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EXISTENCE FOR ONE-DIMENSIONAL NONLINEAR PARABOLIC VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. In this note we consider the global solvability of the nonlinear integrodifferential equation:

$$u_t = a(x,t,u,u_x)u_{xx} + b(x,t,u,u_x) + \int_0^t c(x,t,\tau,u,u_x,u_{xx})\,d\tau$$

subject to appropriate initial and boundary conditions, under suitable assumptions concerning the data and the functions a, b and c.

1. Introduction. Recently, the author of [**7**] considered the following initial-boundary value problem:

$$u_t = a(x,t,u,u_x)u_{xx}+b(x,t,u,u_x)+\int_0^t c(x,\tau,u,u_x)\,d\tau, \quad \text{in } Q_T, \tag{1.1}$$

$$u(0,t) = f_1(t), \qquad u(1,t) = f_2(t), \quad t \in [0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.3}$$

where $T > 0$ is arbitrary and $Q_T = (0,1) \times (0,T]$.

The global solution was obtained under certain growth assumptions on the functions a, b and c. However, a wider class of physical models (cf. [**1, 2, 3, 6**]) requires that the second order derivative of the solution u_{xx} should appear in the integral term. Namely, the function c should be of the form

$$c = c(x,t,\tau,u,u_x,u_{xx}). \tag{1.4}$$

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In this paper we shall illustrate how the argument of [**7**] can actually be extended to this situation in a nontrivial way. The assumptions are more or less the same as those of the previous paper [**7**]. In particular, no assumption on the growth of the derivatives of $c(x,t,\tau,u,u_x,u_{xx})$ is required except for c_t. The above type of equation has been studied extensively. However, there is not much progress in dealing with the global solvability of equations with nonlinear principal parts. The reader may consult the recent survey paper [**3**] and the references therein for motivation and the physical background.

In what follows, the problem (1.1)–(1.3) always means that the function c takes the form (1.4). Without loss of generality, we may assume that $f_1(t)=f_2(t)=0$. The main assumptions are:

Hypotheses. (1) *The functions $a(x,t,u,p)$ and $b(x,t,u,p)$ are twice differentiable while $c(x,t,u,p,r)$ is differentiable with respect to all of their arguments.*

(2) *There exist positive constants A_1, A_2 and A_3 such that*

$$\begin{gathered} a(x,t,u,p)\geq A_1>0, \qquad |b(x,t,u,p)|\leq A_2[1+|u|+|p|];\\ |c(x,t,\tau,u,p,r)|+|c_t(x,t,\tau,u,p,r)|\leq A_3[1+|u|+|p|+|r|]. \end{gathered}$$

(3) $u_0(x)\in C^4[0,1]$ *and the following compatibility conditions hold*:

$$\begin{gathered} u_0(0)=0, \qquad u_0(1)=0;\\ a(0,0,0,u_0'(0))u_0''(0)+b(0,0,0,u_0'(0))=0,\\ a(1,0,0,u_0'(1))u_0''(1)+b(1,0,0,u_0'(0))=0. \end{gathered}$$

Theorem. *Under the above assumptions, the problem* (1.1)–(1.3) *admits a unique classical solution for arbitrary $T>0$.*

Remark 1. The growth condition on b can be relaxed in the following sense:

$$\frac{|b(x,t,u,p)|}{a(x,t,u,p)}\leq A_2[1+|u|+|p|].$$

Remark 2. With some modification in the proof, the result is still true for the equation (1.1) with the homogeneous Neumann conditions:

$$u_x(0,t) = u_x(1,t) = 0.$$

However, it is not known whether the result holds for the Cauchy problem.

2. Proof. As seen in [**7**], the key to obtaining the global existence is to derive an *a priori* estimate in the Banach space $C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$. We shall prove a number of lemmas to achieve this goal. In what follows, various constants which depend only on the known data and the upper bound of T will be denoted by $C_1, C_2, \ldots$.

Lemma 2.1. *There exists a constant* C_1 *such that*

$$\int\int_{Q_T} u_{xx}^2 \, dx \, dt + \sup_{0\le t\le T} \int_0^1 u_x^2 \, dx \le C_1.$$

This can be proved by the same way as in [**7**].

Lemma 2.2. *There exists a constant* C_2 *such that*

$$||u_x(\cdot,t)||_{L^\infty(0,1)} \le C_2.$$

Proof. This is the crucial step in [**7**]. However, the method still works for the present situation. To see this, we perform the same calculation as in [**7**] and then note from page 255 in [**7**] that the quantity I is now equal to

$$I = \int_0^T \int_0^1 u_x^{p-2} \bigg[\int_0^t (1 + |u| + |u_x| + |u_{xx}|) \, d\tau \bigg]^2 dx \, dt.$$

Using Lemma 2.1, we have

$$\begin{aligned} I &\le C \int_0^T \bigg\{ ||u(\cdot,t)||_{L^\infty(0,1)}^{p-2} \bigg[\int_0^1 \int_0^t (1 + u^2 + u_x^2 + u_{xx}^2) \, dx \, d\tau \bigg] \bigg\} dt \\ &\le C \int_0^T ||u(\cdot,t)||_{L^\infty(0,1)}^{p-2} \, dt. \end{aligned}$$

The rest of the proof can then be carried over. □

Since the second order derivative is involved in the function c, we need more *a priori* estimates.

Lemma 2.3. *There exists a constant C_3 such that*

$$\sup_{0\le t\le T}\int_0^1 u_t(x,t)^2\,dx+\int_0^T\int_0^1 u_{xt}^2\,dx\,dt\le C_3.$$

Proof. We rewrite the equation (1.1) into the following divergence form:

$$\begin{aligned}u_t=&\left(\int_0^{u_x}a(x,t,u,s)\,ds\right)_x-\int_0^{u_x}[a_x(x,t,u,s)+a_u(x,t,u,s)]\,ds\\&+b(x,t,u,u_x)+\int_0^t c(x,t,\tau,u,u_x,u_{xx})\,d\tau.\end{aligned}$$

Let $v(x,t)=u_t(x,t)$, $(x,t)\in\bar{Q}_T$. We differentiate the above equation with respect to t to obtain

$$v_t-[a(x,t,u,u_x)v_x]_x=g(x,t)+c(x,t,t,u,u_x,u_{xx})+\int_0^t c_t\,d\tau, \tag{2.1}$$

where

$$\begin{aligned}g(x,t)=&\int_0^{u_x}[a_t+a_u v]\,ds\\&+\frac{d}{dt}\left\{-\int_0^{u_x}[a_x(x,t,u,s)+a_u(x,t,u,s)]\,ds+b(x,t,u,u_x)\right\}.\end{aligned}$$

By Lemma 2.1–2.2, $u(x,t)$ and $u_x(x,t)$ are uniformly bounded, $g(x,t)$ can be written as

$$g(x,t)=d_1(x,t)v_x+d_2(x,t)v+d_3(x,t),$$

where $d_1(x,t),d_2(x,t)$ and $d_3(x,t)$ are bounded functions, whose bounds depend only on the known data.

Now we multiply the equation (2.1) by $v(x,t)$ and then integrate over Q_T; we can easily obtain, after using the growth conditions and applying Gronwall's inequality, that

$$\int_0^1 v^2\,dx + \int\int_{Q_T} v_x^2\,dx\,dt \le C_3. \qquad \square$$

Corollary 2.4. *There exists a constant C_4 such that for each fixed $t \in [0,T]$,*

$$\int_0^1 u_{xx}^2\,dx \le C_4.$$

Proof. We rewrite the equation (1.1) into the following form:

$$u_t - au_{xx} = b + \int_0^t c(x,t,\tau,u,u_x,u_{xx})\,d\tau.$$

Taking the square of both sides of the above equation and performing the integration over $[0,1]$, we have

$$\int_0^1 [u_t^2 + a^2u_{xx}^2]\,dx - \int_0^1 au_tu_{xx}\,dx = \int_0^1 \left(b + \int_0^t c\,d\tau\right)^2 dx.$$

By Lemma 2.1, Lemma 2.2 and the growth conditions on b and c, we know the right-hand side in the above equality is bounded. Moreover, Cauchy's inequality implies

$$\int_0^1 au_tu_{xx}\,dx \le C(\varepsilon)\int_0^1 u_t^2\,dx + \varepsilon\int_0^1 u_{xx}^2\,dx.$$

Therefore, the desired estimate follows from Lemma 2.3. $\square$

Lemma 2.5. *There exists a constant C_5 such that*

$$||u||_{C^{1+1/2,3/4}(\bar{Q}_T)} \le C_5.$$

Proof. Since $u_{xx}(x,t) \in L^2[0,1]$, by Sobolev's imbedding theorem, for each t,

$$||u(\cdot,t)||_{C^{1+1/2}[0,1]} \leq C.$$

By Lemma 2.3, $u(x,t)$ is Hölder continuous with respect to t and the Hölder norm depends only on C_3. Thus, the desired result follows from Lemma 3.1 of Chapter 2 in [**5**]. $\square$

Now we apply the $W_p^{2,1}(Q_T)$-estimate (cf. [**8**]) to have

Corollary 2.6. *There exists a constant C_6 such that for any $p > 1$,*

$$||u||_{W_p^{2,1}(Q_T)} \leq C_6,$$

where C_6 depends only on known data, T and p.

The above estimate is still not enough to establish global existence because of the appearance of a fully nonlinear term in the equation. We need to obtain *a priori* estimates for the higher order derivatives of solutions. To this end, we differentiate the equation (1.1) with respect to t and define $v(x,t) = u_t(x,t)$; then $v(x,t)$ satisfies

$$v_t = av_{xx} + [a_x + a_u v + a_p v_x]u_{xx} + [b_x + b_u v + b_p v_x] + c + \int_0^t c_t \, d\tau;$$
$$v(0,t) = v(1,t) = 0,$$
$$v(x,0) = a(x,0,u_0(x),u_0'(x))u_0'' + b(x,0,u_0,u_0').$$

Since $u_{xx} \in L^p(Q_T)$ for any $p > 1$, one uses the $W_p^{2,1}(Q_T)$-estimate for $v(x,t)$ to obtain

$$||v||_{W_p^{2,1}(Q_T)} \leq C\left[1 + \int\int_{Q_T} [v^p + v_x^p] \, dx \, dt\right],$$

where Hölder's and Young's inequalities have been used. Now we need the following interpolation inequality (cf. [**4**]):

$$\int_\Omega v_x^p \leq \varepsilon \int_\Omega v_{xx}^p \, dx + C(\varepsilon) \int_\Omega v^p \, dx.$$

Hence,

$$||v||_{W_p^{2,1}(Q_T)} \leq C\left[1 + \int\int_{Q_T} v^p \, dx \, dt\right].$$

Consequently, noting that $v = u_t \in L^p(Q_T)$, one has

Lemma 2.7. *There exists a constant C_7 such that for any $p > 1$,*

$$||v||_{W_p^{2,1}(Q_T)} \leq C_7.$$

Next we apply the imbedding theorem to obtain

Corollary 2.8. *There exist two constants C_8 and $\beta \in (0,1)$ arbitrary such that*

$$||u_t||_{C^{1+\beta,(1+\beta)/2}(\bar{Q}_T)} \leq C_8.$$

Finally, we shall show

Lemma 2.9. *There exists a constant C_9 such that*

$$||u||_{C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)} \leq C_9.$$

Proof. We first show

$$||u_{xx}(\cdot,t)||_{L^\infty(0,1)} \leq C. \tag{2.2}$$

Indeed, by the equation (1.1) and Corollary 2.8,

$$||u_{xx}(\cdot,t)||_{L^\infty(0,1)} \leq C + C\int_0^t ||u_{xx}||_{L^\infty(0,1)} \, d\tau.$$

Therefore, the estimate (2.2) follows from Gronwall's inequality. Similarly, we estimate the Hölder norm of u_{xx} with respect to x: for any $t \in [0,T]$ and two arbitrary points $x_1, x_2 \in [0,1]$ as u_{xx} is bounded by (2.2),

$$\frac{|u_{xx}(x_1,t) - u_{xx}(x_2,t)|}{|x_1 - x_2|^\alpha} \leq C + C\int_0^t \frac{|u_{xx}(x_1,\tau) - u_{xx}(x_2,\tau)|}{|x_1 - x_2|^\alpha} \, d\tau.$$

Hence, Gronwall's inequality implies

$$\frac{|u_{xx}(x_1,t)-u_{xx}(x_2,t)|}{|x_1-x_2|^\alpha} \leq C.$$

Again, by applying Lemma 3.1 of Chapter 2 in [**5**], we complete our proof. $\square$

With the above *a priori* estimate, we can prove global existence via the method of continuity (cf. [**7**]). We do not repeat the process. The uniqueness follows from the linear theory of integrodifferential equations (cf. [**8**]).

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