JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 5, Number 2, Spring 1993

A MODIFIED APPROACH TO THE NUMERICAL SOLUTION OF LINEAR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

J. ABDALKHANI

Dedicated to the memory of Paul (Bud) Beesack for his contribution to this work and his love for mathematics.

ABSTRACT. The (unknown) exact solution of a weakly singular Volterra integral equation of the second kind (with smooth kernel and forcing function) normally has unbounded derivatives at the initial point of the interval of integration. Thus, it is not possible to approximate the exact solution with a high rate of convergence while using the ordinary polynomial collocation methods (with uniform meshes) and the resulting Runge-Kutta and block-by-block methods. To improve the accuracy of approximation using these methods, we produce a modified integral equation which is obtained from and closely related to the original equation. This new equation has a singular forcing function but possesses a smoother (unknown) exact solution. We shall prove that it is possible to approximate the solution of this new equation by the above mentioned numerical methods with a high order of convergence and consequently obtain an accurate approximation for the original equation.

1. Introduction. We study Volterra integral equations of the second kind with weakly singular kernels of the form

(1.1)
$$y(t) = g(t) + \int_0^t K(t,s)(t-s)^{-\alpha} y(s)\, ds, \quad 0 < \alpha < 1,\ t \in I = [0,a]$$

where $g \in C^m[I]$ and $K \in C^m(T)$, $T = \{(t,s) : 0 \le s \le t \le a\}$. These equations arise in many practical applications. Specifically, for $\alpha = 1/2$, see [**4**].

Received by the editors on January 7, 1993.
AMS (MOS)*Subject Classification.* 65R20-45Do5.
Key words. Weakly singular integral equations, collocation, Runge-Kutta, block-by-block.

Throughout we assume the existence of a unique continuous solution on the interval I. The existence of a unique solution under proper conditions is discussed, for example, in [**4, 9, 11**]. In general, this solution cannot be found by analytical methods and therefore it is important to find an accurate approximation using numerical techniques. It is well known that the exact analytical solution of equation (1.1), when g and K are smooth, is typically nonsmooth at $t = 0$, see [**4**, Section 1.3.5]. This means that if a numerical method is to possess a high order of convergence on the whole interval $I = [0, a]$, one has to take into account the singular behavior of the exact solution near the origin. In linear multistep methods this is reflected in the special construction of the starting weights, see [**13**]. In collocation methods either graded meshes have been used instead of uniform ones or nonpolynomial spaces were considered instead of polynomial spaces, see [**5**] and [**6**]. We prefer polynomial spline collocation with uniform meshes, since the global convergence results can be established more easily as compared with other methods, see [**4**, Chapter 6]. Moreover, a class of implicit Runge-Kutta (IRK) methods can be obtained from these methods, and RK methods are more practical. For connection of collocation with IRK methods, see [**4**, Section 5.2.1]. Block-by-block methods can be obtained from IRK, see [**11**, pp. 114–116 and 136–137]. These methods, although possessing the same order of convergence as the collocation and IRK methods, showed better stability when solving numerical examples. In passing, we would like to mention that there are simpler numerical methods in the literature which give good accuracy, but on an interval of the form $[\varepsilon_0, a]$, $\varepsilon_0 > 0$, which does not include the origin, see [**9, 12**].

The question is, how do we take into account the singular behavior of $y(t)$ at $t = 0$, to be able to use the ordinary polynomial collocation with uniform meshes and retain a high order of convergence on the whole interval I? We will simply add a known function $f(t)$ to a new unknown function $Y(t)$ such that

$$y(t) = Y(t) + f(t), \tag{1.2}$$

where $Y(t) \in C^m[I]$, and then solve

$$Y(t) = G(t) + \int_0^t K(t,s)(t-s)^{-\alpha} Y(s)\, ds \tag{1.3}$$

where

$$G(t) = g(t) - f(t) + \int_0^t K(t,s)f(s)(t-s)^{-\alpha}\, ds. \tag{1.4}$$

The splitting of y into two functions is not unique, but if $f(t)$ is chosen properly then the exact solution $Y(t)$ of (1.3) is, in fact, smooth of class C^m, and it can be approximated accurately by means of collocation, IRK or block-by-block methods, as explained earlier. Equation (1.2) then can be used to approximate $y(t)$. In Section 2 we demonstrate how to extract $f(t)$, the singular part of $y(t)$.

Section 3 is mainly devoted to the convergence of collocation methods. It is known that when g and K in (1.1) are of continuity class C^m and if we assume $y \in C^m[I]$ and $y(t)$ is approximated (properly) by the collocation methods with uniform meshes, then one would obtain a global convergence of order m. For more details, see [**1**] and [**7**, p. 411, Table 1.1]. However, as mentioned before, $y(t)$ is not smooth at $t=0$ and therefore we are approximating $Y(t)$. Although $Y \in C^m[I]$, the forcing function $G(t)$ given by (1.4) is singular. Moreover, the integral on the right hand side of (1.4) normally must be approximated; therefore, $G(t)$ must be replaced by an approximation function $\hat{G}(t)$. This makes our assumptions different from the known results in literature and we will therefore give a complete proof of the convergence of collocation methods (Theorem 3.2). We also discuss briefly the IRK and block-by-block methods. This discussion is not self-contained and we refer the interested reader to [**4, 11**].

Section 4 contains numerical examples. Our approach has its drawbacks. These are discussed in a conclusion.

2. The singular part of solutions.

Case I. A simple illustration. Consider the most practical case, $\alpha = 1/2$. It has been shown that $y(t)$ in (1.1) can be written as

$$y(t) = u(t) + \sqrt{t}v(t) \quad \text{where } u, v \in C^m,$$

see [**4**, p. 29]. Now we write $v(t)$ as $v(t) = a + bt + ct^2 + Z(t)$, for some $Z(t)$, $Z(t) = dt^3 + \dots$. Thus,

$$\begin{aligned} y(t) &= u(t) + \sqrt{t}[(a+bt+ct^2) + Z(t)] \\ &= u(t) + \sqrt{t}Z(t) + \sqrt{t}(a+bt+ct^2) = Y(t) + f(t) \end{aligned}$$

where

$$Y(t) = u(t) + \sqrt{t}Z(t) \quad \text{and} \quad f(t) = \sqrt{t}(a + bt + ct^2).$$

$Y(t)$ is unknown and $Y \in C^3$, while y is only continuous. $f(t)$ will be a known function when a, b, and c are known. We will find these constants and show that they only depend on $f^{(i)}(0)$, $i = 0, 1, 2$, and $\partial^{i+j}K(0,0)/\partial t^i \partial s^j$, $i, j = 0, 1, 2$, $0 \le i + j \le 2$.

Case II. The general case, $0 < \alpha < 1$. Let $y(t)$ be the exact solution of the integral equation (1.1). Then $y(t) \in C[I] \cap C^m(0, a]$ and has the form

$$y(t) = g(t) + \sum_{j=1}^{\infty} y_j(t) t^{j(1-\alpha)} \tag{2.1}$$

where $y_j(t) \in C^m(I)$. For more details, see [**4**, p. 30, Theorem 1.3.15 with $g_1 = g$ and $g_2 = 0$]. Now we expand $y_j(t)$ by the Taylor series expansion at $t = 0$ as

$$y_j(t) = \sum_{i=0}^{m} y_j^{(i)}(0) \frac{t^i}{i!} + R_j(t), \tag{2.2}$$

where $R_j(t) \to 0$ as $t \to 0$ faster than t^m. Therefore, (2.1) can be written as

$$y(t) = g(t) + \sum_{j=1}^{\infty} \sum_{i=0}^{m} \frac{y_j^{(i)}(0)}{i!} t^{i+j(1-\alpha)} + \sum_{j=1}^{\infty} R_j(t) t^{j(1-\alpha)}. \tag{2.3}$$

Recalling that $g(t) \in C^m[I]$ one can write $y(t)$ as

$$y(t) = Y_m(t) + f_{m,\alpha}(t), \tag{2.4}$$

with

$$Y_m(t) = \sum_{n=0}^{m} Y_m^{(n)}(0) \frac{t^n}{n!} + R(t) \tag{2.5}$$

and

$$f_{m,\alpha}(t) = {\sum_{i,j}}' a_{i,j} t^{i+j(1-\alpha)} \tag{2.6}$$

where Y_m and $f_{m,\alpha}$ are the regular and singular part of y, respectively. $R(t)$ is a remainder function, which is in $C^m[I]$ and approaches zero faster than t^m, as $t \to 0$. Σ' signifies a sum over all i, j so that $i+j(1-\alpha)$ runs through all *distinct non-integer* values greater than 0 and less than m that can be written in this form.

Examples.

$$f_{3,1/2}(t) = \sum_{n=1,3,5} a_{0n} t^{n/2}, \tag{2.7a}$$

$$f_{3,1/3}(t) = \sum_{n=1,2,4} a_{0n} t^{2n/3} + \sum_{n=1,2} a_{1n} t^{1+2n/3}, \tag{2.7b}$$

$$f_{3,2/3}(t) = \sum_{n=1,2,4,5,7,8} a_{0n} t^{n/3}, \tag{2.7c}$$

$$f_{3,0.9}(t) = \sum_{i=0}^{2} \sum_{j=1}^{9} a_{ij} t^{i+j/10}, \tag{2.7d}$$

where $Y_m^{(n)}(0)$ and a_{ij} in (2.5) and (2.6) are constants to be determined. To obtain these constants, we expand g and K in Taylor polynomials of order m plus a remainder, and substitute everything into the integral equation (1.1), which yields the following equation

(2.8)

$$\begin{aligned}
\sum_{n=0}^{m} Y_m^{(n)}(0)\frac{t^n}{n!} + {\sum_{i,j}}' a_{ij} t^{i+j(1-\alpha)} &= \sum_{n=0}^{m} g^{(n)}(0) t^n/n! \\
&+ \sum_{n=0}^{m} \sum_{i=0}^{m} \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix} \frac{Y_m^{(n)}(0)}{n!i!} \frac{\partial^i K(0,0)}{\partial t^j \partial S^{i-j}} t^{i+n+(1-\alpha)} B(i-j+n+1, 1-\alpha) \\
&+ {\sum_{i,j}}' \sum_{l=0}^{m} \sum_{q=0}^{l} \frac{\begin{bmatrix} l \\ q \end{bmatrix}}{l!} a_{ij} \frac{\partial^l K(0,0)}{\partial t^q \partial s^{t-q}} t^{l+i+j(1-\alpha)+(1-\alpha)} \\
&\cdot B(l-q+i+j(1-\alpha)+1, 1-\alpha) + \text{ Remainder.}
\end{aligned}$$

Collecting the coefficients of the various powers of t, one obtains a sequence of equations which can be solved successively, either algebraically or numerically, to give a_{ij} and $Y_m^{(n)}(0)$, $i = 0, \dots, m$.

Although it is possible to derive these coefficients for any m and α from (2.8), we avoid doing this, since the corresponding equations are lengthy. Instead, we give the successive equations obtained from (2.8) for $m = 3$ and $\alpha = 1/2$ and $2/3$, taking advantage of the following notations:

$$K_1 := K(0,0) \tag{2.9a}$$

$$T_1 := \frac{\partial K(0,0)}{\partial t}, \qquad S_1 := \frac{\partial K(0,0)}{\partial S} \tag{2.9b}$$

$$T_2 := \frac{\partial^2 K(0,0)}{\partial t^2}, \qquad S_2 := \frac{\partial^2 K(0,0)}{\partial S^2} \tag{2.9c}$$

$$M_1 := \frac{\partial^2 K(0,0)}{\partial t \partial S} \quad \text{and} \quad Y_{m,0}^{(n)} := Y_m^{(n)}(0) \tag{2.9d}$$

Example 2.1. $m = 3$, $\alpha = 1/2$, $f_{3,1/2}(t)$ given by (2.7a), from (2.8) we obtain

$$Y_{3,0} = g(0), \qquad a_{01} = Y_{3,0} K_1 B(1/2, 1) = 2g(0)K_1,$$

$$Y'_{3,0} = g'(0) + a_{01} K_1 B(1/2, 3/2) = g'(0) + \pi g(0) K_1^2,$$

$$a_{03} = Y_{3,0}[T_1 B(1/2,1) + S_1 B(1/2,2)] + Y'_{3,0} K_1 B(1/2,2),$$

$$Y''_{3,0} = g''(0) + 2a_{01}[S_1 B(1/2,5/2) + T_1 B(1/2,3/2)] + 2a_{03} K_1 B(1/2,5/2),$$

$$\begin{aligned} a_{05} = Y_{3,0}\left[\frac{1}{2} S_2 B\left(\frac{1}{2},3\right) + M_1 B\left(\frac{1}{2},2\right) + \frac{1}{2} T_2 B\left(\frac{1}{2},1\right)\right] \\ + Y'_{3,0}\left[S_1 B\left(\frac{1}{2},3\right) + T_1 B\left(\frac{1}{2},2\right)\right] + \frac{1}{2} Y''_{3,0} K_1 B\left(\frac{1}{2},3\right). \end{aligned}$$

Example 2.2. $m = 3$, $\alpha = 2/3$, $f_{3,2/3}(t)$ given by (2.7c)

$$Y_{3,0} = g(0), \qquad a_{01} = Y_{3,0}K_1B(1, 1/3) = 3K_1g(0),$$

$$a_{02} = a_{01}K_1B(4/3, 1/3), \qquad Y'_{3,0} = g'(0) + a_{02}K_1B(5/3, 1/3),$$

$$a_{04} = Y_{3,0}[S_1B(2, 1/3) + T_1B(1, 1/3)] + Y'_{3,0}K_1B(2, 1/3),$$

$$a_{05} = a_{01}[S_1B(5/3, 1/3) + T_1B(4/3, 1/3)] + a_{04}K_1B(7/3, 1/3),$$

$$Y''_{3,0} = g''(0) + 2a_{02}[S_1B(8/3, 1/3) + T_1B(5/3, 1/3)] + 2a_{05}K_1B(8/3, 1/3),$$

$$\begin{aligned} a_{07} = Y_{3,0}&\left[\frac{1}{2}S_2B\left(3, \frac{1}{3}\right) + M_1B\left(2, \frac{1}{3}\right) + \frac{1}{2}T_2B\left(1, \frac{1}{3}\right)\right] \\ &+ Y'_{3,0}\left[S_1B\left(3, \frac{1}{3}\right) + T_1B\left(2, \frac{1}{3}\right)\right] + Y''_{3,0}\left(\frac{1}{2}K_1B\left(3, \frac{1}{3}\right)\right), \end{aligned}$$

$$\begin{aligned} a_{08} = a_{01}&\left[\frac{1}{2}S_2B\left(\frac{10}{3}, \frac{1}{3}\right) + M_1B\left(\frac{7}{3}, \frac{1}{3}\right) + \frac{1}{2}T_2B\left(\frac{4}{3}, \frac{1}{3}\right)\right] \\ &+ a_{04}\left[S_1B\left(\frac{10}{3}, \frac{1}{3}\right) + T_1B\left(\frac{7}{3}, \frac{1}{3}\right)\right] + a_{07}K_1B\left(\frac{10}{3}, \frac{1}{3}\right). \end{aligned}$$

3. Convergence of collocation, IRK and block-by-block methods. A collocation method is based on the principle of approximating the exact solution of a given functional equation in a suitably chosen finite-dimensional function space such that the approximating element satisfies the functional equation on a certain finite discrete subset of the interval on which the equation is to be solved. This finite subset is called the collocation set. Collocation methods for Volterra integral equations are discussed in detail in [**4**, Chapter 5 and pp. 347–398]. Let $I = [0, a]$ be partitioned by the points $t_k = kh$, $k = 0, \dots, N$, $h = a/N$, $N \geq 1$. Let $Z_N = \{t_0, t_1, \dots, t_N\}$. We denote the set of all piecewise polynomials of degree m, (which may possess finite discontinuities at the knots t_k) by $S_m^{-1}(Z_N)$, where $m + 1$ is the same as the degree of smoothness of Y. We approximate the exact solution of

$$(3.1) \quad Y(t) = G(t) + \int_0^t K(t, s)Y(s)(t-s)^{-\alpha}\, ds, \quad 0 < \alpha < 1, \ t \in [0, a],$$

where

$$G(t) := g(t) - f(t) + \int_0^t K(t,s)f(s)(t-s)^{-\alpha}\,ds \tag{3.2}$$

and $f(t) := f_{m,\alpha}(t)$ and $Y(t) := Y_m(t)$ are given by (2.5) and (2.6), with an element of $S_m^{-1}(Z_N)$. We define our collocation set $X(N)$ by

$$X(N) = \bigcup_{i=0}^{N-1} X_i$$

(3.3)

$$X_i = \{t_n + c_i h : 0 \le c_1 < c_2 < c_3 \cdots < c_{m+1} \le 1\} \quad n = 0, 1, \dots, N-1.$$

Let $u \in S_m^{-1}(Z_N)$ satisfy equation (3.1) on $X(N)$ for $t \in [t_n, t_{n+1}]$ and $t = t_n + c_i h$. Then we have

$$\begin{aligned} u_n(t_n + c_i h) &= G(t_n + c_i h) \\ &\quad + h^{1-\alpha}\Bigg[\sum_{t=0}^{n-1}\int_0^1 \frac{K(t_n+c_i h, t_l+\tau h)u_l(t_l+\tau h)}{(n-l+c_i-\tau)^\alpha}\,d\tau \\ &\qquad + \int_0^{c_i}\frac{K(t_n+c_i h, t_n+\tau h)u_n(t_n+\tau h)}{(c_i-\tau)^\alpha}\,d\tau\Bigg]. \end{aligned} \tag{3.4}$$

We also recall that $G(t)$ is given by (3.2) and even if $f(t)$ is evaluated exactly, the integral in the definition of $G(t)$ must be replaced by a quadrature formula; therefore, $G(t)$ should be replaced by $\hat{G}(t)$ in a numerical approach.

Now approximate the (moment) integrals in (3.4) by quadrature formulas and let $\hat{u}$ be the numerical approximation to u in (3.4). Then we have

(3.5)

$$\begin{aligned} \hat{u}_n(t_n + c_i h) &= \hat{G}(t_n + c_i h) \\ &\quad + h^{1-\alpha}\Bigg[\sum_{l=0}^{n-1}\sum_{j=1}^{m+1} w_{i,j}^{n,l} K(t_n + c_i h, t_l + c_j h)\hat{u}_l(t_l + c_j h) \\ &\qquad + \sum_{j=1}^{q} b_{ij} K(t_n + c_i h, t_n + c_j h)\hat{u}_n(t + c_j h)\Bigg]. \end{aligned}$$

We note here that if $q = i - 1$ we have an explicit numerical method, and if $q = i$ we obtain an implicit method. In (3.5), $w_{i,j}^{n,l}$ and b_{ij} are quadrature weights defined by

$$w_{i,j}^{n,l} = \int_0^1 (n - l + c_i - \tau)^{-\alpha} l_j(\tau)\, d\tau, \tag{3.6a}$$

$$b_{ij} = \int_0^{c_i} (c_i - \tau)^{-\alpha} l_j^*(\tau)\, d\tau, \tag{3.6b}$$

where $l_j(\tau)$ and $l_j^*(\tau)$ are the corresponding Lagrange interpolation polynomials. We note $l_j^*(\tau)$ depends on i, j, and also on q.

Theorem 3.1. *Let $Y(t)$, the exact solution of* (3.1)*, belong to $C^m(I)$, and let $u(t)$ be the approximation to Y which is obtained by collocation method,* (*equation* (3.4))*. Then*

$$|Y(t) - u(t)| \le Ch^m, \quad \text{for all } t \in I \text{ as } h \to 0^+,\ Nh = a \tag{3.7}$$

where C is a constant independent of h and N. This result is valid for any choice of parameters $\{c_i\}_{i=1}^{m+1}$, $0 \le c_1 < \cdots < c_{m+1} \le 1$. See [**1**, *Theorem* 3.1.1].

Remark 3.2. We note here that the proof of Theorem 3.1 depends only on the smoothness of $Y(t)$. In the literature one uses the smoothness of the kernel function K and the forcing function g in equations of the form

$$z(t) = g(t) + \int_0^t K(t,s) z(s)\, ds$$

(i.e., the nonsingular case), to guarantee the smoothness of the exact solution $Z(t)$. However, in the weakly singular case (1.1), the smoothness of K and g leads to nonsmoothness of $y(t)$. The whole purpose of creating the nonsmooth forcing function $G(t)$ in (3.1) is to obtain smoothness for $Y(t)$. Once this smoothness is achieved, we can apply Theorem 3.1.

Theorem 3.2. *Let $u(t)$ be as given in Theorem* 3.1 *and $\hat{u}$ as in* (3.5)*. Suppose the moment integrals in* (3.4) *are evaluated by interpolatory* *quadrature formulas based on* m *(distinct) abscissas with corresponding quadrature errors* $E_{n,l}^i$ *and* E_n^i*, respectively, with* $E_{n,l}^i \leq \tilde{C}_2 h^{r_2}$ *and* $E_n^i \leq \tilde{C}_3 h^{r_3}$*, for some constants* $\tilde{C}_2$ *and* $\tilde{C}_3$*, and for* $0 \leq l \leq n$ *and* $0 \leq n \leq N$*, respectively, and* $i = 1, 2, \dots, m+1$*; in addition, suppose that*

$$|G(t) - \hat{G}(t)| \leq \tilde{C}_1 h^{r_1}.$$

Then for $Y \in C^m[0, a]$ *in* (3.1)*, we have*

$$|Y(t) - \hat{u}(t)| \leq \hat{C} h^p, \quad \text{for } t = t_n + c_i h, \; n = 0, 1, \dots, N \tag{3.8}$$

where $p = \min(m, r_1, r_2 - \alpha, 1 + r_3 - \alpha)$ *and* $\hat{c}$ *is a constant independent of* h *and* N*,* $Nh = a$*.*

To prove this theorem we need the following lemmas.

Lemma 3.2. *Let* x_i*,* $i = 0, 1, \dots, N$*, be a sequence of real numbers satisfying*

$$|x_i| \leq \delta + M h^{1-\alpha} \sum_{j=0}^{i-1} \frac{|x_j|}{(i-j)^\alpha}, \quad i = 1, \dots, N, \tag{3.9}$$

where $0 \leq \alpha < 1$*,* $\delta \geq 0$*,* $M > 0$ *is independent of* h*. Then*

$$|x_i| \leq \delta E_{1-\alpha}[m\Gamma(a-\alpha)(ih)^{1-\alpha}], \quad i = 0, 1, \dots, N, \tag{3.10}$$

where $E_{1-\alpha}(z)$ *is the Mittag-Leffler function defined for any* α *by*

$$E_{1-\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n(1-\alpha)+1)}.$$

For proof, see [**8**]. A somewhat better result is given in Corollary 2 of [**3**]. The improvement consists of replacing $E_{1-\alpha}$ in (3.10) by the finite sum

$$\sum_{n=0}^{i} z^n / \Gamma(n(1-\alpha)+1).$$

Lemma 3.3. *Let $0 \le \alpha < 1$ and $j \ge 0$; then*

$$\int_0^1 (n-l+c_i-\tau)^{-\alpha}\tau^j\,d\tau \le \begin{cases} \frac{1}{1-\alpha} < \frac{2^\alpha}{1-\alpha} & \text{if } l = n-1 \\ 2^\alpha (n-l)^{-\alpha} \le \frac{2^\alpha}{1-\alpha}(n-l)^{-\alpha} & \text{if } 0 \le l \le n-2, \end{cases}$$

where $0 \le c_1 < c_2 < \cdots < c_{m+1} \le 1$. For proof, see [**1**, *Lemma* 3.13].

Proof of Theorem 3.2. To prove this theorem we first introduce some notation. Let

$$\int_0^1 \frac{K(t_n+c_ih, t_l+\tau h)z_l(t_l+\tau h)}{(n-l+c_i-\tau)^\alpha}\,d\tau = I^i_{n,l}(z) \tag{3.11a}$$

$$\int_0^{c_i} \frac{K(t_n+c_ih, t_n+\tau h)x_n(t_n+\tau h)}{(c_i-\tau)^\alpha}\,dt = I^i_n(x) \tag{3.11b}$$

$$\sum_{j=1}^{m+1} w^{n,l}_{i,j} K(t_n+c_iht_l+c_jh)z_l(t_l+c_jh) = S^i_{n,l}(z) \tag{3.12a}$$

$$\sum_{j=1}^{q} b_{i,j} K(t_n+c_ih, t_n+c_jh)x_n(t_n+c_jh) = S^i_{n,q}(x). \tag{3.12b}$$

Now, since

$$|Y(t)-\hat{u}(t)| \le |Y(t)-u(t)| + |u(t)-\hat{u}(t)|,$$

and Theorem 3.1 provides an upper bound for $|Y(t)-u(t)|$, it is enough to find an upper bound for $|u(t)-\hat{u}(t)|$, for $t = t_n + c_ih$. But, recalling that $u(t)$ is the collocation approximation for $Y(t)$, we have (using (3.4) and (3.11))

$$u_n(t_n+c_ih) = G(t_n+c_ih) + h^{1-\alpha}\left[\sum_{l=0}^{n-1} I^i_{n,l}(u) + I^i_n(u)\right] \tag{3.13}$$
$$n = 0,1,\ldots,N,\ \ i = 1,2,\ldots,m+1.$$

Hence, by subtracting equations (3.13) and (3.5) (using (3.12)), we obtain

$$|u_n(t_n + c_i h) - \hat{u}_n(t_n + c_i h)| \le |G(t_n + c_i h) - \hat{G}(t_n + c_i h)| + h^{1-\alpha}\left|\sum_{l=0}^{n-1}[I^i_{n,l}(u) - S^i_{n,l}(\hat{u})] + [I^i_n(u) - S^i_{n,q}(\hat{u})]\right|.$$

Now we add and subtract $S^i_{n,l}(u)$ and $S^i_{n,q}(u)$ inside the absolute value of the right-hand side of the previous inequality. Taking advantage of assumptions of Theorem 3.2 and using the notation

$$\begin{gathered} |u_n(t_n + c_i h) - \hat{u}_n(t_n + c_i h)| := e_n(t_n + c_i h), \\ \text{for } n = 0, 1, \dots, N,\ i = 1, \dots, m+1 \\ |I^i_{n,l}(u) - S^i_{n,l}(u)| := E^i_{n,l}, \qquad |I^i_n(u) - S^i_{n,q}(u)| := E^i_n \\ n = 0, 1, \dots, N,\ 0 \le l \le n-1, \end{gathered}$$

we obtain

$$|e_n(t_n + c_i h)| \le \tilde{c}_1 h^{r_1} + h^{1-\alpha}\sum_{l=0}^{n-1}[|S^i_{n,l}(u - \hat{u})| + E^i_{n,l}] + h^{1-\alpha}[|S^i_{n,q}(u - \hat{u})| + E^i_n].$$

Hence,

(3.14)

$$\begin{aligned} e_n(t_n+c_i h) \le \tilde{c}_1 h^{r_1} + h^{1-\alpha} & \\ \cdot \Bigg[\sum_{l=0}^{n-1}\Bigg(\sum_{j=1}^{m+1}|w^{n,l}_{i,j} K(t_n+c_i h, t_l+c_j h)|e_i(t_l+c_j h) + E^i_{n,l}\Bigg)\Bigg] & \\ + h^{1-\alpha}\Bigg[\sum_{j=1}^{q}|b_{ij} K(t_n+c_i h, t_n+c_j h)|e_n(t_n+c_j h) + E^i_n\Bigg], & \\ n = 0, 1, \dots, N. & \end{aligned}$$

Now we define $b_{ij} = 0$ for $j = q+1, \dots, m+1$ and denote

$$e_l := (e_l(t_l + c_1 h), \dots, e_l(t_l + c_{m+1} h))^T, \quad l = 0, 1, 2, \dots, n-1, n.$$

$$D_n := h^{1-\alpha}[|b_{ij} K(t_n + c_i h, t_n + c_j h)|]_{(m+1)\times(m+1)}$$

$$c_{n,l} := (|w^{n,l}_{i,j} K(t_n + c_i h, t_l + c_j h|)_{(m+1)\times(m+1)}$$

where $n = 1, \ldots, N$, $l = 0, \ldots, n-1$. Then (3.14) can be written as

$$(I - D_n)e_n \le \tilde{c}_1 h^{r_1} e^m + h^{1-\alpha}\left[E_n^i e^m + \sum_{l=0}^{n-1}(c_{n,l}e_l + E_{n,l}^i e^m)\right] \tag{3.15}$$

with $e^m = (1, 1, \ldots, 1)^T$.

The rest of the proof is as follows.

First we show that $I - D_n$ is invertible and $||(I-D_n)^{-1}||_\infty$ is uniformly (in n and h) bounded above for all sufficiently small $h > 0$ with the matrix norm $||\cdot||_\infty$ defined for any matrix $A = (a_{ij})_{(m+1)\times(m+1)}$ by

$$||A||_\infty = \max_{1\le i\le m+1} \sum_{j=1}^{m+1} |a_{ij}|.$$

We note that this norm is subordinate to the vector norm defined by $||(u_1, \ldots, u_{m+1})^T|| = \max_{1\le j\le m+1} |u_j|$. Second, we find an upper bound for

$$h^{1-\alpha}\left[||E_n^i e^m|| + \sum_{l=0}^{n-1} ||E_{n,l}^i e^m||\right].$$

Finally, using these two parts, Lemma 3.2 and Lemma 3.3, we find an upper bound for e_n.

To prove the first part, we have

$$I - D_n = I - h^{1-\alpha}(|b_{ij}K(t_n + c_i h, t_n + c_j h)|)_{(m+1)\times(m+1)}.$$

But the b_{ij}'s are given by

$$b_{ij} = \int_0^{c_i} (c_i - \tau)^{-\alpha} l_j^*(\tau)\, d\tau \text{ with } l_j^*(\tau) = \prod_{\substack{i=1\\ i\ne j}}^{q} \frac{(\tau - c_i)}{(c_j - c_i)}.$$

Now $\int_0^{c_i}(c_i-\tau)^{-\alpha}\tau^k\, d\tau \le \int_0^{c_i}(c_i-\tau)^{-\alpha}\, d\tau \le 1/(1-\alpha)$, since $0 \le c_i \le 1$, and therefore the b_{ij}'s are bounded for any given set of c_i. On the other hand, $|K(t,s)| \le M$ is uniformly bounded for all $(t,s) \in T$, and therefore by Banach's lemma (see [**14**, p. 32]), $I - D_n$ is invertible for sufficiently small $h > 0$ and $||(I - D_n)^{-1}||_\infty$ is uniformly bounded, say

$$||(I - D_n)^{-1}||_\infty \le \Gamma. \tag{3.16}$$

Secondly, we recall that $E^i_{n,l}$ and E^i_n are quadrature errors in evaluating the integrals $I^i_{n,l}(u)$ and $I^i_n(u)$, respectively. Hence, by the hypothesis of the theorem,

$$\begin{aligned} E^i_{n,l} &\leq \tilde{c}_2 h^{r_2}, \quad \text{for } 0 \leq l \leq n-1,\ 1 \leq i \leq m+1 \\ E^i_n &\leq \tilde{c}_3 h^{r_3}, \quad \text{for } n = 0, 1, \ldots, N,\ 1 \leq i \leq m+1. \end{aligned}$$

With vector norm $||(v_1, \ldots, v_{m+1})|| = \max_{1 \leq i \leq m+1} |v_i|$ which implies $||e^m|| = 1$, we have

$$\begin{aligned} h^{1-\alpha}\Big(||E^i_n e^m|| + \sum_{l=0}^{n-1} ||E^i_{n,l} e^m||\Big) &\leq h^{1-\alpha}\Big(\tilde{c}_3 h^{r_3} + \sum_{l=0}^{n-1} \tilde{c}_2 h^{r_2}\Big) \\ &= \tilde{c}_3 h^{1+r_3-\alpha} + h^{-\alpha} \cdot a \cdot \tilde{c}_2 h^{r_2} \end{aligned}$$

where a is the length of the interval $I = [0, a]$. Hence,

$$h^{1-\alpha}\Big(||E^i_n e^m|| + \sum_{l=0}^{n-1} ||E^i_{n,l} e^m||\Big) \leq \tilde{c}_3 h^{1+r_3-\alpha} + \tilde{c} h^{r_2-\alpha} \tag{3.17}$$

where $\tilde{c}$ is a constant independent of n, h and N. Now using (3.15), (3.16) and (3.17), we have

$$||e_n|| \leq \Gamma(\tilde{c}_1 h^{r_1} + \tilde{c} h^{r_2-\alpha} + \tilde{c}_3 h^{1+r_3-\alpha}) + h^{1-\alpha}\Gamma \sum_{l=0}^{n-1} ||c_{n,l}||\, ||e_l||. \tag{3.18}$$

Now if we consider a typical element of $c_{n,l}$. We have

$$|w^{n,l}_{i,j} K(t_n + c_i h, t_l + c_j h)| \leq M \int_0^1 \frac{l_j(\tau)\, d\tau}{(n + c_i - l - \tau)^\alpha}.$$

By Lemma 3.3,

$$\int_0^1 (n + c_i - l - \tau)^{-\alpha} \tau^j \, d\tau \leq \frac{2^\alpha}{1-\alpha}(n-l)^{-\alpha},$$

and therefore, for an appropriate constant $A = A(c_1, \ldots, c_{m+1})$,

$$|w^{n,l}_{i,j} K(t_n + c_i h, t_l + c_j h)| \leq \frac{MA2^\alpha}{1-\alpha}(n-l)^{-\alpha}.$$

Hence, (3.18) can be written as

$$||e_n|| \leq \Gamma(\tilde{c}_1 h^{r_1} + \tilde{c} h^{r_2-\alpha} + \tilde{c}_3 h^{1+r_3-\alpha}) + \Gamma D h^{1-\alpha} \sum_{l=0}^{n-1} \frac{||e_l||}{(n-l)^\alpha},$$

with

$$D = \frac{MA2^\alpha}{1-\alpha}.$$

Now let $r = \min(r_1, r_2 - \alpha, 1 + r_3 - \alpha)$, to obtain

$$||e_n|| \leq ch^r + (D\Gamma) h^{1-\alpha} \sum_{l=0}^{n-1} \frac{||e_l||}{(n-l)^\alpha}.$$

By Lemma 3.2, this implies

$$\begin{aligned} ||e_n|| &\leq ch^r E_{1-\alpha}[M\Gamma(1-\alpha)(nh)^{1-\alpha}], \quad (M = D\Gamma) \\ &\leq ch^r E_{1-\alpha}[M\Gamma(1-\alpha)a^{1-\alpha}] = kh^r, \quad n = 0, 1, \ldots, N, \end{aligned} \tag{3.19}$$

where c and k are constants independent of n, h and N. Hence,

$$|Y(t_n + c_i h) - \hat{u}(t_n + c_i h)| \leq Ch^m + kh^r = \hat{c} h^p,$$

where $p = \min(m, r) = \min(m, r_1, r_2 - \alpha, 1 + r_3 - \alpha)$. This completes the proof. □

We define our IRK method by letting $m + 1 = 3$, $q = i$, $i = 1, 2, 3$, $c_1 = 0$, $c_2 = 0.5$ and $c_3 = 1$ in (3.5). Moreover, we define

$$\begin{aligned} \hat{u}_l(t_l + c_j h) &:= u_l^j, \quad l = 0, \ldots, n-1 \\ \hat{u}_n(t_n + c_i h) &:= u_n^i, \quad n = 0, 1, \ldots, N-1 \end{aligned}$$

together with $u_n^1 := u_{n-1}^3$, $n = 1, \ldots, N$. The coefficients $w_{i,j}^{n,l}$ and b_{ij} are defined by (3.6). It is helpful to note that, for these choices of c_i, m and $\alpha = 1/2$. b_{ij} are given by

$$\begin{gathered} b_{11} = b_{12} = b_{13} = b_{23} = 0, \qquad b_{21} = \sqrt{2}/3, \qquad b_{22} = 2\sqrt{2}/3, \\ b_{31} = 2/15, \qquad b_{32} = 16/15, \qquad b_{33} = 4/5. \end{gathered}$$

A block-by-block method can be obtained as explained in [**11**, pp. 114–116 and 136–137] using the IRK method; we omit the details. Both methods have the same degree of convergence as the collocation method. In (3.5), $\hat{G}$ is an approximation for G, ($G(t)$ given by (3.2)). Since $f(t)$ is a known function, one can use an accurate quadrature to approximate G, see examples below.

4. Numerical examples. We solve

$$y(t) = e^{pt}(1-\lambda^2\pi t) + \lambda\int_0^t e^{p(t-s)}(t-s)^{-1/2}y(s)\,ds, \tag{4.1}$$

with exact solution

$$y(t) = e^{pt}(1+2\lambda\sqrt{t}). \tag{4.2}$$

See [**2**] for more details. The singular part of $y(t)$ for $m=3$ is given by

$$f(t) = 2\lambda\sqrt{t} + 2p\lambda t\sqrt{t} + \lambda p^2 t^2\sqrt{t}. \tag{4.3}$$

We solve these equations for different values of p and λ, once with $f=0$ (i.e., singularity is not extracted) and once with $f(t)$ given by (4.3). In numerically solving these equations, the integrals of the form $\lambda\int_0^t e^{p(t-s)}(t-s)^{-1/2}\,ds$ are replaced by quadrature formulas which were obtained from [**10**].

TABLE 4.1. Maximum (relative) errors are listed.

IRK, $f=0$, $h=0.01$, $0\le t\le 1$

$\lambda=1$, $p=0$	$\lambda=1$, $p=-1$	$\lambda=1$, $p=1$	$\lambda=-1$, $p=0$	$\lambda=-1$, $p=-1$	$\lambda=-1$, $p=1$
1.38	0.19	1.41	2.22 (-3)	2.009 (-3)	2.45 (-3)
at $t=0.9$	at $t=1$	at $t=1$	at $t=0.1$	at $t=0.1$	at $t=0.1$

We did not continue for larger t since the errors for positive kernels are already large.

TABLE 4.2. IRK, f given by (4.3), $h=0.01$, $0\le t\le 2$.

$\lambda=1$, $p=0$	$\lambda=1$, $p=-1$	$\lambda=1$, $p=1$	$\lambda=-1$, $p=0$	$\lambda=-1$, $p=-1$	$\lambda=-1$, $p=1$
2.71 (-6)	0.02	.205	5.58 (-9)	7.75 (-4)	2.62 (-4)
at $t=2$	at $t=2$	at $t=2$	at $t=1.9$	at $t=2$	at $t=2$

TABLE 4.3. Block-by-block method, f given by (4.3), $h = 0.01$, $0 \le t \le 2$.

$\lambda=1$, $p=0$	$\lambda=1$, $p=-1$	$\lambda=1$, $p=1$	$\lambda=-1$, $p=0$	$\lambda=-1$, $p=-1$	$\lambda=-1$, $p=1$
1.97 (-6)	1.28 (-6)	9.96 (-5)	1.16 (-8)	3.16 (-8)	3.72 (-8)
at $t = 2$	at $t = 2$	at $t = 2$	at $t = 0.2$	at $t = 2$	at $t = 1.9$

Conclusion. Extraction of singularity improves the accuracy of numerical solutions in linear equations if we work with "good" equations. That is, if derivatives of g and K are computable at the left end point of the interval of integration and if the numerical cancellation does not present a problem. Moreover, for some α's, e.g., $\alpha = 0.9$, as equation (2.7d) shows that even if we are dealing with a "good" equation finding $f(t)$ is not easy.

Acknowledgment. I am grateful to the late Professor Beesack of Carleton University for his advice and his contribution to this work.

REFERENCES

1. J. Abdalkhani, *Collocation and Runge-Kutta-type methods for Volterra integral equations with weakly singular kernels*, Ph.D. Thesis, Dalhousie University, 1983.

2. ———, *A note on examples of Volterra integral equations with exact solution*, Math. Comput. Simulation **32** (1990), 335–337.

3. P.R. Beesack, *More generalized discrete Gronwall inequalities*, ZAMM **65** (1985), 589–595.

4. H. Brunner and P.J. van der Houwen, *The numerical solution of Volterra equations*, North-Holland, 1986.

5. H. Brunner, *Iterated collocation methods and their discretization for Volterra integral equations*, SIAM J. Numer. Anal. **21** (1984), 1132–1145.

6. ———, *Non-polynomial spline collocation for Volterra equations with weakly singular kernels*, SIAM J. Numer. Anal. **20** (1983), 1106–1119.

7. ———, *On collocation approximations for Volterra equations with weakly singular kernels*, in *Treatment of integral equations by numerical methods* (C.T.H. Baker and G.F. Miller, eds.), Academic Press, New York, 1982, 409–420.

8. J. Dixon and S. McKee, *Singular Gronwall inequalities*, Report NA/83/44, Jan. (1983), University of Oxford.

9. D. Kershaw, *Some results for Abel-Volterra integral equations of the second kind*, in *Treatment of integral equations by numerical methods*, (C.T.H. Baker and G.F. Miller, eds.), Academic Press, New York, 1982, 273–282.

10. V.I. Krylov, V.V. Lugin and L.A. Janavich, *Tablitsy Dlia Chislennogo Integrirovaniia Funksii So Stepennymi Osobennostiami* $\int_0^1 x^\beta(1-x)^\alpha f(x)\,ds$, [Tables of Numerical Integration for $\int_0^1 x^\beta(1-x)^\alpha f(x)\,ds$], Akademiia Nauk Belorusskoi S.S.R. Minsk (1963).

11. P. Linz, *Analytical and numerical methods for Volterra equations*, SIAM Stud. Appl. Math. (1985),

12. C. Lubich, *Runge-Kutta theory for Volterra and Abel integral equations of the second kind*, Math. Comp. **41** (1983), 87–103.

13. ———, *Fractional linear multistep methods for Abel-Volterra integral equations of the second kind*, Math. Comp. **45** (1985), 463–469.

14. J.M. Ortega, *Numerical analysis*: *A second course*, Academic Press, New York, 1972.

Department of Mathematics, Ohio State University at Lima, 4240 Campus Drive, Lima, OH 45804