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ON USING A MODIFIED NYSTRÖM METHOD TO SOLVE THE 2-D POTENTIAL PROBLEM

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ABSTRACT. In this paper the single-layer potential representation is used to solve the 2-D potential problem with either Dirichlet, Neumann, or Robin boundary conditions. We show that the discrete delta-trigonometric method introduced by Cheng and Arnold [**14**] and the discrete Galerkin method introduced by Atkinson [**9**] obtain the same discrete density. Then we introduce another equivalent method called the modified Nyström method, and show that this method requires only $O(n^2)$ simple operations (instead of $O(n^2 \log n)$) to form the matrices. We also discuss previous convergence results for this method and extend the exponential convergence results to the Robin problem. Finally, we present numerical experiments in order to confirm our theory.

1. Introduction. The two-dimensional potential problem is

$$\Delta U = 0 \quad \text{on } \mathbf{R}^2 \backslash \Gamma, \qquad aU + b\frac{dU}{d\nu} = G \quad \text{on } \Gamma, \tag{1.1}$$

where a and b are constants, U is bounded at infinity, ν is the outward normal, G is analytic and Γ is a simple closed analytic curve. For any harmonic U, there exists a unique Φ satisfying the single-layer potential representation,

$$U(z) = \int_\Gamma \Phi(y) \log |z - y| \, d\Gamma_y \quad \text{for } z \in \mathbf{R}^2, \tag{1.2}$$

if the conformal radius of Γ does not equal 1, e.g., [**9, 13, 14, 49, 50**].

Remark. There are two ways to handle the uniqueness problem when the conformal radius equals 1 as in [**15**]. One approach is to add an

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unknown constant to the right side of equation (1.2) and specify

$$\int_\Gamma \Phi(y)\, d\Gamma_y = 0$$

as in [**3, 31**]. The other approach is to scale the domain so that the conformal radius does not equal 1 as in [**9, 13, 14, 50**]. For simplicity, we assume that the conformal radius of Γ does not equal 1.

The density Φ solves the boundary integral equation,

$$a \int_\Gamma \Phi(y) \log|z-y|\, d\Gamma_y \pm \pi b \Phi(z) + b \int_\Gamma \Phi(y) \frac{d}{d\nu_z} \log|z-y|\, d\Gamma_y = G(z) \quad \forall\, z \in \Gamma, \tag{1.3}$$

where the plus and minus signs refer to the exterior and interior boundary value problems, respectively. For the interior Dirichlet and the exterior Neumann problems, equation (1.3) has a unique solution, e.g., [**1, 13, 17, 27, 49**]. For the Robin problem, we would physically expect a and b to be of the same sign for the interior problem, and of opposite signs for the exterior problem. For these physical cases, equation (1.3) has a unique solution, e.g., [**17, 20, 27**]. In cases such as the interior Neumann problem, where equation (1.3) fails to give a unique solution, there are techniques for modifying the kernel so that the modified integral equation has a unique solution, e.g., [**6**]. For simplicity, we only consider problems in which equation (1.3) has a unique solution.

In this paper we seek a numerical method to approximate the density in equation (1.3) such that the potential in equation (1.2) is approximated with exponential convergence based on the mesh discretization. Such a method obtains a very accurate approximate solution using a smaller matrix system, thus reducing the need for tedious matrix operations.

For the Dirichlet problem, equation (1.3) becomes an integral equation of the first kind. Various methods have been analyzed such as the spline-collocation, the spline Galerkin, and the delta-spline methods, i.e., [**5, 19, 24–26, 30, 39, 41–45, 48, 50**]. These methods obtain the potential with an optimal asymptotic convergence rate. Also, iterative methods have been analyzed, e.g., [**1, 2, 19, 21, 34, 37, 38**].

For a good theoretical understanding of the weakly singular integral operator, refer to [**1, 23, 49**].

For the Dirichlet problem, exponential convergence results, based on the mesh discretization using the single-layer potential representation, were obtained only recently. Arnold [**3**] showed that the spline-trigonometric method obtains the potential on compact sets away from the boundary with exponential convergence. His method was also applied for singular integral equations [**4**]. McLean [**31**] showed that the trigonometric-trigonometric method obtained the potential everywhere with exponential convergence. Neither Arnold nor McLean accounted for numerical integration. Cheng [**13**] and Cheng and Arnold [**14**] showed that the discrete delta-trigonometric method obtains the potential on compact sets away from the boundary with exponential convergence. At the same time, Atkinson [**9**] showed that the discrete trigonometric-trigonometric method obtains very rapid convergence for the approximate density function when the boundary and boundary data are smooth. Near the boundary, the potential was improved by using a higher trapezoidal rule after the density is found. McLean, Prossdorf, and Wendland [**32**] investigated this method for singular integral equations. This method is equivalent to a discrete collocation procedure with trigonometric polynomial approximants. In this paper we will refer to the discrete Galerkin method as the discrete trigonometric-delta method because we would like to think of this method as the dual method to the discrete delta-trigonometric method. These two methods require $O(n^2 \log n)$ operations using FFT to form the matrices.

For the Neumann problem, equation (1.3) becomes an integral equation of the second kind. Various methods have been analyzed including the collocation, the Galerkin, and the Nyström methods, e.g., [**7, 8, 10, 12, 18**]. For analytic data and boundary, exponential convergence was obtained long ago. In addition, iterative methods have been widely investigated, e.g., [**2, 7, 28, 35, 36**].

For the mixed problem, the Galerkin method has been investigated the most, e.g., [**29, 33, 46, 47**]. We also note that some work has been done for the case of a nonlinear boundary condition, e.g., [**11, 16, 40**].

In this paper, we show that the discrete trigonometric-delta and the discrete delta-trigonometric methods obtain the same discrete density

function. Then we introduce a modified Nyström method, which is also equivalent to the discrete trigonometric-delta and the discrete delta-trigonometric methods, and show that the matrix formation requires only $O(n^2)$ simple arithmetic operations. The density is first found as a summation of delta functions. Then it can be transformed to a summation of trigonometric functions so that a higher trapezoidal rule can be used to find the potential. This transformation is especially useful for computing the near-field potential.

For the Dirichlet problem, we note that the Nyström method could be used on the boundary integral equation formulated using the double-layer potential representation. For this classical approach, the potential is obtained with exponential convergence, but the matrix system is nonsymmetric. For our approach, the modified Nyström method is used on the boundary integral equation formulated using the single-layer potential representation. For this new approach, the matrix system is symmetric.

For the Dirichlet and Neumann problems, we summarize previous exponential convergence results for the potential on compact set away from the boundary, and rapid convergence results for the density. Then, we extend the exponential convergence results to the Robin problem. Finally, we present numerical results to confirm our theory.

We first review some standard notation and reformulate the boundary integral equation in Section 2. Then we define the discrete delta-trigonometric and the discrete trigonometric-delta methods in Sections 3 and 4, respectively. In Section 5 we show that the two methods compute the same discrete density and then introduce the modified Nyström method. In Section 6 we extend the exponential convergence results to the Robin problem. In Section 7 we present numerical examples for the 2-D potential problem with Dirichlet, Neumann, and Robin boundary conditions.

2. Preliminaries. Define the space of trigonometric polynomials with complex coefficients as

$$T := \operatorname{span}\{\exp(2\pi ikt) \mid k \in \mathbf{Z}\}.$$

Any function f in this space can be represented as

$$f(t) = \sum_{k\in\mathbf{Z}} \hat{f}(k) \exp(2\pi ikt)$$

where

$$\hat{f}(k) := \int_0^1 f(t) \exp(-2\pi i k t)\, dt$$

are Fourier coefficients in which all, but finitely many, are zeros. For $f \in T$, $s \in \mathbf{R}$, and $\varepsilon > 0$, we define the Fourier norm as

$$||f||_{s,\varepsilon} := \sqrt{\sum_{k \in \mathbf{Z}} |\hat{f}(k)|^2 \varepsilon^{2|k|} k^{2s}}$$

where

$$k = \begin{cases} 1, & \text{if } k = 0, \\ 2\pi|k|, & \text{if } k \neq 0. \end{cases}$$

We denote by $X_{s,\varepsilon}$ the completion of T with respect to this norm. As in [**14**], the L^2 inner product extends to a bounded bilinear form on $X_{s,\varepsilon} \times X_{-s,\varepsilon^{-1}}$ for all $s \in \mathbf{R}$, $\varepsilon > 0$ in which $X_{-s,\varepsilon^{-1}}$ is the dual space of $X_{s,\varepsilon}$. For $\varepsilon = 1$, $X_{s,\varepsilon}$ is the usual periodic Sobolev space of order s, H^2, with norm $||\cdot||_s$. See [**3**] for a more complete discussion of these spaces.

We denote the standard Euclidean vector and matrix norms by $||\cdot||$. The constants C and ε are generic and are not necessarily the same in each occurrence.

Let $x : \mathbf{R} \to \Gamma$ be a 1-periodic analytic function which parametrizes Γ and has nonvanishing derivatives, and define

$$\phi(t) := \Phi(x(t))|x'(t)|, \qquad g(t) := G(x(t)).$$

For the single layer potential, define two operators as

$$A\phi(s) := \int_0^1 \phi(t) K(s,t)\, dt,$$
$$V\phi(s) := \int_0^1 \phi(t) \log|2r \sin(\pi(s-t))|\, dt,$$

where $r \neq 0$ and $K : \mathbf{R}^2 \to \mathbf{R}$ is a smooth symmetric kernel defined by

$$K(s,t) := \begin{cases} \log\left|\frac{x(s)-x(t)}{2r \sin(\pi(s-t))}\right|, & \text{if } s \neq t, \\ \log\left|\frac{x'(s)}{2\pi r}\right|, & \text{if } s = t. \end{cases}$$

Typically, $r = 1$ or $\exp(-1/2)$. For the double layer potential component, define two operators as

$$B\phi(s) := \int_0^1 \phi(t)L(s,t)\,dt, \qquad W\phi(s) := \pm\pi\frac{\phi(s)}{|x'(s)|},$$

where $L : \mathbf{R}^2 \to \mathbf{R}$ is a smooth nonsymmetric kernel defined by

$$L(s,t) := \begin{cases} \frac{[x(s)-x(t)]\cdot\nu(s)}{[x(s)-x(t)]^2}, & \text{if } s \neq t, \\ \frac{-x''(s)\cdot\nu(s)}{2|x'(s)|^2}, & \text{if } s = t, \end{cases}$$

and $\nu : \mathbf{R} \to \Gamma$ is the 1-periodic normal pointing into the exterior region. Again, the plus and minus signs refer to the exterior and interior problems, respectively. The single-layer potential representation (1.2) becomes

$$U(z) = \int_0^1 \phi(t)\log|z - x(t)|\,dt \quad \text{for } z \in \mathbf{R}^2\backslash\Gamma,$$

and the boundary integral equation (1.3) becomes

$$a(A+V)\phi(s) + b(B+W)\phi(s) = g(s) \quad \text{for } s \in [0,1]. \tag{2.1}$$

We now proceed to define the spaces for approximating the density. Restrict n to be a positive odd number and define the space of delta functions to be

$$S_n := \operatorname{span}\{\delta(t - j/n) \mid j = 0, \ldots, n-1\},$$

where $\delta(t - j/n)$ is the 1-periodic extension of the Dirac mass at j/n for $j = 0, \ldots, n-1$. Also define the space of trigonometric polynomials with degree $\leq n$ to be

$$T_n := \operatorname{span}\{\xi_k(t) \mid k = 1, \ldots, n\},$$

where

$$\xi_k(t) := \begin{cases} 1, & \text{if } k = 1, \\ \sqrt{2}\sin(k\pi t), & \text{if } k = 2, 4, \ldots, n-1, \\ \sqrt{2}\cos((k-1)\pi t), & \text{if } k = 3, 5, \ldots, n, \end{cases}$$

are orthonormal basis functions of T_n. For theoretical convergence, we use the complex basis functions, $\exp(2\pi ikt)$, and let

$$\Lambda_n := \{k \in \mathbf{Z} \mid |k| \le (n-1)/2\}.$$

3. The delta-trigonometric method. The semi-discrete delta-trigonometric method seeks $\phi_n \in S_n$ such that

$$a\int_0^1 (A+V)\phi_n(s)\psi(s)\,ds + b\int_0^1 (B+W)\phi_n(s)\psi(s)\,ds = \int_0^1 g(s)\psi(s)\,ds \quad \forall\,\psi \in T_n, \tag{3.1}$$

and computes the approximate potential as

$$U_n(z) = \int_0^1 \phi_n(t)\log|z-x(t)|\,dt \quad \text{for } z \in \mathbf{R}^2\backslash\Gamma. \tag{3.2}$$

Here the density is

$$\phi_n(t) = \sum_{j=0}^{n-1} \alpha_j \delta(t - j/n)$$

where α_j's are unknown coefficients, and the test functions are $\psi_k(s) := \xi_k(s)$ for $k = 1, \ldots, n$. The matrices $\mathbf{B}, \mathbf{W}, \mathbf{A}$ and $\mathbf{V}$, and are

$$\begin{aligned}
\mathbf{A}_{kj} &:= \int_0^1 K(s, j/n)\xi_k(s)\,ds,\\
\mathbf{V}_{kj} &:= \int_0^1 \log|2r\sin(\pi(s-j/n))|\xi_k(s)\,ds,\\
\mathbf{B}_{kj} &:= \int_0^1 L(s, j/n)\xi_k(s)\,ds,\\
\mathbf{W}_{kj} &:= \pm\pi\frac{\xi_k(j/n)}{|x'(j/n)|},
\end{aligned}$$

for $k = 1, \ldots, n$ and $j = 0, \ldots, n-1$. The n-vectors $\boldsymbol{\alpha}$ and $\mathbf{g}$ are

$$\boldsymbol{\alpha} := (\alpha_0, \ldots, \alpha_{n-1})^T, \qquad \mathbf{g}_k := \int_0^1 g(s)\xi_k(s)\,ds,$$

for $k = 1, \ldots, n$. Then the matrix form of the semi-discrete delta-trigonometric method (3.1) is

$$a(\mathbf{A} + \mathbf{V})\alpha + b(\mathbf{B} + \mathbf{W})\alpha = \mathbf{g},$$

and the approximate potential (3.2) is

$$U_n(z) = \sum_{j=0}^{n-1} \alpha_j \log|z - x(j/n)| \quad \text{for } z \in \mathbf{R}^2 \backslash \Gamma.$$

Before we discetize the delta-trigonometric method, we note from [**3, 9, 13, 14**] that

$$\mathbf{V}_{kj} := \begin{cases} \log|r|, & \text{if } k = 1, \\ \frac{-\xi_k(j/n)}{|k|}, & \text{if } k = 2, 4, \ldots, n-1, \\ \frac{-\xi_k(j/n)}{|k-1|}, & \text{if } k = 3, 5, \ldots, n, \end{cases}$$

for $k = 1, \ldots, n$ and $j = 0, \ldots, n-1$. The fully discrete method is obtained by using the trapezoidal rule to evaluate $\mathbf{A}, \mathbf{B}$ and $\mathbf{g}$. Define

$$\tilde{A}_{kj} := \frac{1}{n} \sum_{l=0}^{n-1} K(l/n, j/n)\xi_k(l/n), \qquad \tilde{\mathbf{B}}_{kj} := \frac{1}{n} \sum_{l=0}^{n-1} L(l/n, j/n)\xi_k(l/n),$$

$$\tilde{\mathbf{g}}_k := \frac{1}{n} \sum_{l=0}^{n-1} g(l/n)\xi_k(l/n), \qquad \tilde{\boldsymbol{\alpha}} := (\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{n-1})^T,$$

for $k = 1, \ldots, n$ and $j = 0, \ldots, n-1$. The discrete delta-trigonometric method seeks coefficients $\tilde{\alpha}_j$'s such that

$$a(\tilde{\mathbf{A}} + \mathbf{V})\tilde{\alpha} + b(\tilde{\mathbf{B}} + \mathbf{W})\tilde{\alpha} = \tilde{\mathbf{g}},$$

and computes the approximate potential as

$$\tilde{U}_n(z) = \sum_{j=0}^{n-1} \tilde{\alpha}_j \log|z - x(j/n)| \quad \text{for } z \in \mathbf{R}^2 \backslash \Gamma. \tag{3.3}$$

We now express the matrix equation in a reduced form. Define the trigonometric matrix as

$$\Theta_{kj} := \xi_k(j/n)$$

for $k = 1, \ldots, n$ and $j = 0, \ldots, n-1$. Here we note that $\Theta\Theta^T = \Theta^T\Theta = nI$. Also define

$$\hat{\mathbf{A}}_{kj} := K(k/n, j/n), \qquad \hat{\mathbf{B}}_{kj} := L(k/n, j/n),$$

$$\hat{\mathbf{V}}_{ll} := \begin{cases} \log|r|, & \text{if } l = 1, \\ \frac{-1}{|l|}, & \text{if } l = 2, 4, \ldots, n-1, \\ \frac{-1}{|l-1|}, & \text{if } l = 3, 5, \ldots, n, \end{cases}$$

$$\hat{\mathbf{W}}_{kk} := \frac{\pm\pi}{|x'(k/n)|}, \qquad \hat{\mathbf{g}}_k := g(k/n),$$

for $k, j = 0, \ldots, n-1$ and $l = 1, \ldots, n$. Note that $\hat{\mathbf{V}}$ and $\hat{\mathbf{W}}$ are diagonal matrices, $\hat{\mathbf{A}}$ is a full symmetric matirx, and $\hat{\mathbf{B}}$ is a full nonsymmetric matrix. The discrete delta-trigonometric method in reduced matrix form is

$$a\left(\frac{1}{n}\Theta\hat{\mathbf{A}} + \hat{\mathbf{V}}\Theta\right)\tilde{\alpha} + b\left(\frac{1}{n}\Theta\hat{\mathbf{B}} + \Theta\hat{W}\right)\tilde{\alpha} = \frac{1}{n}\Theta\hat{\mathbf{g}}. \tag{3.4}$$

4. The trigonometric-delta method. The semi-discrete trigonometric-delta method seeks $\phi_n \in T_n$ such that

$$a\int_0^1 (A+V)\phi_n(s)\psi(s)\,ds + b\int_0^1 (B+W)\phi_n(s)\psi(s)\,ds = \int_0^1 g(s)\psi(s)\,ds \quad \forall\, \psi \in S_n, \tag{4.1}$$

and computes the approximate potential as

$$\mathbf{U}_n(z) = \int_0^1 \phi_n(t)\log|z - x(t)|\,dt \quad \text{for } z \in \mathbf{R}^2\backslash\Gamma. \tag{4.2}$$

For the trigonometric-delta method, the density is

$$\phi_n(t) = \sum_{j=1}^{n} \beta_j \xi_j(t)$$

where β_j's are unknown coefficients, and the test functions are

$$\psi_k(t) := \delta(t - k/n)$$

for $k = 0, \ldots, n-1$. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{V},$, and $\mathbf{W}$ are

$$\begin{aligned}
\mathbf{A}_{kj} &:= \int_0^1 K(k/n, t)\xi_j(t)\, dt, \\
\mathbf{B}_{kj} &:= \int_0^1 L(k/n, t)\xi_j(t)\, ds, \\
\mathbf{V}_{kj} &:= \int_0^1 \log|2r\sin(\pi(k/n - t))|\xi_j(t)\, dt, \\
\mathbf{W}_{kj} &:= \pm\pi \frac{\xi_j(k/n)}{|x'(k/n)|},
\end{aligned}$$

for $k = 0, \ldots, n-1$ and $j = 1, \ldots, n$. The n-vectors β and $\mathbf{g}$ are

$$\beta := (\beta_1, \ldots, \beta_n)^T, \qquad \mathbf{g}_k := g(k/n),$$

for $k = 0, \ldots, n-1$. Then the matrix form of the semi-discrete trigonometric-delta method (4.1) is

$$a(\mathbf{A} + \mathbf{V})\beta + b(\mathbf{B} + \mathbf{W})\beta = \mathbf{g},$$

and the approximate potential (4.2) is

$$\mathbf{U}_n(z) = \int_0^1 \sum_{j=1}^{n} \beta_j \xi_j(t) \log|z - x(t)|\, dt \quad \text{for } z \in \mathbf{R}^2 \backslash \Gamma.$$

Again, note that

$$\mathbf{V}_{kj} := \begin{cases} \log|r|, & \text{if } j = 1, \\ \frac{-\xi_j(k/n)}{|j|}, & \text{if } j = 2, 4, \ldots, n-1, \\ \frac{-\xi_j(k/n)}{|j-1|}, & \text{if } j = 3, 5, \ldots, n, \end{cases}$$

for $k = 0, \ldots, n-1$ and $j = 1, \ldots, n$. The fully discrete method is obtained by using the trapezoidal rule to evaluate $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ and $\tilde{\mathbf{g}}$. Define

$$\begin{aligned}
\tilde{\mathbf{A}}_{kj} &:= \frac{1}{n} \sum_{l=0}^{n-1} K(k/n, l/n)\xi_j(l/n), \\
\tilde{\mathbf{B}}_{kj} &:= \frac{1}{n} \sum_{l=0}^{n-1} L(k/n, l/n)\xi_j(l/n), \\
\tilde{\beta} &:= (\tilde{\beta}_1, \ldots, \tilde{\beta}_n)^T,
\end{aligned}$$

for $k = 0, \ldots, n-1$ and $j = 1, \ldots, n$. The discrete trigonometric-delta method seeks coefficients $\tilde{\beta}_j$'s such that

$$a(\tilde{\mathbf{A}} + \mathbf{V})\tilde{\beta} + b(\tilde{\mathbf{B}} + \mathbf{W})\tilde{\beta} = \tilde{\mathbf{g}},$$

and computes the approximate potential as

$$\tilde{\mathbf{U}}_n(z) = \frac{1}{n}\sum_{l=0}^{n-1}\sum_{j=1}^{n}\tilde{\beta}_j\xi_j(l/n)\log|z - x(l/n)| \quad \text{for } z \in \mathbf{R}^2\backslash\Gamma. \tag{4.3}$$

In terms of the reduced matrices, the discrete trigonometric-delta method is

$$a\left(\frac{1}{n}\hat{\mathbf{A}}\Theta^T + \Theta^T\hat{\mathbf{V}}\right)\tilde{\beta} + b\left(\frac{1}{n}\hat{\mathbf{B}}\Theta^T + \hat{\mathbf{W}}\Theta^T\right)\tilde{\beta} = \hat{\mathbf{g}}. \tag{4.4}$$

5. The modified Nyström method. Multiply the discrete delta-trigonometric matrix system (3.4) by Θ^T to obtain

$$a(\hat{\mathbf{A}} + \Theta^T\hat{\mathbf{V}}\Theta)\tilde{\alpha} + b(\hat{\mathbf{B}} + n\hat{\mathbf{W}})\tilde{\alpha} = \hat{\mathbf{g}}. \tag{5.1}$$

Also note that the discrete trigonometric-delta matrix system (4.4) is equivalent to

$$a\left(\frac{1}{n}\hat{\mathbf{A}} + \frac{1}{n}\Theta^T\hat{\mathbf{V}}\Theta\right)\Theta^T\tilde{\beta} + b\left(\frac{1}{n}\hat{\mathbf{B}} + \hat{\mathbf{W}}\right)\Theta^T\tilde{\beta} = \hat{\mathbf{g}}.$$

Therefore,

$$\tilde{\alpha} = \frac{1}{n}\Theta^T\tilde{\beta} \quad \text{and} \quad \tilde{\beta} = \Theta\tilde{\alpha}. \tag{5.2}$$

A simple examination of equations (3.3) and (4.3) shows that

$$\tilde{U}_n(z) = \tilde{\mathbf{U}}_n(z) \quad \text{for } z \in \mathbf{R}^2\backslash\Gamma.$$

This implies that we can use one method and obtain the solution of the other method by using equation (5.2). For these two methods, $O(n^2 \log n)$ operations is required to form the matrices using FFT.

We now introduce the modified Nyström method as the method using equation (5.1). In fact, equation (5.1) represents exactly the Nyström method for the Neumann problem except for a multiple of n in the density vector (due to using delta functions instead of actual function values). We proceed to show that matrix formation in equation (5.1) requires only $O(n^2)$ operations.

Define $\mathbf{T} = \Theta^T \hat{\mathbf{V}} \Theta$, and note that

$$\begin{aligned}
\hat{\mathbf{T}}_{kj} &= \sum_{m=1}^{n} \hat{\mathbf{V}}_{mm} \xi_m(k/n) \xi_m(j/n) \\
&= \log|r| + \sum_{m=1}^{(n-1)/2} \frac{-1}{2m} [2\cos(2\pi mk/n)\cos(2\pi mj/n) \\
&\quad + 2\sin(2\pi mk/n)\sin(2\pi mj/n)] \\
&= \log|r| + \sum_{m=1}^{(n-1)/2} \frac{-1}{m} \cos\left(2\pi m \frac{(k-j)}{n}\right)
\end{aligned}$$

for $k, j = 0, \ldots, n-1$. The matrix $\mathbf{T}$ is a circulant Toeplitz matrix, and, therefore, only the first row needs to be computed, which requires $O(n \log n)$ operations using the FFT. The matrix $\hat{\mathbf{W}}$ is a diagonal matrix and requires $O(n)$ operations. The matrices $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are simple kernel matrices and require $O(n^2)$ operations. The density is first found as a summation of delta functions and, then, transformed (in $O(n \log n)$ operations using FFT) to a trigonometric representation using equation (5.2). Then the potential (especially near-field) is found using a higher trapezoidal integration. This requires $O(mn \log n)$ operations to find the discrete density where m is the increased multiplication factor for the higher trapezoidal integration. In our code, we used the LU decomposition which requires $O(n^3)$ operations, but an iterative method could be used.

Remark. For the Dirichlet problem using the single-layer potential representation, the matrix system consists of a symmetric cyclic Toeplitz matrix plus a symmetric matrix. The classical approach is to use the double-layer potential representation and formulate a matrix system which consists of an identity matrix plus a nonsymmetric matrix. The advantages of our approach are that the matrix system is

symmetric and that the single-layer potential may have a better physical representation.

For the Dirichlet problem, the modified Nyström method is equivalent to the discrete delta-trigonometric and the discrete trigonometric-delta methods, and, therefore, the exponential convergence results stated in [**14**, Theorems 3.6 and 4.4] apply. Also, for nonanalytic boundary data, rapid convergence results stated in [**9**, Theorem 9] apply. For the Neumann problem, the modified Nyström method is equivalent to the Nyström method, and, therefore, all classical results for the Nyström method hold for the modified Nyström method. In the following section, we derive exponential convergence results for the Robin problem.

6. Exponential convergence for the Robin problem. In this section we show that, for the Robin problem, the three equivalent methods obtain the potential on compact sets away from the boundary with exponential convergence. Again we assume that a and b are such that the boundary integral equation (1.3) has a unique solution, and that the boundary Γ and boundary data g are analytic. We also assume that $b \neq 0$, otherwise we have the Dirichlet problem in which convergence results have been discussed at the end of the previous section. There are three different approaches: 1) use the delta-trigonometric method, 2) use the trigonometric-delta method, and 3) use a perturbed boundary integral equation. We show a detailed proof for the first approach and then discuss how to prove convergence using the second and third approaches.

For the delta-trigonometric method, we first show the stability of the integral operator associated with equation (1.3).

Theorem 6.1. *Let $s \leq s_0 < -1/2$, $\varepsilon \in (\varepsilon_1, 1]$ (ε_1 being determined by x', K, and L). Then, for sufficiently large n, there exists a constant C depending only on s_0 and Γ such that*

$$\inf_{0\neq\rho\in S_n} \sup_{0\neq\sigma\in T_n} \frac{((a(V+A)+b(W+B))\rho,\sigma)}{||\rho||_{s,\varepsilon}||\sigma||_{-s,\varepsilon^{-1}}} \geq C.$$

Proof. We first show the inf-sup condition for W. From [**13**, Theorem

3.1.1] or [**14**, Theorem 3.1], there exists a constant C_1 depending only on s_0 such that

$$||\rho||_{s,\varepsilon}^2 \leq C_1 \sum_{k \in A_n} |\hat{\rho}(k)|^2 \varepsilon^{2|k|} \underline{k}^{2s} \quad \forall\, \rho \in S_n.$$

Choose $\sigma(t) = \chi(t)\gamma(t)$ where

$$\chi(t) = \pm \frac{1}{\pi} |x'(t)|$$

and

$$\gamma(t) := \sum_{k \in \Lambda_n} \hat{\rho}(k) \varepsilon^{2|k|} \underline{k}^{2s} \exp(2\pi i k t).$$

Then χ is analytic and its Fourier coefficients have exponential decay [**22**, Section 2.1], i.e., there exist constants C_2 and $\varepsilon_2 \in (0,1)$ such that

$$|\hat{\chi}(k)| \leq C_2 \varepsilon_2^{|k|}.$$

Also, it is trivial to show that

$$\hat{\gamma}(k) := \begin{cases} \hat{\rho}(k) \varepsilon^{2|k|} \underline{k}^{2s}, & \text{if } k \in A_n, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} \hat{\sigma}(k) &= \int_0^1 \sum_{l \in \mathbf{Z}} \hat{\chi}(l) \exp(2\pi i l t) \gamma(t) \exp(-2\pi i k t)\, dt \\ &= \sum_{l \in \mathbf{Z}} \hat{\chi}(l) \hat{\gamma}(k-l) \\ &= \sum_{l \in \mathbf{Z}} \hat{\chi}(k-l) \hat{\gamma}(l). \end{aligned}$$

Then

$$
\begin{aligned}
||\sigma||^2_{-s,\varepsilon^{-1}} &= \sum_{k\in\mathbf{Z}} |\hat{\sigma}(k)|^2 \varepsilon^{-2|k|} \underline{k}^{-2s} \\
&= \sum_{k\in\mathbf{Z}} \left| \sum_{l\in\mathbf{Z}} \hat{\chi}(k-l)\hat{\gamma}(l) \right|^2 \varepsilon^{-2|k|} \underline{k}^{-2s} \\
&= \sum_{k\in\mathbf{Z}} \left| \sum_{l\in\mathbf{Z}} \hat{\chi}(k-l)\varepsilon^{|l|}\underline{l}^{s}\hat{\gamma}(l)\varepsilon^{-|l|}\underline{l}^{-s} \right|^2 \varepsilon^{-2|k|} \underline{k}^{-2s} \\
&\le \sum_{k\in\mathbf{Z}} \sum_{l\in\mathbf{Z}} |\hat{\chi}(k-l)|^2 \varepsilon^{2|l|} \underline{l}^{2s} \sum_{m\in\mathbf{Z}} |\hat{\gamma}(m)|^2 \varepsilon^{-2|m|} \underline{m}^{-2s} \varepsilon^{-2|k|} \underline{k}^{-2s} \\
&\le \sum_{k\in\mathbf{Z}} \sum_{l\in\mathbf{Z}} C_2^2 \varepsilon_2^{2|k-l|} \varepsilon^{2|l|} \underline{l}^{2s} \varepsilon^{-2|k|} \underline{k}^{-2s} \sum_{m\in\Lambda_n} |\hat{\rho}(m)|^2 \varepsilon^{2|m|} \underline{m}^{2s} \\
&= C_3 \sum_{m\in\Lambda_n} |\hat{\rho}(m)|^2 \varepsilon^{2|m|} \underline{m}^{2s}
\end{aligned}
$$

for $\varepsilon \in (\varepsilon_2, 1]$ where the Schwarz inequality is used for the inner summation. Therefore,

$$
\begin{aligned}
(W, \rho, \sigma) &= \frac{1}{2} \sum_{k\in\Lambda_n} \overline{\hat{\rho}(k)} \varepsilon^{2|k|} \underline{k}^{2s} \int_0^1 \rho(t) \exp(-2\pi i k t)\, dt \\
&= \frac{1}{2} \sum_{k\in\Lambda_n} |\hat{\rho}(k)|^2 \varepsilon^{2|k|} \underline{k}^{2s} \\
&\ge C_4 ||\rho||_{s,\varepsilon} ||\sigma||_{-s,\varepsilon^{-1}}
\end{aligned}
$$

for $\varepsilon \in (\varepsilon_2, 1]$. From [**13, 14**], we know that there is an $\varepsilon_3 \in (0,1)$ depending on the kernel K such that $V+A$ is an isomorphism from $X_{s,\varepsilon}$ to $X_{s+1,\varepsilon}$ for $\varepsilon \in (\varepsilon_3, 1]$. But $X_{s+1,\varepsilon}$ is compactly contained in $X_{s,\varepsilon}$ [**3**], and, therefore, $V+A$ is a compact operator on $X_{s,\varepsilon}$ for $\varepsilon \in (\varepsilon_3, 1]$. Also, by using similar arguments as in [**3**], there is an $\varepsilon_4 \in (0,1)$ depending on the kernel L such that B is a compact operator on $X_{s,\varepsilon}$ for $\varepsilon \in (\varepsilon_4, 1]$. By setting $\varepsilon_1 := \max(\varepsilon_2, \varepsilon_3, \varepsilon_4)$ and using a compactness argument similar to [**3, 13, 14**], we conclude the theorem. □

The next theorem gives the existence and quasioptimality of the approximate density. Its proof is omitted and can be invoked from the standard theory of Galerkin methods.

Theorem 6.2. *There exists a constant N, depending only on Γ, such that for all $n \geq N$ and g in $\cup\{X_{s,\varepsilon} \mid s \in \mathbf{R}, \varepsilon > 0\}$, the delta-trigonometric method* (3.1) *obtains unique solutions, $\phi_n \in S_n$. Moreover, if $s < -1/2$, $\varepsilon \in (\varepsilon_1, 1]$ (ε_1 being determined in Theorem* 6.1*), $g \in X_{s,\varepsilon}$, and $n \geq N$, then there exists a constant, C, depending only on $\varepsilon, s,$ and Γ such that*

$$||\phi - \phi_n||_{s,\varepsilon} \leq C \inf_{\rho \in S_n} ||\phi - \rho||_{s,\varepsilon}.$$

The next theorem states the convergence of the approximate density and has been proven in [**3, 13, 14**] by using a projection operator and Theorem 6.2. Its proof is omitted.

Theorem 6.3. *Let $s < -1/2$, $t \in [s, 0]$, $n \geq N$, and $\phi \in H^t$. Then for $\varepsilon \in (\varepsilon_1, 1]$ (ε_1 being determined in Theorem* 6.1*), there exists a constant C depending only on $\varepsilon, s,$ and Γ, such that*

$$||\phi - \phi_n||_{s,\varepsilon} \leq C\varepsilon^{n/2}(\pi n)^{s-t}||\phi||_t.$$

Since the single-layer potential was used in the Dirichlet case, the next theorem is similar to [**13**, Theorem 3.1.7] and [**14**, Theorem 3.6], and will not be proved.

Theorem 6.4. *Let $n \geq N$, $\phi \in H^t$, $t \leq 0$, and Ω_K be any compact set away from the boundary. Then, for sufficiently large N and for any multi-index β, there exist constants C and $\varepsilon \in (0, 1)$ depending only on $t, N, \Omega_K,$ and Γ, such that*

$$||\partial^\beta (U - U_n)||_{L^\infty(\Omega_K)} \leq C\varepsilon^n ||\phi||_t$$

where U and U_n are defined in (1.2) *and* (3.2)*, respectively.*

The above theorem shows that the semi-discrete delta-trigonometric method obtains exponential convergence for the potential on compact sets away from the boundary. For the discrete delta-trigonometric

method, we first review a numerical integration result where the trapezoidal rule is used and the integrand is an analytic function times a trigonometric polynomial.

Theorem 6.5. *Let f be an analytic 1-periodic function and define*

$$f_k := \int_0^1 f(s) \exp(2\pi i k s)\, ds$$

and

$$\tilde{f}_k := \frac{1}{n} \sum_{l=0}^{n-1} f(l/n) \exp(2\pi i k l/n).$$

Then there exist constants C and $\varepsilon \in (0,1)$, depending only on f, such that

$$|f_k - \tilde{f}_k| \leq C\varepsilon^n$$

for all $k \in \Lambda_n$.

Proof. See [**13**, Theorem 3.3.2] and [**14**, Theorem 4.2]. □

Theorem 6.6. *Let $n \geq N$, $\phi \in H^t$, $t \leq 0$, and Ω_K be any compact set away from the boundary. Then, for sufficiently large N and for any multi-index β, there exist constants C and $\varepsilon \in (0,1)$ depending only on t, N, Ω_K, g, and Γ, such that*

$$||\partial^\beta (U - \tilde{U}_n)||_{L^\infty(\Omega_K)} \leq C\varepsilon^n ||\phi||_t$$

where U and $\tilde{U}_n$ are defined in (1.2) *and* (3.3)*, respectively.*

Proof. By Theorem 6.4, it suffices to show that

$$||\partial^\beta (U_n - \tilde{U}_n)||_{L^\infty(\Omega_K)} \leq C\varepsilon^n ||\phi||_t$$

where U_n is defined in (3.2). Using the same argument as in [**13**, Theorem 3.3.3] and [**14**, Theorem 4.3], there exist constants C and $\varepsilon \in (0,1)$, depending only on g and Γ, such that

$$||\mathbf{g} - \tilde{\mathbf{g}}|| < C\varepsilon^n, \qquad ||\mathbf{A} - \tilde{\mathbf{A}}|| < C\varepsilon^n, \qquad ||\mathbf{B} - \tilde{\mathbf{B}}|| < C\varepsilon^n.$$

Define

$$\mathbf{S} := a(\mathbf{A} + \mathbf{V}) + b(\mathbf{B} + \mathbf{W}),$$
$$\tilde{\mathbf{S}} := a(\tilde{\mathbf{A}} + \mathbf{V}) + b(\tilde{\mathbf{B}} + \mathbf{W}).$$

Using a similar argument as in [**13**, Theorem 3.2.2] and $b \neq 0$, we note that $||\mathbf{S}^{-1}|| \leq C$. Also, note that

$$\begin{aligned}\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}} &= \mathbf{S}^{-1}[\mathbf{g} - \tilde{\mathbf{g}} - (\mathbf{S} - \tilde{\mathbf{S}})\tilde{\boldsymbol{\alpha}}] \\ &= \mathbf{S}^{-1}[\mathbf{g} - \tilde{\mathbf{g}} - (\mathbf{A} - \tilde{\mathbf{A}} + \mathbf{B} - \tilde{\mathbf{B}})\tilde{\boldsymbol{\alpha}}]\end{aligned}$$

so that $||\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}|| \leq C\varepsilon^n$ where $C > 0$ and $\varepsilon \in (0, 1)$ depend only on g and Γ. Using the same argument as in [**13**, Theorem 3.3.4] and [**14**, Theorem 4.4], we conclude the theorem. □

The above theorem implies that, for the Robin problem, the discrete delta-trigonometric method obtains exponential convergence for the potential on compact sets away from the boundary. Therefore, the modified Nyström and discrete trigonometric-delta methods also obtain exponential convergence for the potential on compact sets away from the boundary for the Robin problem.

The second approach is to extend the exponential convergence results for the trigonometric-trigonometric method in [**31**] to the Robin problem. Then a similar technique as for the delta-trigonometric method can be used to show that the effect of numerical integration is exponential. Also, a similar technique as in [**9**] can be used to show rapid convergence for the density. Although no proof is presented for this approach, we conjecture that the proofs can be derived without much difficulty.

The third approach is to consider a perturbation of the original integral equation (1.3). No proof is presented for this approach, but a brief explanation is provided. Define V_n such that

$$V_n\phi(s) = \sum_{k=1}^{n} \widehat{\mathbf{V}\phi}(k)\xi_k(s) = \sum_{k=1}^{n} \hat{\mathbf{V}}_{kk}\hat{\phi}(k)\xi_k(s).$$

The perturbed integral equation is

$$a(\mathbf{A} + \mathbf{V}_n)\gamma_n(s) + b(\mathbf{B} + \mathbf{W})\gamma_n(s) = g(s) \quad \text{for } s \in [0, 1].$$

For sufficiently large n, equation (6.1) has a unique solution, and the error of γ_n is exponential in the Fourier norm. Here, the definition of the Fourier norm is used to bound $||(V - V_n)\phi||_{s,\varepsilon}$, and then the perturbation theorem [**7**, p. 94] is used to bound $||\phi - \gamma_n||_{s,\varepsilon}$. As in the previous approaches, Theorem 6.6 is derived by showing that the effect of numerical integration is exponentially small.

7. Numerical results. We present two examples using the modified Nyström, the discrete delta-trigonometric, and the discrete trigonometric-delta methods. All three methods obtain the same solution except for machine errors; however, they require different computational time.

Example 7.1. Dirichlet data. This example is the same ellipse example as in [**13, 14**]. Here we compare the CPU times for all three equivalent methods.

Boundary: $x_1^2/4 + x_2^2 = 1/25$

Data: $g = 5x_1/2$

Exact solution:

$$u := \begin{cases} 5x_1/2, & \text{if } (x_1, x_2) \in \Omega, \\ 5x_1 - w, & \text{if } (x_1, x_2) \in \Omega_c \text{ and } x_1 \geq 0, \\ 5x_1 + w, & \text{if } (x_1, x_2) \in \Omega_c \text{ and } x_1 \leq 0, \end{cases}$$

where

$$w = \sqrt{\frac{25(x_1^2 + x_2^2) - 3 + \sqrt{(25(x_1^2 + x_2^2) - 3)^2 + 2500x_1^2x_2^2}}{2}}.$$

Figure 1A shows the logarithmic error of the approximate potential for the modified Nyström method on the line $x_1 = 2x_2$. Here the line crosses the boundary at $x_1 = 0.2\sqrt{2}$. Note that the convergence is only linear across the boundary. Although not shown, we noted that the delta-trigonometric and trigonometric-delta methods obtain the same approximate potential except when the errors are near machine errors. In Figure 1B, we transform the density from a delta representation to a trigonometric representation using equation (5.2), and then evaluate approximate potential using a (8X) higher trapezoidal rule, i.e.,

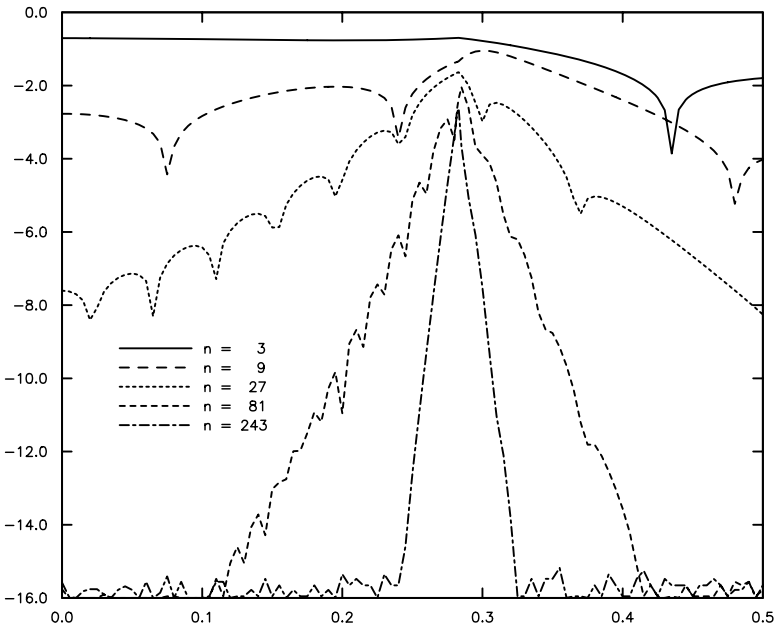

FIGURE 1A. Log(error) versus x_1 on the line $x_1 = 2x_2$ for Example 7.1. No higher trapezoidal integration is used.

trapezoidal rule with 8 times more integration points. We note that the approximate potential at the boundary converges only slightly better than linear, but the approximate near-field potential has improved considerably.

In Table 1A, a comparison in CPU times (APOLLO DN 3000) is shown for $n = 243$ using the three equivalent methods. Here we noted a significant CPU saving, especially in creating the matrices. The LU decomposition was used to solve for the density and requires similar CPU for all three methods. However, further computational saving can be achieved using an iteration method. No higher trapezoidal integration was used for these runs.

TABLE 1A. CPU times for $n = 243$ in Example 7.1.

	modified Nyström	delta-trig	trig-delta
preliminary	1.0187	1.0280	1.0204
g	0.0109	0.7790	0.0109
V	3.0475	7.0117	6.8817
A	14.6405	200.3261	198.5745
W	0.0004	0.0025	0.0037
B	0.0010	0.0046	0.0011
$a(\mathbf{A}+\mathbf{V})+b(\mathbf{B}+\mathbf{W})$	2.2929	2.7085	2.6887
$\tilde{\alpha}$ or $\tilde{\beta}$	126.2318	127.3710	132.7381
$\tilde{U}_n$	3.6433	3.6411	3.7771
$\lvert\tilde{U}_n - U\rvert$	0.8273	0.7901	0.7981
total	151.7141	343.6626	346.4942

Example 2. Robin data. In this example we consider the 2-D potential problem with Robin boundary condition in the interior region. As in Example 1, we compare the computational time between the three analogous methods and show that the numerical solution can be improved by transforming the density from a delta representation to a trigonometric representation and, then, using a higher trapezoidal integration.

Boundary: $x_1^2/a^2 + x_2^2/b^2 = 1$

Data:

$$g(t) = \log|x(t) - (c,d)| + ((x(t) - (c,d)) \cdot \nu(t))/|x(t) - (c,d)|^2.$$

Exact interior solution: $u(z) := \log|z - (c,d)|$

For this example we used $a = 2$, $b = 1$, $c = 4$, and $d = 3$. As in the previous example, Figure 2A shows the logarithmic error of the approximate potential on the line $x_1 = 2x_2$ without a higher trapezoidal rule, and Figure 2B is the same with an (8X) higher trapezoidal rule. Here the line meets the boundary at $x_1 = 0.1\sqrt{2}$. Again, note that the

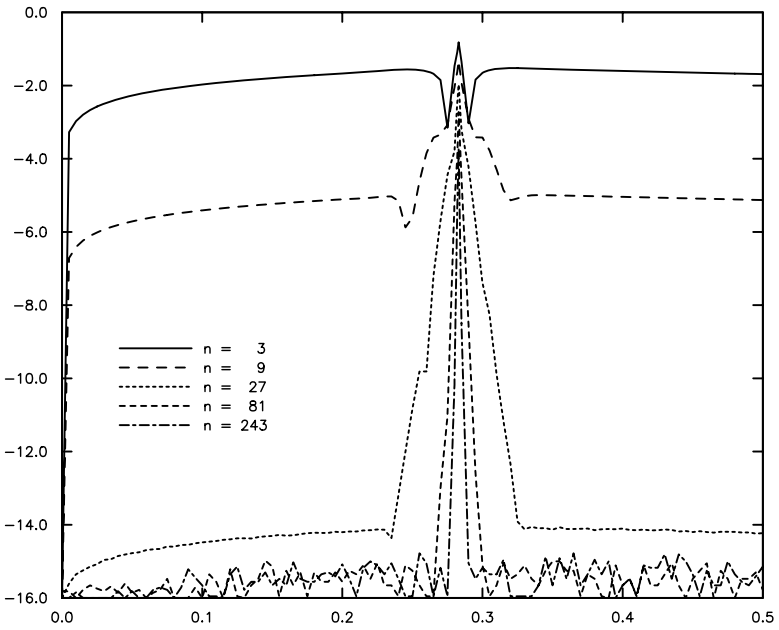

FIGURE 1B. Log(error) versus x_1 on the line $x_1 = 2x_2$ for Example 7.1. 8X-higher trapezoidal integration is used.

approximate near-field potential improved, but the convergence rate is only slightly better than linear on the boundary. As in Example 1, Table 2 shows the CPU time for the three analogous methods. Again, the modified Nyström method uses significantly less time to create the matrices.

8. Summary. In this paper we showed that the discrete delta-trigonometric and the discrete trigonometric-delta methods obtain the same discrete density. Moreover, we can interchanged between the delta and trigonometric representations of the density. The trigonometric representation of the density is more desirable so that a higher trapezoidal integration can be used to obtain better near-field results. Then we introduced another equivalent method called the modified Nyström method which is computationally quicker and requires a simple $O(n^2)$ operation to form the matrices. Thus, our technique is to

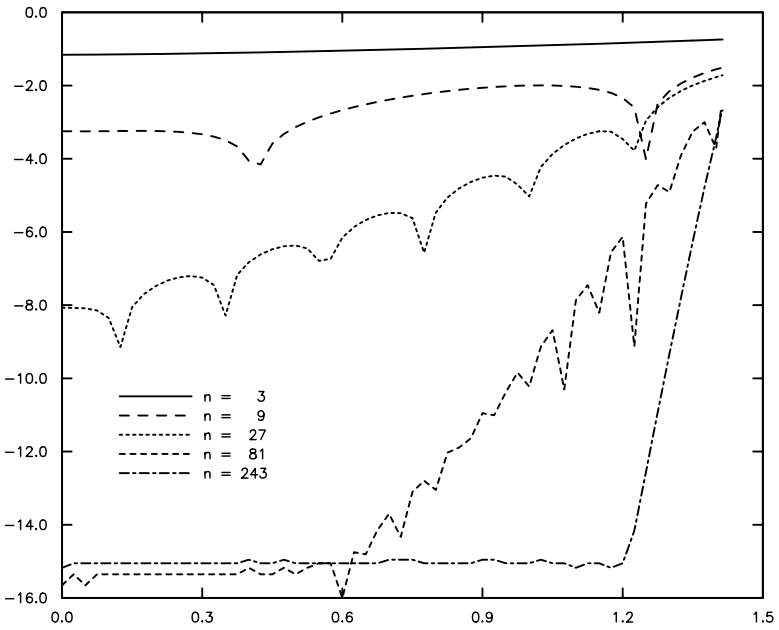

FIGURE 2A. Log(error) versus x_1 on the line $x_1 = 2x_2$ for Example 6.2. No higher trapezoidal integration is used.

find the delta representation of the density using the modified Nyström method, convert to the trigonometric representation of the density, and use a higher trapezoidal integration to find the approximate potential.

Second, for the Dirichlet problem, we noted that using the single-layer potential representation instead of the double-layer potential representation to formulate the boundary integral equation is more feasible because the matrix system is symmetric.

Third, we proved that the three equivalent methods obtain the potential on compact sets away from the boundary with exponential convergence for the Robin problem.

Fourth, we noted that the perturbed integral equation (6.1) is more suitable than the original equation since the logarithmic singularity problem is removed. The usage of this equation should be examined further in the future.

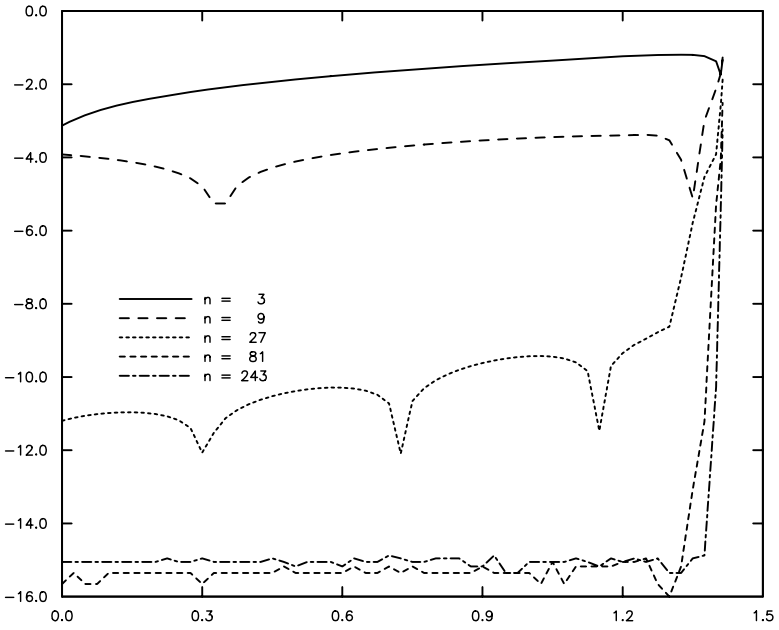

FIGURE 2B. Log(error) versus x_1 on the line $x_1 = 2x_2$ for Example 6.2. 8X-higher trapezoidal integration is used.

TABLE 2. CPU times for $n = 243$ in Example 7.1.

	modified Nyström	delta-trig	trig-delta
preliminary	0.7120	0.6997	0.7028
$\mathbf{g}$	0.0487	0.8114	0.0488
$\mathbf{V}$	3.0675	6.5015	6.1772
$\mathbf{A}$	14.5537	199.1406	198.0494
$\mathbf{W}$	1.5918	6.9492	7.1744
$\mathbf{B}$	8.8122	193.0474	194.2434
$a(\mathbf{A}+\mathbf{V})+b(\mathbf{B}+\mathbf{W})$	4.3190	5.3929	5.1352
$\tilde{\alpha}$ or $\tilde{\beta}$	126.8790	129.6312	132.0138
$\tilde{U}_n$	2.1204	2.1836	2.1324
$\lvert\tilde{U}_n - U\rvert$	0.5252	0.5099	0.5300
total	162.6295	544.8674	546.2072

Finally, we note that n, the number of mesh, and m, the increased multiplication factor of the higher trapezoidal integration, can be chosen to optimize the numerical results. If we are interested in the potential at near-field with a specified error criterion, then we would choose n such that the far-field results satisfy a little better than the error criterion, and then increase m until the near-field results satisfy the error criterion.

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