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COUNTEREXAMPLES FOR ABSTRACT LINEAR VOLTERRA EQUATIONS

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ABSTRACT. The following objects exist:

1) An abstract Volterra equation in a Banach space with an exponentially bounded scalar kernel, such that the resolvent exists but is not exponentially bounded.

2) An analytic resolvent operator corresponding to a self adjoint negative definite operator in a Hilbert space, and a scalar kernel a, such that $1/|\hat{a}|$ grows faster than polynomially.

3) An analytic function F defined on the right half plane and satisfying $|F(s)| \leq M/|s|$ on the right half plane, such that F is not the Laplace transform of an L^∞ function on the positive half axis.

1. Introduction. This paper deals with the resolvent operator of an abstract Volterra integral equation

$$u(t) = \int_0^t a(t-s)Au(s)\,ds + f(t). \tag{1.1}$$

Here A is an unbounded linear operator in some Banach space X, f is an X-valued function, and a is a scalar valued function. By a *resolvent operator* we mean a strongly continuous family $\{S(t) : t \geq 0\}$ of bounded linear operators in X satisfying

$$S(t)Ax = AS(t)x \quad \text{for all } x \in \operatorname{dom}(A),\ t \geq 0;$$
$$S(t)x = x + A\int_0^t a(t-s)S(s)x\,ds \quad \text{for all } x \in X,\ t \geq 0.$$

The solution to (1.1) is then given—at least formally—by

$$u(t) = \frac{d}{dt}\int_0^t S(t-s)f(s)\,ds.$$

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The existence of a resolvent guarantees in some sense existence and uniqueness of solutions to (1.1).

We say that the resolvent operator is *analytic* of type (ω, θ), if it exists and it depends analytically on t in a sector $\Sigma(0, \theta)$, where

$$\Sigma(\gamma, \theta) = \{s \in \mathbf{C} : |\arg(s - \gamma)| < \theta\},$$

and if for each $\theta_1 < \theta$, $\omega_1 > \omega$, and all $t \in \Sigma(\omega_1, \theta_1)$ the resolvent $S(t)$ satisfies an exponential estimate

$$||S(t)|| \leq M e^{\omega_1 \Re(t)}.$$

Similar to analytic semigroups, analytic resolvent operators exhibit particularly strong smoothness properties in time and space.

Abstract Volterra equations and their resolvents have been thoroughly investigated throughout the last two decades. For an exposition of the theory and a bibliography we refer to the forthcoming monograph [**8**]. The aim of this paper is to fill some of the little gaps in the theory by negative answers to the following conjectures:

i) If a is Laplace transformable, and (1.1) admits a resolvent operator, the resolvent operator is exponentially bounded.

ii) If (1.1) admits an analytic resolvent, then $1/\hat{a}$ grows at most polynomially in the right half plane.

iii) If F is analytic and $sF(s)$ is bounded in the right half plane, then F is the Laplace transform of an L^∞ function.

Each of the following sections will be devoted to one item, with a short explanation of its significance for the theory and disproving the conjecture.

2. Exponentially unbounded resolvent. We say that a resolvent S to (1.1) is exponentially bounded if there exist $M > 0$ and ω such that for all $t \geq 0$ the following estimate holds:

$$||S(t)|| \leq M e^{\omega t}.$$

Exponentially bounded resolvents are nice, since they can be tackled by Laplace transform methods. For instance, the Volterra equation version of the Hille-Yosida theorem from [**6**] (see also [**8**, Theorem 1.3]) characterizes the equations that admit an exponentially bounded resolvent.

Not every resolvent needs to be exponentially bounded. This is what one expects for kernels a which do not satisfy

$$\int_0^\infty e^{-\gamma t}|a(t)|\,dt < \infty \tag{2.1}$$

with any $\gamma > 0$. It is also known by a counterexample for integrable operator valued kernels [**4**]. By now it has been open whether the resolvent corresponding to a Laplace transformable scalar kernel has to be exponentially bounded, if it exists. We close this gap by the answer "no."

Proposition 2.1. *There exists a scalar function $a \in C^\infty([0,\infty),\mathbf{C}) \cap L^1([0,\infty),\mathbf{C})$ with $a(0) = 1$, such that for any unbounded infinitesimal generator A of a C_0-semigroup in some Banach space the resolvent S to* (1.1) *exists but is not exponentially bounded.*

The idea of the proof—somewhat hidden in the abstract framework of Baire's Category Theorem—resembles the proof in [**4**]. The kernel starts out nice, and as time passes by, worse and worse parts are switched on. While we leave the kernel integrable, the frequencies increase rapidly, so that the derivative of the kernel is far from being exponentially bounded. (Otherwise one could prove an exponential bound for the resolvent, since it can be obtained by semigroup methods, e.g., [**4**].) Let us now do the technicalities. The proof is based on the following lemma.

Lemma 2.1. *There exists a function $a \in C^\infty([0,\infty),\mathbf{C}) \cap L^1([0,\infty),\mathbf{C})$ with $a(0) = 1$ whose Laplace transform $\hat{a}$ admits zeros with arbitrary large real part.*

Proof. Let

$$X = \{a \in C^\infty([0,\infty),\mathbf{C}) \cap L^1([0,\infty),\mathbf{C}) : a(0) = 1\}$$

topologized by the metric

$$d(a_1, a_2) = \int_0^\infty |a_1(t) - a_2(t)|\, dt + \frac{1}{1 + \inf\{s \in [0, \infty] : a_1(s) \neq a_2(s)\}}.$$

We claim that X is a complete metric space. Evidently, $d(a_1, a_2) \geq 0$ with equality if and only if $a_1 = a_2$, and $d(a_1, a_2) = d(a_2, a_1)$. Also, the triangle inequality is obvious for the integral part, for the other term we have even better

$$\frac{1}{1 + \inf\{s \in [0, \infty] : a_1(s) \neq a_3(s)\}} \leq \max\left(\frac{1}{1 + \inf\{s \in [0, \infty] : a_1(s) \neq a_2(s)\}}, \frac{1}{1 + \inf\{s \in [0, \infty] : a_2(s) \neq a_3(s)\}}\right).$$

Notice that convergence, $a_n \to a$ in this metric, means that $a_n \to a$ in L^1 and for each compact interval $[0, T]$ and sufficiently large n, $a_n|_{[0,T]} = a|_{[0,T]}$. A Cauchy sequence a_n in X is a fortiori a Cauchy sequence in L^1, so that it admits an L^1-limit a. On each compact interval, the sequence is constant after finitely many n, so that $a(t) = a_n(t)$ throughout that interval. Thus $a \in C^\infty$ and $a_n \to a$ in X. This shows completeness.

For $M > 0$, we define

$$X_M = \{a \in X : \hat{a}(s) \neq 0 \text{ for all } \Re(s) > M\}.$$

The assertion of the lemma states that

$$X \neq \bigcup_{M=1}^{\infty} X_M.$$

We assume the contrary and achieve a contradiction.

We prove first that X_M is closed. Let $a \in X \backslash X_M$; thus there is some s_0 with $\Re(s_0) > M$ and $\hat{a}(s_0) = 0$. As $a(0) = 1$, the Laplace transform $\hat{a}$ cannot vanish identically, and by the principle of isolated zeros, there is a circle B around s_0 entirely contained in the half plane $\{s : \Re(s) > M\}$ and some $\delta > 0$ so that $|\hat{a}(s)| > \delta$ on the boundary of

B. If $a_n \to a$ in X, by the L^1-convergence, $\hat{a}_n \to \hat{a}$ uniformly on B, so that for sufficiently large n, $|\hat{a}_n(s) - \hat{a}(s)| \leq |\hat{a}(s)|/2$ on the boundary of B. By Rouché's Theorem, $\hat{a}_n$ admits a zero s_n in the interior of B, in particular, $\Re(s_n) > M$. Consequently, $a_n \notin X_M$. This shows that X_M is closed.

Assuming $X = \cup_{M=1}^{\infty} X_M$, we invoke Baire's theorem and obtain that some X_M contains an open ball with center a. This means that there are some $\varepsilon > 0$ and $T > 0$ such that any $b \in X$ satisfying

$$b(t) = a(t) \quad \text{for } t \in [0,T], \qquad \int_0^\infty |b(t) - a(t)|\, dt < \varepsilon, \tag{2.2}$$

is contained in X_M.

We start out constructing some $c \in L^1$ satisfying (2.2), but such that $\hat{c}$ admits a zero s_0 with $\Re(s_0) = 2M$. We set

$$c(t) = \begin{cases} a(t) & \text{if } t \leq 2T, \\ a(t) + \alpha e^{(2T-t)+i\nu t} & \text{if } t > 2T. \end{cases}$$

The numbers α and ν are still to be chosen. As

$$\int_0^\infty |c(t) - a(t)|\, dt = |\alpha|,$$

we will take care that $|\alpha| < \varepsilon$. Moreover,

$$\hat{c}(2M + i\nu) - \hat{a}(2M + i\nu) = \frac{\alpha e^{-4MT}}{1 + 2M}.$$

Thus $\hat{c}(2M + \nu) = 0$ if

$$\alpha = \alpha(\nu) := -(2M+1)e^{4MT}\hat{a}(2M + i\nu).$$

As $\hat{a}(2M + i\nu) \to 0$ for $\nu \to \infty$, we may choose some ν such that the corresponding $\alpha(\nu) < \varepsilon$. This completes the construction of c.

Since C^∞ is dense in L^1, we may construct a sequence b_n in X such that $b_n \to c$ in L^1 and $b_n(t) = c(t)$ for $t \leq T$. By the argument used above to show closedness of X_M, we see that for sufficiently large n, the Laplace transform, $\hat{b}_n$, admits a zero s_n with $\Re(s_n) > M$, so that

$b_n \notin X_M$. Moreover, for large n, b_n satisfies (2.2). This yields the desired contradiction, and Lemma 2.1 is proved. $\square$

Proof of Proposition 2.1. We take the function a from Lemma 2.1. Let A be the unbounded infinitesimal generator of an arbitrary C_0-semigroup in some Banach space. By the smoothness of a and since $a(0) = 1$, we may rewrite (1.1) in differentiated form

$$u'(t) = Au(t) + \int_0^t a'(t-s)Au(s)\,ds. \tag{2.3}$$

It is standard [**8**, Corollary 1.4] that (2.3) admits a resolvent operator $S(t)$. If S is exponentially bounded, one knows [**8**, Theorem 1.3] that $\hat{a}(s) \neq 0$ for s with sufficiently large real part. This contradicts the construction of a according to Lemma 2.1. $\square$

3. Lower bounds for the transform of the kernel. This section deals with bounds for $1/\hat{a}(s)$ when (1.1) generates an analytic resolvent operator. To understand why such estimates are important, let us review some facts about spatial regularity.

If A generates an analytic semigroup $S(t)$ in some Banach space, for each $t > 0$ and each initial vector x, $S(t)x$ is contained in $\cap_{k=1}^{\infty} \mathrm{dom}\,(A^k)$, which means infinite spatial smoothness in PDE applications. Moreover, there is an estimate

$$||AS(t)x|| \leq \frac{M}{t} e^{\omega t} ||x||. \tag{3.1}$$

This is no longer true for analytic resolvent operators. Let the resolvent operator $S(t)$ to (1.1) be analytic of type (ω_0, θ_0). It is discussed in [**8**, Theorem 2.2] that in general $S(t)x \notin \mathrm{dom}\,(A^2)$. There is an estimate analogous to (3.1), namely

$$||AS(t)|| \leq Me^{\omega t}(t^{\beta} + e^{c/t^{\beta}}) \tag{3.2}$$

where $\omega > \omega_0$ and $\beta > \pi/(2\theta_0)$. Compared to (3.1) this looks very coarse. In fact, the estimate can be improved ([**7, 8**]) to

$$||AS(t)|| \leq Me^{\omega t}(1 + t^{-\alpha}), \tag{3.3}$$

provided

$$\left|\frac{1}{\hat{a}(s)}\right| \le c(|s-\omega|^{\alpha}+1) \tag{3.4}$$

with suitable $c > 0$, $\alpha > 0$.

Unfortunately, the estimate available in general is much weaker (see again [**8**, (2.19) in the Proof of Theorem 2.2]): Let $\theta < \theta_0$ and $\omega > \omega_0$, pick

$$\alpha \in \left(\frac{1}{1+\frac{2\theta}{\pi}}, 1\right).$$

Then there exists some constant $c > 0$ such that for all $s \in \Sigma(\omega, \pi/2+\theta)$

$$\left|\frac{1}{\hat{a}(s)}\right| \le e^{c(1+|s-\omega|^{\alpha})}. \tag{3.5}$$

In this section we show that this estimate is in fact the best one to be obtained in general. Before we construct a pertinent example, however, we mention that (3.4) holds, roughly speaking, whenever generation of an analytic resolvent can be proved without a very close look on the structure of A:

Proposition 3.1. 1) *Suppose that a is such that* (1.1) *admits an analytic resolvent operator for each negative definite self-adjoint operator A in any Hilbert space. Then there exist some $M > 0$, $\omega > 0$, and $\theta, \varepsilon \in (0, \pi/2)$ such that for all $s \in \Sigma(\omega, \pi/2+\theta)$*

$$\left|\frac{1}{\hat{a}(s)}\right| \le M(|s-\omega|^{1+2\varepsilon/\pi}+1). \tag{3.6}$$

2) *Suppose that a is such that* (1.1) *admits an analytic resolvent operator for each generator of an analytic semigroup A in any Hilbert space. Then for each $\varepsilon \in (0, \pi/2)$ there exist some $M > 0$, $\omega > 0$, and $\theta \in (0, \pi/2)$ such that* (3.6) *holds for all $s \in \Sigma(\omega, \pi/2+\theta)$.*

Proof. The proof is the same for both cases. We pick a sufficiently large Hilbert space, and a normal operator A whose spectrum is

$$\sigma(A) = \mathbf{C}\backslash\Sigma\left(0, \frac{\pi}{2}+\eta\right).$$

For case (1) we have to pick $\eta = \pi/2$, while for case (2), $\eta \in (0, \pi/2])$ may be taken arbitrarily small. From the Hille-Yosida Theorem for analytic resolvent operators [**6**], [**8**, Theorem 2.1], we infer that there exist $\delta \in (0, \pi/2)$, $\omega \geq 0$, $N > 0$ such that for each $s \in \Sigma(\omega, \pi/2 + \delta)$,

$$\left\| \frac{1}{\hat{a}(s)} \left(\frac{1}{\hat{a}(s)} - A \right)^{-1} \right\| \leq N. \tag{3.7}$$

Since A is normal,

$$\|(\lambda - A)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(A))}.$$

Since $\sigma(A)$ is a closed sector, (3.7) implies that there exists some $\varepsilon \in (0, \eta)$ such that for $s \in \Sigma(\omega, \pi/2 + \delta)$,

$$\frac{1}{\hat{a}(s)} \in \Sigma(0, \pi/2 + \varepsilon).$$

Put

$$g(z) = \left(\frac{1}{\hat{a}(\omega + z^{1+2\delta/\pi})} \right)^{\frac{1}{1+2\varepsilon/\pi}}.$$

Identifying

$$z = (s - \omega)^{\frac{1}{1+2\delta/\pi}}$$

we see that g maps the right half plane into the right half plane. According to a theorem by Carathéodory, Julia and Wolff [**2**, Theorem 6.23], there exists a finite limit

$$\frac{g(z)}{z} \to c \quad \text{when } |z| \to \infty,\ z \in \Sigma(0, \xi)$$

for any $\xi \in (0, \pi/2)$. We pick ξ large enough, such that

$$\theta := \xi(1 + 2\delta/\pi) - \pi/2 > 0.$$

Returning to terms of s, we infer that there exists a bound M such that for each $s \in \Sigma(\omega, \pi/2 + \theta)$

$$\left| \frac{1}{\hat{a}(s)} \right| \leq M_1(1 + |s - \omega|^{(1+2\varepsilon/\pi)/(1+2\delta/\pi)}) \leq M(1 + |s - \omega|^{1+2\varepsilon/\pi}).$$

Notice that in case (1) $\eta \in (0, \pi/2)$ somewhere, while in case (2) A can be chosen such that η is arbitrarily small, and hence ε can be picked arbitrarily small. □

We now construct an example which shows that (3.5) is sharp in the general case.

Proposition 3.2. *Let $\theta \in (0, \pi/2)$, $\alpha \in (0, 1/(1 + 2\theta/\pi))$, and $\rho \in [0, \pi/2]$. Then there exists a separable Hilbert space H, a negative definite, self-adjoint unbounded operator A in H, and a kernel $a \in L^2(0, \infty)$ such that* (1.1) *admits an analytic resolvent of type* $(0, \theta)$, *but there is a sequence s_n with $\arg(s_n) = \rho$, $|s_n| \to \infty$, such that for any constants $c > 0$ and $\omega > 0$, one can find arbitrarily large n with*

$$\left| \frac{1}{\hat{a}(s_n)} \right| \geq e^{c(1+|s_n-\omega|^\alpha)}. \tag{3.8}$$

We give a short overview over the idea before we start with the technicalities. In order that (1.1) admits an analytic resolvent, $1/\hat{a}(s)$ needs to stay away from the negative axis if $-|1/\hat{a}(s)|$ comes close to an eigenvalue of A. Thus, as $s \to \infty$ along the ray $\arg(s) = \pi/2 + \theta$, $\hat{a}(s)$ has to come back to some sector $\Sigma(0, \pi/2 + \eta)$ infinitely often to leave space for the spectrum of A. On the other hand, there is a relation between the decrease of the absolute value and the argument (e.g., [**5**, Section 1.7]). Strong decrease of $|\hat{a}(s)|$ along that ray implies a large negative argument, which would push $\hat{a}(s)$ out of the sector across the negative axis. Our problem is to find a reasonable tradeoff, allowing for a small phase angle sometimes and a strong decrease of the absolute value on other intervals. We solve it by constructing $\hat{a}$ as an infinite product of phase lag compensators. These provide the decrease of amplitude, with the disadvantage of a large negative phase angle for s close to their corner frequencies. However, for very large and very small s, the phase angle of a phase lag compensator is almost 0, so all we have to do is to give sufficient space between the corner frequencies of subsequent compensators to let the argument recover in between, before we push again for the absolute value. These considerations explain also, why a far better estimate works when the spectrum of A is the whole negative half line (cf. Proposition 3.1). In this case the argument is always confined to some sector, and there is no frequency interval left to achieve decrease of the absolute value. Let us now do the details.

Lemma 3.1. *Let* $\phi \in [0, \pi/2)$ *and* $g : [0, \infty) \to [0, \infty)$ *be a nondecreasing function such that*

$$\lim_{r \to \infty} \frac{\ln(g(r))}{r} = 0. \tag{3.9}$$

There exists a function f *analytic on the open positive half plane and continuously extensible to its closure with the following properties*:

i)

$$|f(s)| \leq \left| \frac{1}{1+s} \right| \quad \text{for } \Re(s) \geq 0. \tag{3.10}$$

ii) *There is a sequence* $0 < S_1 < T_1 < S_2 < T_2 \cdots \to \infty$

$$-3\pi/4 < \arg(f(s)) < 3\pi/4 \quad \text{for } |s| \in [S_j, T_j],\ \Re(s) \geq 0. \tag{3.11}$$

iii) *If*

$$m_j = \sup\{|f(s)| : \Re(s) \geq 0,\ |s| \geq T_j\},$$

and

$$M_j = inf\{|f(s)| : \Re(s) \geq 0,\ |s| \leq S_j\},$$

then

$$m_j \leq \frac{1}{2} M_j. \tag{3.12}$$

iv) *There is a sequence* $r_j \to \infty$ *such that*

$$|f(r_j e^{i\phi})| \leq 1/g(r_j). \tag{3.13}$$

Proof. We fix some $\gamma > 1$ and define for $r_0 > 0$

$$H_{r_0}(s) = \frac{1 + s/(r_0\gamma)}{1 + s\gamma/r_0}$$

The following properties of H_{r_0} are well known and can be checked by straightforward computation:

$$0 < |H_{r_0}(s)| \le 1 \quad \text{if } \Re(s) \ge 0,$$
$$\lim_{|s|\to\infty, \Re(s)\ge 0} H_{r_0}(s) = 1/\gamma^2,$$
$$|H_{r_0}(r_0 e^{i\phi})| = \left| \frac{1 + \frac{1}{\gamma} e^{i\phi}}{1 + \gamma e^{i\phi}} \right| = Q < 1 \quad \text{(independent of } r_0).$$

Starting with $f_0(s) = 1/(1+s)$, we construct inductively S_n, T_n, r_n, k_n,

$$f_n = \frac{1}{1+s} H_{r_1}^{k_1} \cdots H_{r_n}^{k_n}$$

with the following properties:

(3.14)
$$-3\pi/4 < \arg f_n(s) < 3\pi/4 \quad \text{for } |s| \in [S_j, T_j],\ \Re(s) \ge 0,\ j = 1, \ldots, n;$$

(3.15) $$|f_n(r_j e^{i\phi})| < 1/g(r_j) \quad \text{for } j = 1, \ldots, n;$$

(3.16)
$$|f_n(s)| < \frac{1}{2} \inf\{|f_n(t)| : \Re(t) \ge 0,\ |t| \le S_j\}, \quad \text{for } |s| \ge T_j,\ j = 1, \ldots, n;$$

(3.17) $$|f_n(s) - f_{n-1}(s)| \le 2^{-n} \quad \text{for } \Re(s) \ge 0,\ |s| \le n.$$

Given these objects up to index n, we have to find r_{n+1} and an integer k_{n+1} to specify f_{n+1}, as well as an interval $[S_{n+1}, T_{n+1}]$. By its construction, $f_n(s)$ converges to 0 as $|s| \to \infty$, while $\limsup_{|s|\to\infty} |\arg(f_n)(s)| \le \pi/2$; also, f_n has no zeros in the closed right half plane. Therefore, we can find S_{n+1} and T_{n+1} such that $T_n < S_{n+1} < T_{n+1}$ such that

(3.18) $$-3\pi/4 < \arg(f_n(s)) < 3\pi/4 \quad \text{for } |s| \ge S_{n+1},$$

and
(3.19)

$$|f_n(s)| < \frac{1}{2}\inf\{|f_n(t)| : \Re(t) \geq 0,\ |t| \leq S_{n+1}\}, \quad \text{for } |s| \geq T_{n+1},$$
$$j = 1, \dots, n.$$

Pick some $R > \max(n+1, T_{n+1}, r_n)$. By construction, f_n satisfies (3.14), (3.18), (3.16), (3.19), (3.15) and is bounded by $|f_n(s)| \leq 1$. Thus there exists some $\varepsilon > 0$ such that f_{n+1} satisfies (3.14), (3.16), (3.15), (3.17) (each with n replaced by $n+1$, if

$$|f_{n+1}(s) - f_n(s)| \leq \varepsilon \quad \text{for } \Re(s) \geq 0,\ |s| \leq R; \tag{3.20}$$

and

$$|f_{n+1}(r_{n+1}e^{i\phi})| < 1/g(r_{n+1}). \tag{3.21}$$

To obtain (3.21) notice that

$$|f_{n+1}(r_{n+1}e^{i\phi})| = |f_n(r_{n+1}e^{i\phi})H_{r_{n+1}}^{k_{n+1}}(r_{n+1}e^{i\phi})| \leq Q^{k_{n+1}}.$$

Therefore, once r_{n+1} is known, we determine k_{n+1} to be the least integer satisfying

$$k_{n+1} > \frac{\ln(g(r_{n+1}))}{\ln(1/Q)}.$$

Now we have to take care of (3.20). For $|s| \leq R$, $\Re(s) \geq 0$,

$$\begin{aligned}
|f_{n+1}(s) - f_n(s)| &= |f_n(s)|\,|H_{r_{n+1}}^{k_{n+1}}(s) - 1| \\
&\leq |H_{r_{n+1}}(s) - 1| \sum_{j=0}^{k_{n+1}-1} |H_{r_{n+1}}^{j}(s)| \\
&\leq k_{n+1} \left| \frac{\frac{s}{r_{n+1}\gamma} - \frac{s\gamma}{r_{n+1}}}{1 + \frac{s\gamma}{r_{n+1}}} \right| \\
&\leq \left(1 + \frac{\ln(g(r_{n+1}))}{\ln(1/Q)}\right)\left(\gamma - \frac{1}{\gamma}\right)\frac{R}{r_{n+1}}
\end{aligned}$$

which tends to 0 as $r_{n+1} \to \infty$. Therefore we may pick some large r_{n+1} and the corresponding k_{n+1} to have (3.20) and (3.21) satisfied. This completes the inductive construction of the functions f_n.

By (3.17), the sequence f_n converges to some analytic f uniformly on each bounded subset of the closed right half plane. f now evidently fulfills the assertions i) through iv), thus Lemma 3.1 is proved. □

Proof of Proposition 3.2. Put $\phi = \rho/(1 + 2\theta/\pi)$ and choose $\beta \in (\alpha, 1/(1+2\theta/\pi))$. Put

$$g(r) = e^{r^{(\beta(1+2\theta/\pi))}}.$$

Evidently, g satisfies (3.9), so that we may find some f according to Lemma 3.1. We define

$$\hat{a}(s) = f(s^{1/(1+2\theta/\pi)}).$$

This defines $\hat{a}$ as an analytic function on the sector $\Sigma(0, \pi/2 + \theta)$. As

$$|\hat{a}(s)| \le |1+s|^{-1/(1+2\theta/\pi)},$$

we infer that $\hat{a}$ lies in the Hardy-Lebesgue space H^2, thus it is the Laplace transform of some $a \in L^2(0,\infty)$. Putting $s_n = r_n^{1+2\theta/\pi} e^{i\rho}$, we have

$$\left|\frac{1}{\hat{a}(s_n)}\right| = \left|\frac{1}{f(r_n e^{i\phi})}\right| \ge g(|s_n|^{1/(1+2\theta/\pi)}) = e^{|s_n|^\beta}.$$

As $\beta > \alpha$ and $|s_n| \to \infty$, this easily implies (3.8).

We now have to find an operator A such that (1.1) with our kernel a admits an analytic resolvent. For that purpose, let H be a separable Hilbert space and A be a self-adjoint operator whose spectrum consists of the eigenvalues $\mu_n = -1/(2M_n) - 1/(2m_n)$. By Lemma 3.1, (ii and iii), we have that for each $s \in \Sigma(0, \pi/2+\theta)$ one of the following holds:

1. $|1/\hat{a}(s)| \ge 1/m_n$, then $|\mu_n| \le (3/4)|1/\hat{a}(s)|$, thus $|(1/\hat{a}(s))(1/\hat{a}(s) - \mu_n)^{-1}| \le 4$.
2. $|1/\hat{a}(s)| \le 1/M_n$, then $|\mu_n| \ge (3/2)|1/\hat{a}(s)|$, thus $|(1/\hat{a}(s))(1/\hat{a}(s) - \mu_n)^{-1}| \le 2$.
3. $|\arg(1/\hat{a}(s))| \le 3\pi/4$, then $|(1/\hat{a}(s))(1/\hat{a}(s) - \mu_n)^{-1}| = |(1 - \hat{a}(s)\mu_n)^{-1}| \le \sqrt{2}$.

Consequently,

$$||(1/\hat{a}(s))(1/\hat{a}(s) - A)^{-1}|| \le 4.$$

By the generation theorem for analytic resolvent operators [**8**, Theorem 2.1] (1.1) admits an analytic resolvent operator of type $(0, \theta)$. □

4. Growth $1/s$ in the right half plane does not imply an inverse transform in L^∞_{loc}. If (1.1) admits a resolvent operator, then there are constants M and ω such that

$$||s^{-1}(I - \hat{a}(s)A)^{-1}|| \leq \frac{M}{\Re(s) - \omega}$$

for all s with $\Re(s) > \omega$. If the estimate can be improved to

$$||s^{-1}(I - \hat{a}(s)A)^{-1}|| \leq M/|s - \omega| \tag{4.1}$$

for all $s \in \Sigma(\omega, \theta + \pi/2)$ with some $\theta > 0$, then (1.1) generates an analytic resolvent. In case of a semigroup, (4.1) holds in a sector, if it holds in a half plane $\Re(s) > \omega$.

In the general case, we call (1.1) parabolic, if (4.1) holds with $\omega = 0$ in the right half plane. To assure generation of a resolvent one seems to need additional hypotheses on the kernel [**8**, Theorem 3.1]. An example of a parabolic equation which does not admit a resolvent is not yet known. One is tempted to conjecture that parabolicity by itself is sufficient to guarantee wellposedness. This could be easily proved, if one could show that an estimate

$$||F(s)|| \leq M/|s|$$

for an operator valued function in the right half plane implies that F is the Laplace transform of a function in L^∞. A result like this would also improve the regularity estimates for parabolic resolvents with k-regular kernels [**8**, (3.13)] and might be useful in many other respects. However, it has the disadvantage of being wrong even in the scalar case:

Proposition 4.1. *There is a function F analytic in the right half plane with $(1 + |s|)|F(s)|$ bounded in the right half plane, such that F is not the Laplace transform of some function $f \in L^\infty_{\text{loc}}(0, \infty)$.*

Proof. Notice first, that it is sufficient to show existence of some F which is not the Laplace transform of some $f \in L^\infty_{\text{loc}}$ although $|sF(s)|$ is bounded. Then $F_1(s) = F(s+1)$ evidently has the properties claimed in the proposition.

We prove Proposition 4.1 by contradiction. Let Y be the space of functions analytic in the right half plane such that

$$||F||_Y = \sup_{\Re s>0} |sF(s)| < \infty.$$

For shorthand, let $X = L^\infty_{\text{loc}}(0,\infty)$. X is a Frechet space with the seminorms

$$p_j(f) = \operatorname*{ess\,sup}_{t\in(0,j)} |f(t)|.$$

Here $j = 1, 2, 3, \ldots$. If Proposition 4.1 is wrong, then the Laplace transform admits an inverse $T : Y \to X$. We show that T is bounded. For this purpose we fix some $\varepsilon > 0$ and define the right shift

$$(\tau_\varepsilon F)(s) = F(s+\varepsilon) \quad \text{for } \Re(s) > 0,$$

and observe that

$$(T\tau_\varepsilon F)(t) = e^{-\varepsilon t}(TF)(t) \quad \text{for } t > 0,\ \varepsilon > 0.$$

Let $F_n \to F$ in Y and $TF_n \to g$ in X. Obviously, $\tau_\varepsilon F_n \to \tau_\varepsilon F$ in the Hardy-Lebesgue space H^2, and since T is an isomorphism from H^2 into $L^2(0,\infty,\mathbf{C})$, we infer that $T\tau_\varepsilon F_n$ converges to $T\tau_\varepsilon F$ in L^2. Consequently, TF_n converges to TF in L^2_{loc}. On the other hand, $TF_n \to g$ in L^∞_{loc} so that $g = TF$. This shows that T is closed. By the closed graph theorem in Frechet spaces [**1**, Corollary (48.6)] $T : Y \to X$ is continuous. We infer in particular that there exists some $M > 0$ such that for all $f \in X$ with $\hat{f} \in Y$

$$\operatorname*{ess\,sup}_{t\in(0,1)} |f(t)| \le M||\hat{f}||_Y. \tag{4.2}$$

Now we pick some positive integer n and functions

$$f(t) = \sum_{j=0}^{n-1} a_j \chi_{j/n}(t),$$

where χ_ε is the indicator function of the interval $[\varepsilon, \infty)$. Then

$$\hat{f}(s) = \frac{1}{s} \sum_{j=0}^{n-1} a_j e^{-sj/n}.$$

From (4.2) we infer for every $k = 1 \cdots n$

$$\left| \sum_{j=0}^{k-1} a_j \right| \leq \operatorname*{ess\,sup}_{t \in (0,1)} |f(t)| \leq M ||\hat{f}||_Y \leq \sup_{\Re s > 0} \left| \sum_{j=0}^{n-1} a_j e^{-sj/n} \right|.$$

With $z = e^{-s/n}$ and using the maximum principle, we obtain

$$\left| \sum_{j=0}^{k-1} a_j \right| \leq M \sup_{|z|<1} \left| \sum_{j=0}^{n-1} a_j z^j \right| = M \sup_{|z|=1} \left| \sum_{j=0}^{n-1} a_j z^j \right|.$$

Let $n = 2m + 1$, $k = m$ and $|u| = 1$. Replacing a_j by $a_j u^j$, we obtain

$$\left| \sum_{j=-m}^{-1} a_{j+m} u^j \right| \leq M \sup_{|z|=1} \left| \sum_{j=-m}^{m} a_{j+m} z^j \right|,$$

thus

$$\left| \sum_{j=-m}^{m} \operatorname{sgn}(j) a_{j+m} u^j \right| \leq (1 + 2M) \sup_{|z|=1} \left| \sum_{j=-m}^{m} a_{j+m} z^j \right|, \tag{4.3}$$

Let C_{per} denote the space of 2π-periodic, complex valued continuous functions on $\mathbf{R}$, normed by the maximum norm. For a trigonometric polynomial

$$g(x) = \sum_{j=-m}^{m} a_{j+m} e^{ijx}$$

let Hg denote the Hilbert transform

$$(Hg)(u) = \sum_{j=-m}^{m} \operatorname{sgn}(j) a_{j+m} e^{iju}.$$

It is known [**3**, Proposition 9.3.1] that

$$(Hg)(u) = \frac{1}{2\pi} PV\text{-} \int_{-\pi}^{\pi} \cot\left(\frac{x}{2}\right) g(u-x)\, dx. \tag{4.4}$$

This is to be understood as a principle value integral. If estimate (4.3) is valid, then H is continuous on the space of trigonometric polynomials, which is dense in C_{per}. Consequently, H would be a bounded operator $C_{\text{per}} \to C_{\text{per}}$. As the kernel $\cot(x/2)$ does not correspond to a measure of bounded variation, it is seen from (4.4) that H is unbounded on C_{per}. Therefore we have arrived at a contradiction. □

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