large. For this new solution, we can develop a rapidly convergent approximating solution.

Because collocation and Galerkin methods are less practical in the sense that the evaluation of matrix elements is costly, there has been a demand for Nyström methods. Graham and Chandler [**8**] and Atkinson and Graham [**5**] use a Nyström method with a locally approximating quadrature method on each graded subinterval. With this method, the order of convergence depends on the quadrature method if sufficient mesh grading is used. However, because the integral equation we encounter in reality has an integrand that is usually smooth except at corner points, it is desirable to use a quadrature method based on a smooth global approximation over each smooth section of the boundary. Kress [**13**] adopts this idea and obtains successful numerical results. The convergence depends on the order of mesh grading. In [**5, 8, 13**], a slight modification around corner is needed in the approximating system to achieve stability. But no modification is necessary in the actual numerical implementation.

In this paper, we suggest a new Nyström method. Basically, our method follows the framework of Kress [**13**]. But we adopt a new quadrature method, one with an 'infinite order' mesh grading. Through this, we expect a convergence of infinite order; and this is shown in the numerical results. With this method, some cutoff is necessary in the numerical approximation to overcome the instability of our approximating system. The cutoff method used in this paper is similar to that used in [**5, 8**], rather than that of [**13**].

2. Numerical quadrature method. We first introduce a class of transformations,

$$w_{\alpha,\beta}(s) = \frac{\gamma_{\alpha,\beta}(v(s))}{\gamma_{\alpha,\beta}(v(s)) + \gamma_{\alpha,\beta}(1-v(s))} \tag{2.1}$$

where

$$\gamma_{\alpha,\beta}(s) = \begin{cases} (s^{\alpha})^{(1/s)^{\beta}} & \text{if } 0 < s \le 1 \\ 0 & \text{if } s = 0 \end{cases} \tag{2.2}$$

and

$$v(s) = \left(\frac{1}{2} - \frac{1}{p}\right)(2x-1)^3 + \frac{1}{p}(2x-1) + \frac{1}{2} \tag{2.3}$$

with

$$p = 2^{\beta}(\alpha + \beta \log 2) \ge 2. \tag{2.4}$$

Here α and β are positive real numbers which satisfy (2.4). (2.3) with (2.4) is considered to make mesh equally distributed over interval $[0,1]$, and it is easy to see that $w'_{\alpha,\beta}(1/2) = 2$. $w_{\alpha,\beta}(s)$ satisfies the following additional properties.

$$\text{(P1)} \quad \lim_{s\to 0, s\to 1} s^{-k_1}(1-s)^{-k_2}[w_{\alpha,\beta}(s)(1-w_{\alpha,\beta}(s))]^{\mu} = 0 \tag{2.5}$$

for any $k_1, k_2 > 0$ and $\mu > 0$.

(2.6)

$$\text{(P2)} \quad w^{(n)}_{\alpha,\beta}(s) = f_n(s)\frac{[\log(s) + \log(1-s)]^n}{[s(1-s)]^{(1+\beta)n}} w_{\alpha,\beta}(s)(1-w_{\alpha,\beta}(s)) \quad \text{for } n \ge 1$$

where $f_n(s)$ is continuous on $[0,1]$. Furthermore, f_1 is nonzero.

For the detailed properties and proofs, see [**11**]. In [**11**], (2.3) and (2.4) are not considered to define $w_{\alpha,\beta}$, but, because v is analytic and $|v'(s)| \ne 0$, we easily see that $w_{\alpha,\beta}$ defined in this paper satisfies (2.5) and (2.6) by simple generalization. Using (2.5), (2.6), it is easy to check that $w_{\alpha,\beta}^{(n)}(0) = w_{\alpha,\beta}^{(n)}(1) = 0$ for all $n \geq 1$. From here on, denote

$$\gamma(s) = \gamma_{\alpha,\beta}(s), \qquad w(s) = w_{\alpha,\beta}(s) \tag{2.7}$$

for simplicity of notation.

Now it is time to introduce a numerical quadrature method. Before doing that, we introduce a class of functions in which we are interested. Define

$$\mathcal{S}^{\gamma,q} = \{g \in C^q(0,1) | \int_0^1 [t(1-t)]^{j-\gamma} |g^{(j)}(t)|\, dt < \infty \text{ for } j = 0, \dots, q\} \tag{2.8}$$

and

$$\mathcal{S}^{\gamma,\infty} = \bigcap_{q=1}^{\infty} \mathcal{S}^{\gamma,q}. \tag{2.9}$$

Here $0 < \gamma < 2$, and q is a positive integer. Let

$$||g||_{\gamma,q} = \max_{0 \leq j \leq q} \int_0^1 [t(1-t)]^{j-\gamma} |g^{(j)}(t)|\, dt. \tag{2.10}$$

Then $|| \cdot ||_{\gamma,q}$ is a norm on $\mathcal{S}^{\gamma,q}$. If $g \in \mathcal{S}^{\gamma,q}$, it is easy to see that

$$[t(1-t)]^{j+1-\gamma} |g^{(j)}(t)| \leq 2 \max_{0 \leq j \leq q-1} (||g||_{\gamma,q}, g^{(j)}(1/2)) \tag{2.11}$$

for $0 \leq j \leq q-1$. Especially, when $1 < \gamma < 2$, note that $g(0) = g(1) = 0$, and

$$[t(1-t)]^{1-\gamma} |g(t)| \leq ||g||_{\gamma,q}. \tag{2.12}$$

For details, see [**13**].

Now we propose a numerical quadrature method for the integral

$$I(g) = \int_0^1 g(t)\, dt \tag{2.13}$$

where $g \in \mathcal{S}^{\gamma,q}$. By the substitution $t = w(s)$, we obtain

$$I(g) = \int_0^1 h(s)\,ds \tag{2.14}$$

where

$$h(s) = w'(s)g(w(s)), \quad 0 \le s \le 1.$$

Applying the trapezoidal rule to (2.14), we have the numerical quadrature method.

$$I_n(g) = \frac{1}{2n}\sum_{j=1}^{2n-1} a_j g(t_j) \tag{2.15}$$

where

$$a_j = w'(s_j), \qquad t_j = w(s_j), \qquad s_j = j/2n \quad \text{for } 1 \le j \le 2n-1.$$

According to [**11**], $h(s)$ has the following property

$$h^{(m)}(s) = \sum_{j=0}^{m} u_j^m(s) g^{(j)}(\tau)|_{\tau=w(s)} \tag{2.16}$$

where

$$u_j^m(s) = r_j^m(s)\frac{[\log(s)+\log(1-s)]^{m+1}}{[s(1-s)]^{(1+\beta)(m+1)}}[\tau(1-\tau)]^{j+1}|_{\tau=w(s)} \tag{2.17}$$

$$u_j^m(s) = \Lambda_j^m(s)\frac{[\log(s)+\log(1-s)]^{m}}{[s(1-s)]^{(1+\beta)m}}[\tau(1-\tau)]^{j} w'(s)|_{\tau=w(s)} \tag{2.18}$$

and $r_j^m(s)$ and $\Lambda_j^m(s)$ are continuous. From (2.5),

$$\frac{[\log(s)+\log(1-s)]^{k}}{[s(1-s)]^{(1+\beta)k}}[\tau(1-\tau)]^{\mu}|_{\tau=w(s)}$$

is continuous for any μ, $k > 0$. Then u_j^m can be rewritten as

$$u_j^m(s) = \tilde{r}_j^m(s)[\tau(1-\tau)]^{j+1-\mu}|_{\tau=w(s)} \tag{2.19}$$

$$= \tilde{\Lambda}_j(s)[\tau(1-\tau)]^{j-\mu} w'(s)|_{\tau=w(s)} \tag{2.20}$$

with some continuous functions $\tilde{r}_j^m$ and $\tilde{\Lambda}_j^m$, depending on μ.

Theorem 2.1. *If* $g \in \mathcal{S}^{\gamma,2q+2}$ *for* $0 < \gamma < 2$,

$$|I(g) - I_n(g)| \le \frac{C}{n^{2q+2}} ||g||_{\gamma,2q+2}.$$

Proof. From (2.16) and (2.19), for any $\mu > 0$,

$$\begin{aligned} h^{(m)}(s) &= \sum_{j=0}^{m} \tilde{r}_j^m(s)[\tau(1-\tau)]^{j+1-\mu} g^{(j)}(\tau)|_{\tau=w(s)} \\ &= \sum_{j=0}^{m} \tilde{r}_j^m(s)[\tau(1-\tau)]^{\gamma-\mu}[\tau(1-\tau)]^{j+1-\gamma} g^{(j)}(\tau)|_{\tau=w(s)}. \end{aligned}$$

Taking $\mu < \gamma$, and by (2.11), $h^{(m)}(s) = 0$ at $s = 0$ and $s = 1$ for $0 \le m \le 2q+1$. If $m = 2q+2$, using (2.16) with (2.20), we rewrite $h^{(m)}(s)$ as follows:

$$h^{(m)}(s) = \sum_{j=0}^{m} \tilde{\Lambda}_j^m(s)[\tau(1-\tau)]^{\gamma-\mu}[\tau(1-\tau)]^{j-\gamma} g^{(j)}(\tau) w'(s)|_{\tau=w(s)}.$$

Take $\gamma > \mu$. Then we have

$$\int_0^1 |h^{(m)}(s)| \le C||g||_{\gamma,m}.$$

Applying the Euler-Maclaurin formula for the trapezoidal rule [**2**], we immediately have the theorem.

3. A class of singular integral equations. Let us consider the following integral equation of the second kind.

$$\psi(t) + \int_0^1 K(t,\tau)[\psi(\tau) - \psi(0)]\, d\tau + r(t)\psi(0) = f(t), \quad 0 \le t \le 1. \tag{3.1}$$

Here $r(t)$ is a sufficiently smooth function. We assume that the solution ψ is a 1-periodic continuous function, and $\psi - \psi(0)$ belongs to $\mathcal{S}^{\gamma,q}$ for some γ, $1 < \gamma < 2$. The kernel $K(t,\tau)$ is assumed to be a periodic function with period 1, and it is to be smooth in both variables except at the four corners of $[0,1] \times [0,1]$. Furthermore, $K(t,\tau) = L(t,\tau) + M(t,\tau)$, where M is continuous on $[0,1] \times [0,1]$ and L is either a nonpositive or nonnegative function with compact support in $([0,T] \cup [1-T,1]) \times ([0,T] \cup [1-T,1])$. For our mathematical analysis, we need additional assumptions on L.

$$\text{(A1)} \qquad \int_0^1 |L(t,\tau)|\, d\tau < 1$$

$$\text{(A2)} \qquad \begin{gathered} [(1-\tau)\tau]^{m+1} \left| \frac{\partial^m L(t,\tau)}{\partial \tau^m} \right| < C_m, \\ \int_0^1 [(1-\tau)\tau]^m \left| \frac{\partial^m L(t,\tau)}{\partial \tau^m} \right| d\tau < D_m, \quad m \geq 0 \end{gathered}$$

for constants C_m and D_m.

$$\text{(A3)} \qquad L(0,\cdot) = L(1,\cdot) = 0$$

Remark 1. In the double layer representation of Laplace's equation with the Dirichlet boundary data, the kernel function satisfies the above assumptions as will be seen in Section 5.

On $C[0,1]$ we define a norm, $||\cdot||_{\infty,0}$, as follows

$$||\psi||_{\infty,0} = |\psi(0)| + \max_{0 \leq t \leq 1} |\psi(t) - \psi(0)|.$$

It is easy to see that $||\cdot||_{\infty,0}$ is equivalent to $||\cdot||_{\infty}$.

Introduce the integral operators,

$$(\mathcal{A}\psi)(t) = \int_0^1 L(t,\tau)[\psi(\tau) - \psi(0)]\, d\tau$$

and

$$(\mathcal{B}\psi)(t) = \int_0^1 M(t,\tau)[\psi(\tau) - \psi(0)]\,d\tau + r(t)\psi(0).$$

Then (3.1) can be rewritten as

$$\psi + \mathcal{A}\psi + \mathcal{B}\psi = f.$$

The operator $\mathcal{B}$ is compact from $C[0,1]$ to $C[0,1]$ because $M(t,\tau)$ is continuous. Usually, $\mathcal{A}$ is a noncompact operator [**5**], but we will prove in the next theorem that $\mathcal{A}$ is a bounded operator with $||\mathcal{A}|| < 1$.

Theorem 3.1. *The integral operator $\mathcal{A}$ is bounded from $C[0,1]$ to $C[0,1]$ with norm,*

$$||\mathcal{A}|| < 1.$$

Proof. The proof follows a similar way shown in [**13**].

First, we will prove that $\mathcal{A}\psi$ is continuous. Because L is continuous except at four corners, it is obvious that $\mathcal{A}\psi$ is continuous on $(0,1)$. We will show that $\mathcal{A}\psi$ is continuous at $t = 0$ and $t = 1$.

Splitting the integral into three parts and using the assumptions on L, we can estimate

$$\begin{aligned}
|\mathcal{A}\psi(t)| &\le \int_0^1 |L(t,\tau)|\,|\psi(\tau) - \psi(0)|\,d\tau \\
&\le \sup_{0\le t\le\sigma} |\psi(\tau) - \psi(0)| \int_0^{\sigma} |L(t,\tau)|\,d\tau \\
&\quad + \sup_{1-\sigma\le\tau\le 1} |\psi(\tau) - \psi(0)| \int_{1-\sigma}^{1} |L(t,\tau)|\,d\tau \\
&\quad + ||\psi||_{\infty,0} \int_{\sigma}^{1-\sigma} |L(t,\tau)|\,d\tau \\
&\le \sup_{0\le\tau\le\sigma} |\psi(\tau) - \psi(0)| + \sup_{1-\sigma\le\tau\le 1} |\psi(\tau) - \psi(0)| \\
&\quad + ||\psi||_{\infty,0} \int_{\sigma}^{1-\sigma} |L(t,\tau)|\,d\tau.
\end{aligned}$$

Given $\varepsilon > 0$, there is $\sigma > 0$ so that

$$\max\left(\sup_{0\le t\le\sigma}|\psi(\tau)-\psi(0)|, \sup_{1-\sigma\le t\le 1}|\psi(\tau)-\psi(0)|\right) < \varepsilon/3$$

because ψ is a 1-period continuous function. Because $\int_\sigma^{1-\sigma}|L(t,\tau)|\,d\tau$ is continuous and converges to 0 as $t\to 0$ or $t\to 1$ by (A3), there is a $\delta > 0$ so that

$$\int_\sigma^{1-\sigma}|L(t,\tau)|\,d\tau < \frac{\varepsilon}{3||\psi||_{\infty,0}}$$

for $0 < t < \delta$ and $1-\delta < t < 1$. Then

$$|\mathcal{A}\psi(t)| < \varepsilon$$

for $0 < t < \min\{\sigma,\delta\}$ and $1-\min\{\sigma,\delta\} < t < 1$. Moreover, $\mathcal{A}\psi(0) = \mathcal{A}\psi(1) = 0$ by (A3). Now we have proved the continuity of $\mathcal{A}\psi$. It is straightforward that

$$||\mathcal{A}|| \le \int_0^1 |L(t,\tau)|\,d\tau < 1. \qquad \square$$

4. Nyström method. For Nyström's method, we approximate (3.1) by using the quadrature method (2.15) as follows:

$$\psi_n(t) + \frac{1}{2n}\sum_{j=1}^{2n-1} a_j K(t,\tau_j)[\psi_n(\tau_j)-\psi_n(0)] + r(t)\psi_n(0) = f(t) \tag{4.1}$$

for $0\le t\le 1$. Define

$$(\mathcal{A}_n\psi)(t) = \frac{1}{2n}\sum_{j=1}^{2n-1} a_j L(t,\tau_j)[\psi(\tau_j)-\psi(0)]$$

and

$$(\mathcal{B}_n\psi)(t) = \frac{1}{2n}\sum_{j=1}^{2n-1} a_j M(t,\tau_j)[\psi(\tau_j)-\psi(0)] + r(t)\psi(0).$$

Then (4.1) can be rewritten as

$$\psi_n + \mathcal{A}_n\psi_n + \mathcal{B}_n\psi_n = f.$$

Lemma 4.1. *Assume* $\psi - \psi(0) \in \mathcal{S}^{\gamma,2q+2}$ *for* $1 < \gamma < 2$. *Then*

$$||(\mathcal{A} - \mathcal{A}_n)\psi||_\infty \le \frac{C}{n^{2q+2}}||\psi - \psi(0)||_{\gamma,2q+2}$$

for some constant C.

Proof. Define

$$h(s) = w'(s)L(t,\tau)\psi(\tau)|_{\tau=w(s)}.$$

By (2.16), (2.19) and (2.20),

$$h^{(m)}(s) = \sum_{j=0}^{m} u_j^m(s)\frac{\partial^j L(t,\tau)\psi(\tau)}{\partial\tau^j}\Big|_{\tau=w(s)}$$

where

$$u_j^m(s) = \tilde{r}_j^m(s)\{\tau(1-\tau)\}^{j+1-\mu}|_{\tau=w(s)}$$

or, alternatively,

$$u_j^m(s) = \tilde{\Lambda}_j^m(s)\{\tau(1-\tau)\}^{j-\mu}w'(s)|_{\tau=w(s)}$$

with some continuous functions $\tilde{r}_j^m$ and $\tilde{\Lambda}_j^m$, depending on $\mu > 0$.

Because

$$\frac{\partial^j L(t,\tau)\psi(\tau)}{\partial\tau^j} = \sum_{j'=0}^{j}\binom{j}{j'}\frac{\partial^{j'} L(t,\tau)}{\partial\tau^{j'}}\psi^{(j-j')}(\tau),$$

$$\begin{aligned}(4.2)\quad [\tau(1-\tau)]^{j+1-\mu}\frac{\partial^j L(t,\tau)\psi(\tau)}{\partial\tau^j} &= \sum_{j'=0}^{j}\binom{j}{j'}[\tau(1-\tau)]^{j'+1} \\ &\cdot \frac{\partial^{j'} L(t,\tau)}{\partial\tau^{j'}}[\tau(1-\tau)]^{j-j'+1-\gamma}\psi^{(j-j')}(\tau)[\tau(1-\tau)]^{\gamma-1-\mu}.\end{aligned}$$

Take $\mu < \gamma - 1$, and replace ψ with $\psi - \psi(0)$. By the assumption (A2) on L, and (2.11), $h^{(m)}(s) = 0$ at $s = 0$, $s = 1$ if $m \leq 2q + 1$. When $m = 2q + 2$, as in the proof of Theorem 2.1, we need to rewrite above (4.2) slightly differently.

$$[\tau(1-\tau)]^{j-\mu} \frac{\partial^j L(t,\tau)\psi(\tau)}{\partial \tau^j} w'(s)|_{\tau=w(s)} = \sum_{j'=0}^{j} \binom{j}{j'} [\tau(1-\tau)]^{j'+1} \cdot \frac{\partial^{j'} L(t,\tau)}{\partial \tau^{j'}} [\tau(1-\tau)]^{j-j'-\gamma} \psi^{(j-j')}(\tau) [\tau(1-\tau)]^{\gamma-1-\mu} w'(s).$$

Taking $\mu < \gamma - 1$, we have the desired result by the Euler-Maclaurin formula. $\square$

When (4.1) is used for the numerical approximation, some possible instability happens in our approximating system, as in [**5, 8, 13**]. Because this instability is caused by $\mathcal{A}_n$, we need to modify just $\mathcal{A}_n$. But in the actual numerical implementation, we also modify $\mathcal{B}_n$ because it is difficult and cumbersome to separate the kernel K into L and M. Then we approximate (3.1) as

$$\psi_n(t) + \frac{1}{2n} \sum_{j=d(n)}^{2n-d(n)}{}'' a_j K(t,\tau_j)[\psi_n(\tau_j) - \psi_n(0)] + r(t)\psi_n(0) = f(t) \tag{4.3}$$

instead of (4.1), where

$$d(n) = \text{ the greatest integer less than } n^\eta \tag{4.4}$$

for some $0 < \eta < 1$. Here $\sum''$ represents the sum of terms obtained by halving the first and the last terms.

Solve for $\psi_n(0), \psi_n(t_{d(n)}), \ldots, \psi_n(t_{2n-d(n)})$ by collocation equation (4.3) at $t = 0, t_{d(n)}, \ldots, t_{2n-d(n)}$. Then

$$\psi_n(t) = -\frac{1}{2n} \sum_{j=d(n)}^{2n-d(n)}{}'' a_j K(t,\tau_j)[\psi_n(\tau_j) - \psi_n(0)] - r(t)\psi_n(0) + f(t)$$

is the solution we are looking for.

Introduce the modified approximating integral operators,

$$(\mathcal{A}_n^\eta \psi)(t) = \frac{1}{2n} \sum_{j=d(n)}^{2n-d(n)}{}'' a_j L(t,\tau_j)[\psi(\tau_j) - \psi(0)]$$

and

$$(\mathcal{B}_n^\eta \psi)(t) = \frac{1}{2n} \sum_{j=d(n)}^{2n-d(n)}{}'' a_j M(t,\tau_j)[\psi(\tau_j) - \psi(0)] + r(t)\psi(0).$$

Then (4.3) can be written as

$$\psi_n + \mathcal{A}_n^\eta \psi_n + \mathcal{B}_n^\eta \psi_n = f.$$

Corollary 4.2. *Under the same assumption as in Lemma* 4.1*,*

$$||(\mathcal{A} - \mathcal{A}_n^\eta)\psi||_\infty \le \frac{C}{n^{2q+2}} ||\psi - \psi(0)||_{\gamma,2q+2}$$

for some constant C and sufficiently large n.

Proof. First, let us look at $|(\mathcal{A}_n - \mathcal{A}_n^\eta)\psi(t)|$.

$$\begin{aligned} |(\mathcal{A}_n - \mathcal{A}_n^\eta)\psi(t)| &\le \frac{1}{2n} \sum_{j=1}^{d(n)} a_j |L(t,\tau_j)(\psi(\tau_j) - \psi(0))| \\ &\quad + \frac{1}{2n} \sum_{j=2n-d(n)}^{2n-1} a_j |L(t,\tau_j)(\psi(\tau_j) - \psi(0))|. \end{aligned} \tag{4.5}$$

Because $\tau_j = w(s_j)$ and $s_j = j/2n$, and using (2.5) and (2.6)

$$a_j = w'(s_j) \le C\tau_j \frac{\log(s_j)}{s_j^{(1+\beta)}} \le C\tau_j^{1-\delta}, \quad s_j < 1/2$$

for any $\delta > 0$. Also, note that

$$|\psi(\tau_j) - \psi(0)| \le C\tau_j^{\gamma-1} ||\psi - \psi(0)||_{\gamma,2q+2}, \quad \tau_j < 1/2$$

from our assumption on ψ and (2.12), and

$$\tau_j|L(t,\tau_j)| < C_0, \quad \tau_j < 1/2$$

by (A2). Then

$$\begin{aligned}\frac{1}{2n}\sum_{j=1}^{d(n)} a_j|L(t,\tau_j)(\psi(\tau_j)-\psi(0))| &\le \frac{C}{2n}\sum_{j=1}^{d(n)}(\tau_j)^{\gamma-1-\delta}||\psi-\psi(0)||_{\gamma,2q+2}\\ &\le C(\tau_{d(n)})^{\gamma-1-\delta}||\psi-\psi(0)||_{\gamma,2q+2}\\ &\le CM(n)||\psi-\psi(0)||_{\gamma,2q+2}\end{aligned}$$

where

$$M(n) = \left(\frac{1}{2n^{1-\eta}}\right)^{\alpha(2n^{1-\eta})^{\beta}(\gamma-1-\delta)}.$$

Note that $0 \le \eta < 1$. We will also have the same kind of bound for

$$\frac{1}{2n}\sum_{j=2n-d(n)}^{2n-1} a_j|L(t,\tau_j)(\psi(\tau_j)-\psi(0))|$$

by the same way.

Choose $0 < \delta < \gamma - 1$ and sufficiently large n. Then

$$|(\mathcal{A}_n - \mathcal{A}^{\eta})\psi| \le \frac{C}{n^{2q+2}}||\psi-\psi(0)||_{\gamma,2q+2}.$$

Combined with Lemma 4.1, the main result is straightforward. □

Now we turn to the stability of the approximating system, beginning with the following lemma.

Lemma 4.3. *Let $\varepsilon(n) = (1/2)(d(n)/n)$. For arbitrary $\sigma > 0$, we have*

$$\frac{1}{2n}\sum_{j=d(n)}^{2n-d(n)} a_j|L(t,\tau_j)| \le ||\mathcal{A}|| + \frac{1}{n^2}\frac{C}{\varepsilon(n)^{(1+\beta)2+\sigma}} \tag{4.6}$$

for some constant C, depending on σ.

Proof. Because we assume that L is nonnegative or nonpositive, we consider $L(t,\tau)$ is nonnegative without loss of generality. Define $\delta(n) = w(\varepsilon(n))$. Then

$$\left| \int_{\delta(n)}^{1-\delta(n)} L(t,\tau)\,d\tau - \frac{1}{2n} \sum_{j=d(n)}^{2n-d(n)} {}'' a_j L(t,\tau_j) \right| \leq C \frac{1}{n^2} \int_{\varepsilon(n)}^{1-\varepsilon(n)} |h''(s)|\,ds$$

where $h(s) = L(t,w(s))w'(s)$, by the error formula for the trapezoidal rule [**2**].

Rewrite h'' as follows, using (2.16) with (2.18).

$$\begin{aligned} h''(s) &= \sum_{j=0}^{2} u_j^2(s) \frac{\partial^j L(t,\tau)}{\partial \tau^j}\bigg|_{\tau=w(s)} \\ &= \sum_{j=0}^{2} \Lambda_j^2(s) \frac{[\log(s)+\log(1-s)]^2}{[s(1-s)]^{(1+\beta)2}} [\tau(1-\tau)]^j \frac{\partial^j L(t,\tau)}{\partial \tau}\bigg|_{\tau=w(s)} w'(s). \end{aligned}$$

Then, by our assumptions on L,

$$\int_{\varepsilon(n)}^{1-\varepsilon(n)} |h''(\tau)|\,d\tau \leq \frac{C}{\varepsilon(n)^{(1+\beta)2+\sigma}} \quad \text{for any } \sigma > 0$$

for some constant C, depending on σ. By a simple calculation, we get the desired result. $\square$

Remark 2. From Lemma (4.3), if $\eta > 1 - 2/((1+\beta)2+\sigma)$ for some $\sigma > 0$, we have $(1-\eta)[(1+\beta)2+\sigma] < 2$. Then

$$\frac{1}{n^2} \frac{C}{\varepsilon(n)^{(1+\beta)2+\sigma}} \to 0$$

as $n \to \infty$, and we have

$$||\mathcal{A}_n^\eta|| < 1$$

for sufficiently large n.

Remark 3. In [**5, 8, 13**], the number of subintervals cut off does not depend on n. Here we need a cutoff which is dependent on n.

Theorem 4.4. *If* $\eta > 1 - 2/((1+\beta)2+\sigma)$ *for some* $\sigma > 0$,

$$||\mathcal{A}\psi - \mathcal{A}_n^\eta \psi||_{\infty,0} \to 0, \quad \text{as } n \to \infty$$

for all $\psi \in C[0,1]$.

Proof. $\mathcal{A}_n^\eta \psi$ converges uniformly to $\mathcal{A}\psi$ on $[0,1]$ for all polynomials ψ by Corollary 4.2. Lemma 4.3 says that $||\mathcal{A}_n^\eta||$ is uniformly bounded in n. Then, by the Banach-Steinhaus theorem, we are done. □

Now we state the main theorem which shows the unique solvability of our approximating system and the error estimate.

Theorem 4.5. *Assume* $I+\mathcal{A}+\mathcal{B}$ *is bijective from* $C[0,1]$ *to* $C[0,1]$. *Let* ψ *be a solution of* $(I+\mathcal{A}+\mathcal{B})\psi = f$ *and* $\psi - \psi(0) \in \mathcal{S}^{\gamma,2q+2}$, $1 < \gamma < 2$. *Then, for*

$$\eta > 1 - \frac{2}{(1+\beta)2+\sigma} \quad \text{for some } \sigma > 0$$

and sufficiently large n,

$$\psi_n + \mathcal{A}_n^\eta \psi_n + \mathcal{B}_n^\eta \psi_n = f$$

is uniquely solvable, and we have the error estimate

$$||\psi_n - \psi||_\infty \le \frac{C}{n^{2q+2}} ||\psi - \psi(0)||_{\gamma,2q+2}.$$

Proof. First, we prove that $\{\mathcal{B}_n^\eta\}$ is collectively compact. This is a standard type of proof; for example, see [**1, 12**].

Because the operator $\mathcal{B}$ is better behaved than the operator $\mathcal{A}$, it is easy to show that $\mathcal{B}_n^\eta \psi(t) \to \mathcal{B}\psi(t)$ uniformly on $[0,1]$ as $n \to \infty$.

Since the quadrature method as defined by (2.15) converges, the weights satisfy

$$\sum_{k=1}^{2n-1} |a_k| \le C \quad \text{for all } n$$

by Theorem 12.4 in [**12**]. Then we have the estimate

$$||\mathcal{B}_n^\eta \psi||_\infty \le C \max_{t,\tau\in[0,1]} |M(t,\tau)| \cdot ||\psi||_\infty \tag{4.7}$$

and

$$|(\mathcal{B}_n^\eta \psi)(t_1) - (\mathcal{B}_n^\eta \psi)(t_2)| \le C \max_{\tau\in[0,1]} |M(t_1,\tau) - M(t_2,\tau)| \cdot ||\psi||_\infty. \tag{4.8}$$

From (4.7) and (4.8), $\{\mathcal{B}_n^\eta \psi : ||\psi||_{\infty,0} \le 1, n \in N\}$ is bounded and equicontinuous because M is uniformly continuous on $[0,1]\times[0,1]$. By the Arzela-Ascoli theorem [**12**], $\{\mathcal{B}_n^\eta\}$ is collectively compact.

Now we will show that $\{(I+\mathcal{A}_n^\eta)^{-1}\mathcal{B}_n^\eta\}$ is collectively compact. By a simple algebraic manipulation,

$$(I+\mathcal{A}_n^\eta)^{-1}\mathcal{B}_n^\eta = (I+\mathcal{A})^{-1}\mathcal{B}_n^\eta + (I+\mathcal{A}_n^\eta)^{-1}(\mathcal{A}-\mathcal{A}_n^\eta)(I+\mathcal{A})^{-1}\mathcal{B}_n^\eta.$$

Let $U = \{(I+\mathcal{A})^{-1}\mathcal{B}_n^\eta \psi : ||\psi||_{\infty,0} < 1\}$. Then $\overline{U}$ is compact. Because $\{I+\mathcal{A}_n^\eta\}^{-1}$ is uniformly bounded by Remark 2, $(I+\mathcal{A}_n^\eta)^{-1}(\mathcal{A}_n^\eta - \mathcal{A})$ converges uniformly to 0 on the compact set $\overline{U}$. So $\{(I+\mathcal{A}_n^\eta)^{-1}\mathcal{B}_n^\eta\}$ is collectively compact. Then $(I+(I+\mathcal{A}_n^\eta)^{-1}\mathcal{B}_n^\eta)^{-1}$ exists and is uniformly bounded for sufficiently large n. Because

$$(I+\mathcal{A}_n^\eta+\mathcal{B}_n^\eta)^{-1} = (I+(I+\mathcal{A}_n^\eta)^{-1}\mathcal{B}_n^\eta)^{-1}(I+\mathcal{A}_n^\eta)^{-1},$$

$(I+\mathcal{A}_n^\eta+\mathcal{B}_n^\eta)^{-1}$ is invertible and is uniformly bounded with respect to n. By a standard argument, we have an error bound,

$$||\psi_n - \psi||_\infty \le C[||(\mathcal{A}-\mathcal{A}_n^\eta)\psi||_{\infty,0} + ||(\mathcal{B}-\mathcal{B}_n^\eta)\psi||_{\infty,0}].$$

The proof of the theorem is now straightforward. □

5. The Dirichlet problem in plane domains with corners. We consider the Dirichlet problem for the Laplace's equation, given as follows:

$$\Delta u = 0 \quad \text{in } D$$

$$u = g \quad \text{on } \Gamma = \partial D$$

where D is a bounded simply connected region, and Γ is at least twice continuously differentiable, except at a corner $x = x_0$. Here we have only one corner. With a little extra work, we can extend the results to domains with a finite number of corners.

Let ν_y be the outward unit normal vector at $y \in \Gamma$. We use a double layer potential representation for u,

$$u(x) = \frac{1}{2\pi}\int_\Gamma \phi(y)\frac{\partial}{\partial \nu_y}\log|x-y|\,dS_y, \quad x \in D. \tag{5.1}$$

By Green's theorem,

$$\frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial \nu_y}\log|x-y|\,dS_y = 1, \quad x \in D,$$

we can rewrite (5.1) as follows:

$$u(x) = \frac{1}{2\pi}\int_\Gamma [\phi(y)-\phi(x_0)]\frac{\partial}{\partial \nu_y}\log|x-y|\,dS_y + \phi(x_0), \\ x \in D.$$

On the boundary we have

$$\phi(x) + \frac{1}{\pi}\int_\Gamma [\phi(y)-\phi(x_0)]\frac{\partial}{\partial \nu_y}\log|x-y|\,dS_y + \phi(x_0) \\ = 2g(x), \quad x \in \Gamma. \tag{5.2}$$

Parametrize the boundary curve Γ as follows:

$$x(t) = (x_1(t), x_2(t)), \quad 0 \le t \le 1$$

where $[x_1'(t)]^2 + [x_2'(t)]^2 > 0$. The corner x_0 of Γ corresponds to $t = 0$. By this parametrization, (5.2) can be transformed into the form,

$$\psi(t) + \int_0^1 K(t,\tau)[\psi(\tau)-\psi(0)]\,d\tau + \psi(0) = f(t), \quad 0 \le t \le 1. \tag{5.3}$$

Here $\psi(t) = \phi(x(t))$, $f(t) = 2g(x(t))$, and

$$K(t,\tau) = \begin{cases} \dfrac{1}{\pi} \dfrac{x_2'(\tau)[x_1(t) - x_1(\tau)] - x_1'(\tau)[x_2(t) - x_2(\tau)]}{[x_1(t) - x_1(\tau)]^2 + [x_2(t) - x_2(\tau)]^2}, & t \neq \tau \\ \dfrac{1}{2\pi} \dfrac{x_2'(t)x_1''(t) - x_1'(t)x_2''(t)}{[x_1'(t)]^2 + [x_2'(t)]^2}, & t = \tau. \end{cases}$$

Note that $K(t,\tau)$ is continuous on $(0,1) \times (0,1)$ because Γ is at least twice continuously differentiable, except at the corners.

As shown in [**4, 6, 13**], $K(t,\tau)$ satisfies the following property around $t = \tau = 0$

$$K(t,\tau) = L(t,\tau) + M(t,\tau)$$

where

$$|L(t,\tau)| \leq \frac{1}{\pi} \frac{t \sin\theta}{\tau^2 - 2t\tau\cos\theta + t^2}$$

around $t = \tau = 0$ and M is continuous on $[0,1] \times [0,1]$. Here θ is the interior angle at the corner. We can easily check that $L(t,\tau)$ satisfies the assumptions in Section 3. Furthermore,

$$\int_0^\infty |L(t,s)|\, ds \leq \frac{|\pi - \theta|}{\pi} < 1.$$

Using the regularity results for solutions of Laplace's equations in nonsmooth domain [**7, 9, 10**], we have

$$\begin{aligned} u^+ &= h^+(\vartheta) r^{\pi/\theta} + \text{smoother terms} \\ u^- &= h^-(\vartheta) r^{\pi/(2\pi - \theta)} + \text{smoother terms} \end{aligned}$$

around the corner with some smooth functions $h^{\pm}$, where u^+ is harmonic in D and u^- is harmonic in $R^2/\overline{D}$. Here (r,ϑ) represents the polar coordinate system with the origin at the corner. Because the double layer density function satisfies $\psi = (u^+ - u^-)|_\Gamma$ [**3**], we have

$$\psi = Cr^{\alpha^*} + \text{smoother terms}, \qquad \alpha^* = \frac{\pi}{\pi + |\pi - \theta|}$$

around the corner. Then, by the definition of $\mathcal{S}^{1+\gamma,\infty}$ in (2.9),

$$\psi - \psi(0) \in \mathcal{S}^{1+\gamma,\infty} \quad \text{with} \quad 0 < \gamma < \frac{\pi}{\pi + |\pi - \theta|}$$

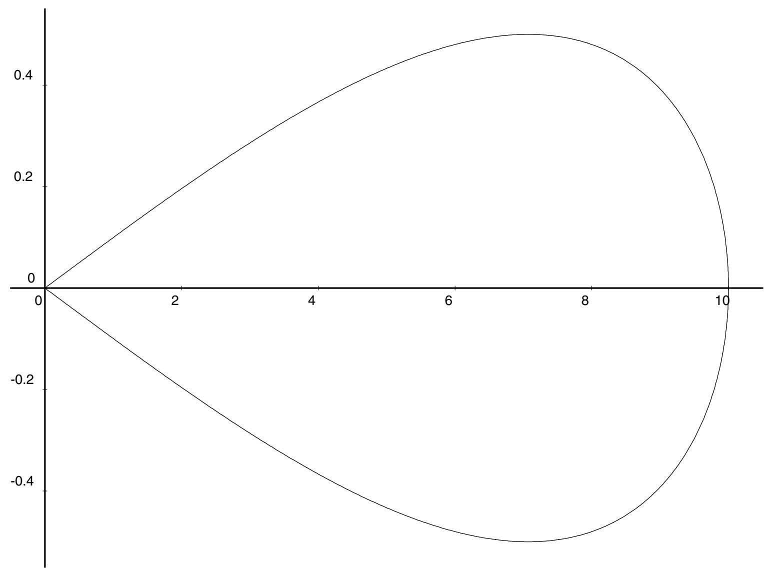

FIGURE 1.

for any smooth boundary data f.

6. Numerical examples. We consider solving Laplace's equations on two domains with a corner, which are defined as follows:

$$\text{Domain 1: } X(t) = ((2/\sqrt{3})\sin(\pi t), -\sin(2\pi t))$$
$$\text{Domain 2: } X(t) = (-(2/3)\sin(3\pi t), -\sin(2\pi t))$$

for $0 \le t \le 1$. Domain 1 is a drop-shaped domain with interior angle $2\pi/3$, and Domain 2 has a reentrant corner with interior angle $3\pi/2$. Figures 1 and 2 show Domains 1 and 2, respectively.

Let us consider the following test harmonic functions to give boundary conditions.

Example 1. $u(r,\vartheta) = r^{3/2}\cos(3/2)\vartheta$ on Domain 1.

Example 2. $u(r,\vartheta) = r^{2/3}\cos(2/3)\vartheta$ on Domain 2.

Let us look at $w_{\alpha,\beta}$ as defined in Section 2. When $\beta = 0$, it is an algebraic mesh grading considered in [**13**]. If $\beta \neq 0$, it is an exponential mesh grading. Table 1 and Table 2 show the effect of

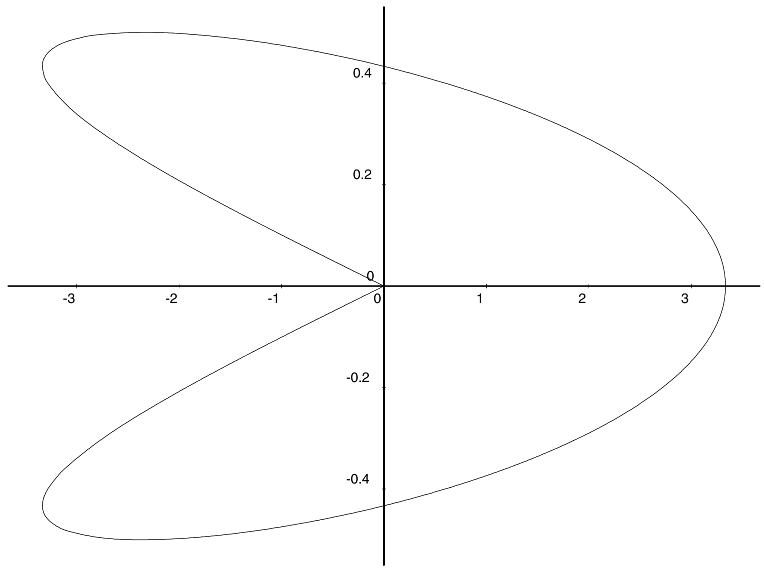

FIGURE 2.

various mesh gradings on convergence of density functions. Because the mesh tends to be heavily toward a corner point when a larger β is used, our numerical experiments concentrate on suitable size of α's and small β's.

TABLE 1. E_n with various mesh grading for Example 1.

n	$(2,0)$	$(2,1/5)$	$(2,1/3)$	$(4,0)$	$(4,1/5)$	$(4,1/3)$
4	.17E 00	.15E 00	.16E 00	.15E 00	.13E 00	.14E-00
8	.78E-02	.81E-02	.76E-02	.80E-02	.92E-02	.13E-01
16	.15E-04	.77E-05	.10E-04	.12E-04	.21E-04	.37E-04
32	.22E-05	.58E-06	.29E-07	.82E-07	.33E-08	.29E-08
64	.31E-06	.68E-08	.13E-09	.15E-09	.65E-11	.31E-11

In the tables, the first column is the number of node points in $[0, 1/2]$, the first row represents the mesh grading parameters $\{\alpha, \beta\}$, and

$$E_n = \sup_{1 \le i \le n} \{|\psi_n(x_i) - \psi_{2n}(x_i)| : x_i = w_{\alpha,\beta}(i/2n)\}$$

where ψ_n is an approximating double layer density function. Note that the second and the fifth columns in each table represent the case when an algebraic mesh grading is used. It is clear that convergence improves as the mesh grading parameter β becomes bigger with the fixed α.

TABLE 2. E_n with various mesh grading for Example 2.

n	$(2,0)$	$(2,1/5)$	$(2,1/3)$	$(4,0)$	$(4,1/5)$	$(4,1/3)$
4	.11E 01	.12E 01	.12E 01	.13E 01	.14E 01	.17E 01
8	.39E-01	.44E-01	.59E-01	.63E-01	.64E-01	.11E 00
16	.25E-02	.25E-01	.32E-02	.22E-02	.14E-02	.52E-02
32	.90E-03	.48E-03	.29E-03	.33E-03	.74E-04	.58E-04
64	.34E-03	.50E-04	.15E-04	.51E-04	.31E-05	.16E-05
128	.13E-03	.35E-05	.54E-06	.80E-05	.80E-07	.22E-07

The cutoff criteria is, as in (4.4) with η in *Remark* 2,

$$d(n) = \text{integer part of } n^\eta, \quad \eta = 1 - \frac{2}{(1+\beta)2+\sigma}, \quad \sigma > 0.$$

In [**13**], the cutoff around the corner is not necessary in the actual numerical implementation. We also observe that the above cutoff criteria is not strictly applied to our method in our numerical experiments. It is interesting to note that $d(n)$ does not depend on α, and $d(n) = 1$ when $\beta = 0$, which would mean that we don't need cutoff when an algebraic mesh grading is used.

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