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STABILITY OF EQUILIBRIA IN A CLASS-AGE-DEPENDENT EPIDEMIC MODEL

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ABSTRACT. An SIS model with subpopulations and where the infectivity of an infected person is a function of age in the infected class is considered. The existence of equilibria and threshold conditions are established. The global asymptotic stability of the disease-free (zero) equilibrium below the threshold and, under certain conditions, the local asymptotic stability of the endemic (positive) equilibrium above the threshold are proved. Perturbations of the equilibrium together with a stability theorem in nonlinear integral equations are used to prove the stability of the endemic equilibrium.

1. Introduction. For most infectious disease models the infectious contact number, which is the average number of contacts of an infective during his infectious period, has been identified as one of the threshold quantities which determines the behavior of the infectious disease [**14**]. In this work we obtain thresholds for some models for which infection does not confer immunity. Models for gonorrhea and AIDS can be included in this type of models.

The population under consideration is divided into classes. The susceptible class consists of those individuals who can incur the disease but are not yet infective. The infective class consists of those who are transmitting the disease to others. Usually $S(t)$, $I(t)$, denote the number of individuals, respectively, in each of the classes. The total population $S(t) + I(t) = N$ is usually assumed to be constant. It is convenient to normalize so that $S(t) + I(t) = 1$; in this way, the functions measure fractions of the total population. If recovery does not confer immunity, then the model is called an SIS model, since the individuals move from the susceptible class to the infective class and then back to the susceptible class upon recovery. SIS models are appropriate for some bacterial agent diseases such as meningitis, plague, and venereal diseases, and for protozoan agent disease such as malaria and sleeping sickness. SI models can be used for modeling AIDS.

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To say that an infectious disease is endemically stable corresponds to saying that there is an equilibrium point of the model with a positive number (fraction) $0 < I_\infty < 1$ of infectious people, called the endemic equilibrium; and that starting from any initial condition with nonzero number of infective, the number (fraction) $I(t)$ of infectives approaches the equilibrium number (fraction) I_∞ as time goes to infinity. The technical term for this behavior of the model is global stability of the endemic equilibrium. The equilibrium where the number of infective is 0 is called the disease-free equilibrium. Similarly, if from any positive initial condition the number of infectives approaches 0 when time goes to infinity, we say that the disease free-equilibrium of the model is globally stable of that the disease dies out. Where the global stability has been impossible to prove, we usually settle for local asymptotic stability, where the approach to the equilibrium is required only for initial conditions in a neighborhood of the equilibrium.

For nearly all models the disease dies out or remains endemic depending on whether the threshold quantity is below or above 1. Recently, it has been recognized that some models with nonlinear contact rates or with variable population size can have more than one important threshold quantity. In this particular model we consider only the threshold quantity which determines the stability (in the sense we have defined it) of the equilibria. For SIS models with the most commonly accepted nonlinear disease transmission terms, the addition of vital dynamics (births and deaths) only changes the thresholds and the equilibrium points as long as the population size remains constant. Of course, the behavior would be different if the total population were growing.

Sometimes it is convenient to subdivide the population into subpopulations to account for different contact rates. Each group is homogeneous in the sense that individuals belonging to it have the same transmission coefficients. Moreover, the contacts between individuals depend only on the group to which they belong. These models are referred to as heterogeneous or n-subpopulations models. Some useful models consider one or more time delays. References for models of diseases without immunity can be found in [**15**].

Age structure has been incorporated into epidemic models for several reasons, one of which is that in some infectious diseases like childhood diseases and in influenza contact rates depend strongly on age; another is the necessity of having models closer to reality than the traditional epidemic models; finally, age-dependent models arise in diseases that are vertically transmitted from parent to newborn offspring. To formulate an SIS age-dependent model the population is usually divided into susceptibles and infected classes, where $x(a,t)$ and $y(a,t)$ are the densities in these respective classes, so that $\int_a^b x(s,t)\,ds$, $\int_a^b y(s,t)\,ds$, denote the proportions of the population in each class that have age in the age-interval (a,b) at time t. The earliest age-structured models were stated by Bernoulli [**3**] and McKendrick [**19**] and more recently by Hoppensteadt [**17**]. Busenberg and Cooke [**4**] and El Doma [**10**] considered vertically transmitted diseases. Childhood diseases were studied by Anderson and May [**1**] and by Dietz and Schenzle [**9**]. Influenza, where contacts rates between individuals in different age groups vary has been modelled by Gripenberg [**13**] and Greenhalgh [**12**]. Age-dependent models have also been studied in [**2, 5, 7, 24**]. Much of this previous work has been built upon finding threshold conditions for the disease to become endemic and describing the stability of steady-state solutions, often under the assumption that the age structure of the total population is fixed.

Recently, Busenberg, et al. [**6**] ruled out the possibility of oscillations in these models by proving, for a quite general SIS model, that there is a threshold quantity ρ (the spectral radius of certain map), depending on the parameters of the equation, such that if $\rho \leq 1$ the only equilibrium is the disease-free equilibrium which is globally asymptotically stable and if $\rho > 1$ there is a unique endemic equilibrium which is globally asymptotically stable.

Class-age-dependent models differ from the usual chronological-age-dependent models in that they consider an infectivity that depends on the age in the infected class. The infected class may include both exposed (latent) and infectious people. This allows, as the individual ages, a latent period with zero infectivity followed by an infectious period with positive infectivity and then by a removed period (if there is one) with zero infectivity. Hethcote and Thieme [**16**] considered an SIRS model of this type with subpopulations. They proved the existence and local stability of an endemic equilibrium. For further references on class-age dependent models, see [**16**].

2. A class-age-dependent model. In this section we formulate the SIS model with class-age dependent infectivity and with n subpopulations. This model is a modification of one formulated by Hethcote and Thieme [**16**]. In class-age-dependent models it is assumed that the infectivity depends on the age in the infected class. The total infectivity of the infected people is the integral over all class ages of the infectivity at each age times the density of infected persons with class age a. As usual, S_j denotes the fraction of susceptibles in subpopulation j and I_j is the fraction of infected individuals in subpopulation j. Thus

$$I_j(t) = \int_0^\infty x_j(t,a)\,da \tag{2.1}$$

where $x_j(t,a)$ denotes the density of infected individuals at time t with class age a, the lapse of time since infection.

$\lambda_{jk}(a)$ denotes the effective contact rate with individuals in subpopulation j by an infected individual in subpopulation k with class age a. The normalized infectivity impact $J_j(t)$ in subpopulation j at time t is given by

$$J_j(t) = \sum_{k=1}^{n} \int_0^\infty \beta_{jk}(a) x_k(t,a)\,da \tag{2.2}$$

where

$$\beta_{jk}(a) = (N_k/N_j)\lambda_{jk}(a). \tag{2.3}$$

Let μ_j denote the birth and mortality rate in subpopulation j, and $\gamma_j(a)$ the class-age-dependent rate of movement from the infected to the susceptible class in the j subpopulation.

The equations for the densities of the infected subpopulations are

$$(\partial_t + \partial_a)x_j(t,a) = -[\mu_j + \gamma_j(a)]x_j(t,a) \tag{2.4}$$

$$x_j(t,0) = S_j(t)J_j(t) \tag{2.5}$$

for t, $a > 0$, $t \neq a$, $a \neq a_j \leq \infty$. Here a_j is the maximum class age at removal, $\gamma_j(a)$ is continuous on $[0, a_j)$ and $\gamma_j(a) = 0$ for $a > a_j$. $S_j(t)$ and $I_j(t)$ are the fractions of the population in susceptible and infected classes; hence,

$$S_j(t) + I_j(t) = 1 \quad \text{for } t \geq 0.$$

For the infected subpopulation the following equation can be obtained from (2.4) by integrating with respect to a

$$I_j'(t) = (1 - I_j)J_j(t) - \mu_j I_j - \int_0^\infty \gamma_j(a) x_j(t,a)\,da. \tag{2.6}$$

All parameters in the equations are nonnegative. We assume $\mu_j > 0$ or $\int_0^\infty \gamma_j(a)\,da = \infty$ which implies that no individual may stay infected forever. The expression

$$h_j(a) = \exp\left(- \mu_j a - \int_0^a \gamma_j(s)\,ds \right) \tag{2.7}$$

is the probability of still being infected at time a after infection began, and $[\mu_j + \gamma_j(a)]h_j(a)$ is the rate at which infected individuals leave the infected class at time a after infection began. Our assumptions imply that $h_j(a) \to 0$ as $a \to \infty$ and

$$\int_0^\infty [\mu_j + \gamma_j(a)] h_j(a)\,da = 1. \tag{2.8}$$

The conditions $a_j < \infty$, $\int_0^{a_j} \gamma_j(a)\,da = \infty$ with $\gamma_j(a) = 0$ for $a > a_j$ hold for a disease with finite period of infectivity. The condition $\int_0^\infty \gamma_j(a)\,da < \infty$ would correspond to a disease with lifelong carriers.

The product $\gamma_j(a)h_j(a)$ gives the rate at which a just-infected individual of the j-th class will be susceptible again at time a after infection began. If $a_j = \infty$ we impose the following conditions on γ_j:

(a) $\gamma_j(a)h_j(a)$ is monotone nonincreasing for large $a > 0$.

(b) $\mu_j > 0$ or $\liminf a\gamma_j(a) > 1$ as $a \to \infty$.

If γ_j is absolutely continuous, (a) is equivalent to $\gamma_j'(a) \le \gamma_j(a)\mu_j + \gamma_j(a)]$ for large a. Furthermore, since $\int_0^\infty \gamma_j(a)h_j(a)\,da \le 1$, $\gamma_j(a)h_j(a) \to 0$ as $a \to \infty$. Also (a) implies $\gamma_j(t+a)h_j(t+a) \le \text{const}\,\gamma_j(a)h_j(a)$ for $t, a \ge 0$ and t large. Assumption (b) implies that $ah_j(a) \to 0$ for $a \to \infty$ and $\int_0^\infty h_j(a)\,da < \infty$. Note that $H_j = \int_0^\infty a[\mu_j + \gamma_j(a)]h_j(a)\,da = \int_0^\infty h_j(a)\,da$ is the average class age at leaving the infected class and $H_j < \infty$. These implications of (a) and (b) are also satisfied if $a_j < \infty$. We use the condition $\int_0^\infty \gamma_j(a)\,da < \infty$ if we want to model diseases with lifelong carriers.

Furthermore, we assume

i) $\beta_{jk}(a) \geq 0$ and β_{jk} is continuous on $[0, a_k]$ and (a_k, ∞)

ii) β_{jk} is bounded on $[0, \infty)$

iii) the matrix with components $\int_0^\infty \beta_{jk}(a)\, da$ is irreducible.

Assumption iii) means that the population is epidemiologically connected or that the infection will spread over all subpopulations. If both $\beta_{jk}(a)$ and $\gamma_j(a)$ are constants, then this model reduces to an SIS model with n subpopulations with births and deaths.

3. Existence and uniqueness of the endemic equilibrium. To the model (2.1)–(2.5) we add the initial conditions

$$x_j(0,a) \geq 0, \qquad I_j(0) = \int_0^\infty x_j(0,a)\, da < 1. \tag{3.1}$$

For equilibrium (i.e., time-independent) solutions $x_j^*(a)$, the model takes the form

$$\begin{aligned} I_j^* &= \int_0^\infty x_j^*(a)\, da \\ x_j^{*\prime}(a) &= -[\mu_j + \gamma_j(a)]x_j^*(a) \\ x_j^*(0) &= (1 - I_j^*)J_j^* \end{aligned}$$

$$J_j^* = \sum_{k=1}^n \int_0^\infty \beta_{jk}(a) x_k^*(a)\, da. \tag{3.2}$$

We can obtain a fixed point equation for $U_j = x_j^*(0)$. Integrating the x_j^*-equations, we have

$$x_j^*(a) = h_j(a)U_j \tag{3.3}$$

$$I_j^* = \left(\int_0^\infty h_j(a)\, da\right)U_j \tag{3.4}$$

$$U_j = (1 - I_j^*)J_j^* \tag{3.5}$$

$$J_j^* = \sum_{k=1}^n \left(\int_0^\infty \beta_{jk}(a)h_k(a)\, da\right)U_k, \tag{3.6}$$

where $h_j(a)$ is given by (2.7). Hence

$$U_j = [1 - \eta_j U_j] \sum_{k=1}^{n} \alpha_{jk} U_k \tag{3.7}$$

where

$$\eta_j = \int_0^\infty h_j(a)\,da, \qquad \alpha_{jk} = \int_0^\infty \beta_{jk}(a) h_k(a)\,da. \tag{3.8}$$

Equation (3.7) can be transformed into

$$U_j = F_j(U) = \frac{\sum_{k=1}^{n} \alpha_{jk} U_k}{1 + \eta_j \left(\sum_{k=1}^{n} \alpha_{jk} U_k\right)}. \tag{3.9}$$

Thus, the endemic equilibrium is a fixed point of F given by $U = F(U)$, where $U = (U_1, \dots, U_n)$, $F = (F_1, \dots, F_n)$. We use a fixed point theorem mentioned in the Appendix to prove the following theorem.

Theorem 3.1. *Let $\alpha_{jk} = \int_0^\infty \beta_{jk}(a) h_k(a)\,da$ and assume that the conditions of Section* 2 *are satisfied, then*

(a) *If the spectral radius $\rho([\alpha_{jk}]) \le 1$, then $x_j^*(a) = 0$, $I_j^* = 0$ is the unique nonnegative equilibrium of the epidemic model* (2.1)–(2.5).

(b) *If $\rho([\alpha_{jk}]) > 1$, then there exists a unique positive endemic equilibrium solution of* (2.1)–(2.5). *Moreover, $0 < I_j^* < 1$ for $j = 1, \dots, n$.*

Proof. (a) If $\rho[\alpha_{jk}] \le 1$, by Theorem in the Appendix, $U = (0, 0, \dots, 0)$ is the only nonnegative solution of (3.9); hence from (3.3) and (3.4), $x_j^* = 0$, $I_j^* = 0$ is the unique nonnegative equilibrium solution of the epidemic model.

(b) If $\rho[\alpha_{jk}] > 1$, again Theorem in the Appendix implies that there exists a unique positive solution of (3.9), i.e., $U_j > 0$, $j = 1, \dots, n$. Moreover, $\eta_j U_j < 1$, $j = 1, \dots, n$. Hence, from (3.3) and (3.4) there exists a unique positive endemic equilibrium solution of the model. The result $0 < I_j^* < 1$ follows from (3.4) and (3.5). $\square$

4. A stability result on integral equations. To prove the local asymptotic stability of the endemic equilibria we need a result on the stability of integral equations. Consider the equation

$$u + A * u = f(u, v) \tag{4.1}$$

which is to be solved for $u : [0, \infty) \to \mathbf{R}^m$. A is an $m \times m$ matrix of functions continuous on $[0, \infty)$ called the integral kernel of (4.1) and $\int_0^\infty ||A(t)||\, dt < \infty$. $A * u$ is the convolution

$$(A * u)(t) = \int_0^t A(s)u(t - s)\, ds.$$

The forcing function f maps $C([0, \infty), \mathbf{R}^m) \times C([0, \infty), \mathbf{R}^p)$ into $C([0, \infty), \mathbf{R}^m)$ when $v : [0, \infty) \to \mathbf{R}^p$ is a given, continuous function. Assume

$$f(0, 0) = 0. \tag{4.2}$$

Hence $v = 0$, $u = 0$ is an equilibrium of (4.1). Let $||v||_t = \sup\{||v(s)|| : 0 \le s \le t\}$, where $||v(s)|| = \sup\{|v_j(s)|, 1 \le j \le n\}$.

Definition 4.1. The zero solution of (4.1) is *stable* if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any continuous solution u of (4.1), $||v||_\infty \le \delta$ implies $||u||_\infty \le \varepsilon$.

Definition 4.2. The zero solution of (4.1) is *locally asymptotically stable* if it is stable and if there exists $\delta > 0$ such that $||v||_\infty < \delta$ and $v(t) \to 0$ as $t \to \infty$ implies $u(t) \to 0$ as $t \to \infty$.

We make the following assumptions:

Assumption (1). $\det(E + \hat{A}(z)) \neq 0$ for $z \in \mathbf{C}$ and $\operatorname{Re} z \ge 0$ where $\hat{A}(t) = \int_0^\infty \exp(-zt)A(t)\, dt$ is the Laplace transform of A and E is the identity matrix.

Note: Usually the solution R of the matrix convolution equation

$$R + A * R = A = R + R * A \tag{4.3}$$

is called the resolvent kernel associated with A. Any solution of (4.1) can be expressed as

$$u = -R * f(u,v) + f(u,v). \tag{4.4}$$

According to a result of Paley and Wiener, if $\int_0^\infty ||A(t)||\, dt < \infty$, the previous assumption is a necessary and sufficient condition for the existence of an integrable resolvent kernel $R(t)$, [**20**].

Assumption (2). There exist $\rho > 0$ and functions $g_i : [0,\rho] \rightarrow [0,\infty)$, $i = 1, 2$, such that

$$\begin{aligned} &(a)\ \ g_i(r) \rightarrow 0 \text{ as } r \rightarrow 0 \text{ and } g_i \text{ is nondecreasing} \\ &(b)\ \ \text{for any } t > 0,\ ||u||_t,\ ||v||_\infty \leq \rho \text{ implies} \end{aligned} \tag{4.5}$$

$$|f(u,v)(t)| \leq ||u||_t g_1(||u||_t) + g_2(||v||_\infty) \tag{4.6}$$

(c) $\overline{\lim}_{t\rightarrow\infty} ||f(u,v)(t)|| \leq \overline{\lim}_{t\rightarrow\infty} ||u(t)|| g_1(\overline{\lim}_{t\rightarrow\infty} ||u(t)||) + g_2(\overline{\lim}_{t\rightarrow\infty} ||v(t)||)$ if

$$||u||_\infty, ||v||_\infty \leq \rho. \tag{4.7}$$

The main idea in the proof of the next theorem was communicated by Horst Thieme.

Theorem 4.2. *Under the previous assumptions* $(0,0)$ *is a locally asymptotically stable solution of* (4.1).

Proof. First we prove the local stability of the equilibrium. Let $g = \max(g_1, g_2)$. Then $g : [0,\rho] \rightarrow [0,\infty)$, $g(r) \rightarrow 0$ as $r \rightarrow 0$ and g is nondecreasing. Further, for any $t > 0$

$$||f(u,v)(t)|| \leq ||u||_t g(||u||_t) + g(||v||_\infty) \tag{4.8}$$

if

$$||u||_t, ||v||_\infty \leq \rho.$$

Let $v_0 = \int_0^\infty ||R(t)||\, dt$. Choose $\varepsilon_0 > 0$, $\varepsilon_0 < \rho$ such that

$$g(r) \leq \eta \quad \text{if } 0 \leq r \leq \varepsilon_0 \tag{4.9}$$

where

$$\eta(v_0 + 1) < 1.$$

Now let $\varepsilon > 0$, $\varepsilon < \varepsilon_0 < \rho$. Choose $\delta > 0$ such that

$$||u||_\delta \leq \varepsilon_0 \quad \text{and} \quad g(r) < \frac{1 - \eta(v_0 + 1)}{1 + v_0}\varepsilon \quad \text{for } |r| \leq \delta. \tag{4.10}$$

Assume that $||v||_\infty \leq \delta$. From (4.4)

$$||u||_\delta \leq ||R * f(u, v)||_\delta + ||f(u, v)||_\delta.$$

Using the definitions of the convolution and of v_0:

$$||u||_\delta \leq (v_0 + 1)||f(u, v)||_\delta.$$

From (4.8) and the assumption that g is nondecreasing, we have

$$||f(u, v)||_\delta \leq ||u||_\delta g(||u||_\delta) + g(||v||_\infty).$$

It follows that

$$||u||_\delta \leq (v_0 + 1)||u||_\delta g(||u||_\delta) + (v_0 + 1)g(||v||_\infty).$$

From (4.9) and (4.10), we obtain

$$||u||_\delta \leq (v_0 + 1)\eta||u||_\delta + (v_0 + 1)g(||v||_\infty),$$

therefore,

$$||u||_\delta \leq \frac{(v_0 + 1)g(||v||_\infty)}{1 - (v_0 + 1)\eta} < \varepsilon.$$

We have verified that for some $\delta > 0$, $||v||_\infty \leq \delta$ implies $||u||_\delta < \varepsilon$. Moreover, if $||u||_t = \varepsilon$ for some t, say the first one, it follows by a similar argument as above that $||u||_t < \varepsilon$ which would be a contradiction. Hence, $||u||_t < \varepsilon$ for all t and, consequently, $||u||_\infty \leq \varepsilon$. This proves the local stability of $(0, 0)$.

Secondly, we prove the asymptotic stability of $(0,0)$. Using Fatou's Lemma, it follows from (4.4) that

$$\overline{\lim_{t\to\infty}}\,||u(t)|| \le (v_0+1)\,\overline{\lim_{t\to\infty}}\,||f(u(t),v(t))||$$

and, from assumption (4.7),

$$\overline{\lim_{t\to\infty}}\,||u(t)|| \le (v_0+1)[\eta\,\overline{\lim_{t\to\infty}}\,||u(t)|| + g(\overline{\lim_{t\to\infty}}\,||v(t)||)].$$

If $v(t)\to 0$ as $t\to\infty$, then $\overline{\lim}_{t\to\infty}\,||u(t)|| \le (v_0+1)\eta\,\overline{\lim}_{t\to\infty}\,||u(t)||$ and, since $(v_0+1)\eta<1$, we conclude $u(t)\to 0$ as $t\to\infty$. This proves the asymptotic stability of $(0,0)$. □

5. Local stability of the endemic equilibrium.

Definition 5.1. An equilibrium $x_j^*(a)$ of model (2.1)–(2.5) is called *locally stable* if for any $\varepsilon>0$ there exists $\delta>0$ such that if

$$\int_0^\infty |x_j(0,a)-x_j^*(a)|\,da \le \delta \tag{5.1}$$

for $j=1,\dots,n$, then

$$\int_0^\infty |x_j(t,a)-x_j^*(a)|\,da < \varepsilon. \tag{5.2}$$

In addition, if the last integral approaches 0 as $t\to\infty$, we say that the equilibrium is locally asymptotically stable.

Let us assume that there is an endemic equilibrium, hence $\rho\ge 1$. To find conditions for local asymptotic stability to hold, we translate the equilibrium to the origin by letting

$$\begin{aligned} x_j(t,a) &= x_j^*(a)+y_j(t,a)\\ I_j(t) &= I_j^*+u_j(t)\\ J_j(t) &= J_j^*+w_j(t). \end{aligned} \tag{5.3}$$

Then

$$(\partial_t + \partial_a) y_j(t,a) = -[\mu_j + \gamma_j(a)] y_j(t,a) \tag{5.4}$$

$$y_j(t,0) = -u_j(t)[J_j^* + w_j(t)] + w_j(t)(1 - I_j^*) \tag{5.5}$$

$$u_j(t) = \int_0^\infty y_j(t,a)\, da \tag{5.6}$$

$$w_j(t) = \sum_{k=1}^n \int_0^\infty \beta_{jk}(a) y_k(t,a)\, da. \tag{5.7}$$

Solving (5.4) along characteristics, we obtain

$$y_j(t,a) = \begin{cases} h_j(a) z_j(t-a) & \text{for } t > a \\ \frac{h_j(a)}{h_j(a-t)} y_j(0, a-t) & \text{for } t < a \end{cases} \tag{5.8}$$

where $h_j(a)$ is given by (2.7) and

$$z_j(t) = y_j(t,0). \tag{5.9}$$

We now transform the problem (5.4)–(5.7) into an integral equation for $z_j(t)$ separating u_j and w_j into two parts which depend on $z_j(t) = y_j(t,0)$ and on $y_j(0,a)$.

Thus,

$$u_j = h_j * z_j + \mathring{u}_j \tag{5.10}$$

with

$$\mathring{u}_j(t) = \int_0^\infty \frac{h_j(a+t)}{h_j(a)} y_j(0,a)\, da \tag{5.11}$$

where $f * g = \int_0^t f(t-s) g(s)\, ds$ is the convolution of f and g.

Furthermore,

$$w_j = \sum_{k=1}^n (\beta_{jk} h_k) * z_k + \mathring{w}_j \tag{5.12}$$

with

$$\overset{\circ}{w}_j(t) = \sum_{k=1}^{n} \int_0^\infty \beta_{jk}(a+t) \frac{h_k(a+t)}{h_k(a)} y_k(0,a)\, da. \tag{5.13}$$

Substituting into (5.5), we obtain the integral equation

$$\begin{aligned} z_j = -(h_j * z_j + \overset{\circ}{u}_j)\left(J_j^* + \sum_{k=1}^{n} b_{jk} * z_k + \overset{\circ}{w}_j \right) \\ + \left(\sum_{k=1}^{n} b_{jk} * z_k + \overset{\circ}{w}_j \right)(1 - I_j^*) \end{aligned} \tag{5.14}$$

with

$$b_{jk} = \beta_{jk} h_k. \tag{5.15}$$

Equation (5.14) can be written as

$$z + A * z = f(z, \overset{\circ}{u}, \overset{\circ}{w}) \tag{5.16}$$

where $z = [z_1, \ldots, z_n]^T$, $\overset{\circ}{u} = [\overset{\circ}{u}_1, \ldots, \overset{\circ}{u}_n]^T$, $\overset{\circ}{w} = [\overset{\circ}{w}_1, \ldots, \overset{\circ}{w}_n]^T$, $A(s) = [a_{jk}(s)]$ with

$$a_{jk}(s) = J_j^* h_j(s)\delta_{jk} - (1 - I_j^*) b_{jk}(s) \tag{5.17}$$

and

$$\begin{aligned} f_j(z, \overset{\circ}{u}, \overset{\circ}{w}) = -(h_j * z_j + \overset{\circ}{u}_j)\left(\sum_{k=1}^{n} b_{jk} * z_j + \overset{\circ}{w}_j \right) \\ - J_j^* \overset{\circ}{u}_j + \overset{\circ}{w}_j (1 - I_j^*). \end{aligned} \tag{5.18}$$

The next lemma states the local stability of the 0 solution of the integral equation (5.14) with unknown function $u = z$ and with function $v = (\overset{\circ}{u}, \overset{\circ}{w})$.

Lemma 4.2. *Assume that A given by* (5.17) *has an integrable resolvent kernel; then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if*

$$|\overset{\circ}{u}_j(t)| + |\overset{\circ}{w}_j(t)| \le \delta \quad \textit{for } j = 1, \dots, n,\ t \ge 0 \textit{ and}$$
$$|\overset{\circ}{u}_j(t)| + |\overset{\circ}{w}_j(t)| \to 0 \quad \textit{for } t \to \infty$$

then $|z_j(t)| \le \varepsilon$ for $t \ge 0$ and $z_j(t) \to 0$ for $t \to \infty$, $j = 1, \dots, n$.

Proof. To apply Theorem 4.2 to equation (5.16), let $u = z$ and $v = (\overset{\circ}{u}, \overset{\circ}{w})$. Note that $\int_0^\infty ||A(t)||\, dt < \infty$ since each entry of A is integrable, hence by Paley and Wiener's result, $\det(E + \hat{A}(z)) \ne 0$ for $z \in \mathbf{C}$ and $\operatorname{Re} z \ge 0$. We need only to verify (4.5), (4.6) and (4.7). We can see that

$$\begin{aligned} |f_j(z, \overset{\circ}{u}, \overset{\circ}{w})(t)| \le & \left(||z_j||_t \int_0^\infty h_j(s)\, ds + |\overset{\circ}{u}_j(t)| \right) \\ & \cdot \left(||z_j||_t \sum_{k=1}^n \int_0^\infty b_{jk}(s)\, ds + |\overset{\circ}{w}_j(t)| \right) \\ & + J_j^* |\overset{\circ}{u}_j(t)| + (1 - I_j^*) |\overset{\circ}{w}_j(t)|. \end{aligned}$$

Let

$$M_1 = \max \left\{ \int_0^\infty h_j(s)\, ds \right\}, \quad M_2 = \max \left\{ \sum_{k=1}^n \int_0^\infty b_{jk}(s)\, ds \right\}$$

and

$$J^* = \max\{J_j^*, 1 - I_j^*\} \quad \text{for } 1 \le j \le n.$$

Since $||z_j||_t = ||u_j||_t \le ||u||_t$ and $|\overset{\circ}{u}_j(t)|$, $|\overset{\circ}{w}_j(t)| \le ||v||_\infty$, we have

$$|f_j(u, v)(t)| \le (M_1 ||u||_t + ||v||_\infty)(M_2 ||u||_t + ||v||_\infty) + J^* ||v||_\infty.$$

Hence, for $||u||_t$, $||v||_\infty \le \rho$,

$$|f_j(u, v)(t)| \le ||u||_t g_1(||u||_t) + g_2(||v||_\infty)$$

where

$$g(r) = M_1 M_2 r, \qquad g_2(r) = [(M_1 + M_2 + 1)\rho + J^*] r.$$

This shows that conditions (4.5) and (4.6) hold.

Similarly, from (5.18)

$$
\begin{aligned}
|f_j(u,v)(t)| \le & \left(\int_0^\infty h_j(s)|z_j(t-s)|\,ds + |\mathring{u}(t)| \right) \\
& \cdot \left(\sum_{k=1}^{n} \int_0^\infty b_{jk}(s)|z_j(t-s)|\,ds + |\mathring{w}_j(t)| \right) \\
& + J_j^*|\mathring{u}_j(t)| + (1-I_j^*)|\mathring{w}_j(t)|.
\end{aligned}
$$

For simplicity, let $\bar{z}_j$ denote $\overline{\lim}_{t\to\infty} |z_j(t)|$. Note that

$$
\bar{z}_j \le \overline{\lim_{t\to\infty}}\, ||u||_t = \bar{u}, \qquad \overline{\lim_{t\to\infty}}\, |\mathring{u}(t)| \le \overline{\lim_{t\to\infty}}\, ||v(t)|| = \bar{v} \quad \text{and}
$$
$$
\overline{\lim_{t\to\infty}}\, |\mathring{w}(t)| \le \overline{\lim_{t\to\infty}}\, ||v(t)|| = \bar{v}.
$$

Hence, applying Fatou's Lemma,

$$
\begin{aligned}
\overline{\lim_{t\to\infty}}\, |f_j(u,v)(t)| \le & \left(\bar{z}_j \int_0^\infty h_j(s)\,ds + \bar{v} \right) \left(\bar{z}_j \left(\sum_{k=1}^{n} \int_0^\infty b_{jk}(s)\,ds \right) + \bar{v} \right) \\
& + J_j^*\bar{v} + (1-I_j^*)\bar{v},
\end{aligned}
$$

hence,

$$
\overline{\lim_{t\to\infty}}\, |f_j(u,v)(t)| \le (M_1\bar{u}+\bar{v})(M_2\bar{u}+\bar{v}) + J^*\bar{v};
$$

therefore,

$$
\overline{\lim_{t\to\infty}}\, ||f_j(u,v)(t)|| \le \bar{u}g_1(\bar{u}) + g_2(\bar{v}),
$$

hence condition (4.7) holds.

To finish the proof, we remark that we now can apply Theorem 4.2 since $|\mathring{u}_j(t)| + |\mathring{w}_j(t)| \le \delta$ for every j implies $||(\mathring{u}, \mathring{w})||_\infty = ||v||_\infty \le \delta$ and $|\mathring{u}_j(t)| + |\mathring{w}_j(t)| \to 0$ as $t \to \infty$ implies $v(t) \to 0$ as $t \to \infty$. □

Lemma 5.3.

(a) $|\mathring{u}_j(t)| + |\mathring{w}_j(t)| \le \text{const} \left(\sum_{k=1}^{n} \int_0^\infty |y_k(0,a)|\,da \right)$

(b) $|\mathring{u}_j(t)| + |\mathring{w}_j(t)| \to 0$ *as* $t \to \infty$ *if* $\int_0^\infty |y_k(0,a)|\,da < \infty$ *for all* k.

Proof. Result (a) follows from the definitions of $\overset{\circ}{u}_j(t)$ and $\overset{\circ}{w}_j(t)$, the boundedness of $\beta_{jk}(a)$ and $h_j(a+t)/h_j(a) \le 1$. Result (b) is a consequence of the definitions and $h_j(a+t)/h_j(a) \to 0$ as $t \to \infty$. □

As we already remarked in the note after assumption (1), $\det(E + \hat{A}(z)) \neq 0$ for $z \in \mathbf{C}$, $\operatorname{Re} z \ge 0$ is equivalent to the existence of an integrable resolvent kernel $R(t)$.

The next theorem whose proof is in [**16**] gives a sufficient condition for the previous property to hold.

Theorem 5.4. *Consider an integral kernel* $A(s) = [a_{jk}(s)]$ *with*

$$a_{jk}(s) = \delta_{jk} g_j(s) - f_{jk}(s)$$

where

i) g_j, f_{jk} *are nonnegative integrable functions*,

ii) $(\hat{f}_{jk}(0))$ *is a nonnegative irreducible matrix with eigenvalue* 1 *and a corresponding positive eigenvector* $U > 0$.

Then

$$\det(E + \hat{A}(z)) \neq 0 \quad \textit{for } z \in \mathbf{C},\ \operatorname{Re} z \ge 0$$

provided $\operatorname{Re} \hat{g}_j(z) \ge 0$ *for all pure imaginary* $z \in \mathbf{C}$, $j = 1, \dots, n$.

In our case $g_j(z) = J_j^* h_j(s)$ and $f_{jk}(s) = (1 - I_j^*) b_{jk}(s)$. We can see that assumption i) in this theorem is clearly satisfied and assumption ii) follows if we note that the conclusion of Theorem 3.1 (b) says that equation (3.7), which is equivalent to

$$U_j = (1 - I_j^*) \sum_{k=1}^{n} \widehat{\beta_{jk} h}(0) V_k$$

has a positive vector solution $U > 0$.

The condition $\operatorname{Re} \hat{g}_j(z) \ge 0$ for pure imaginary z is equivalent to

$$\operatorname{Re} \int_0^\infty \exp(-\mu_j a - \int_0^a \gamma_j(s)\, ds) \exp(-za)\, da \ge 0 \quad \text{for } z = ir,\ r \in \mathbf{R}$$

or to

$$\int_0^\infty \exp(-\mu_j a - \int_0^a \gamma_j(s)\, ds) \cos ra\, da \ge 0 \quad \text{for } r \in \mathbf{R}. \tag{5.19}$$

This condition is satisfied if the exponential appearing in the integrand is a convex function since, in general, if we assume that $f(a)$ is any integrable nonincreasing convex positive function, by partitioning the interval $[0,\infty)$ at the points $a_k = k\pi/r$, $k = 0, 1, \ldots$, we can write

$$\int_0^\infty f(a) \cos ra\, da = \sum_{k=0}^\infty (-1)^k b_k$$

where $b_0 \geq 0$ and $0 \leq b_{k+1} \leq b_k$, $k = 0, 1, 2, \ldots$.

We can see that condition (5.19) holds if

a) $\gamma_j(a)$ is constant or

b) $\gamma_j'(a) \leq (\mu + \gamma_j(a))^2$ for $a \geq 0$ in case $\gamma_j(a)$ is absolutely continuous on $[0, a_j]$.

Note that $d/dt(1/\gamma_j(a)) \geq 0$ implies this last inequality, hence the condition is satisfied if the mean periods of infectivity are nondecreasing functions of the age. If condition (5.19) holds, all conditions of Theorem 5.4 are met and we can conclude that $\det(E + \hat{A}(z)) \neq 0$ and A has an integrable resolvent kernel.

Now we can state the following

Theorem 5.5. *If* $\int_0^\infty \exp(-\mu_j a - \int_0^a \gamma_j(s)\, ds) \cos ra\, da \geq 0$ *for* $r \in \mathbf{R}$*, the endemic equilibrium of the epidemic model* (2.1)–(2.5) *is locally asymptotically stable.*

Proof. By Lemma 5.3(b), $|\mathring{u}_j(t) + |\mathring{w}_j(t)| \to 0$, for all j. Hence, by Lemma 5.2 given $\varepsilon > 0$, we can find $\delta_1 > 0$ such that if $|\mathring{u}_j(t)| + |\mathring{w}_j(t)| < \delta_1$ for all j, then $|z_j(t)| < \varepsilon/(2M_1)$ where, as before, $M_1 = \max_j \left\{\int_0^\infty h_j(a)\, da\right\}$, and $z_j(t) \to 0$ as $t \to \infty$. Let $\delta = \min\{\delta_1/n \times C, \varepsilon/2\}$ where C is the constant in Lemma 5.3 (a). If $\int_0^\infty |y_j(0,a)|\, da \leq \delta$ for all j, then by Lemma 5.3 (a), $|\mathring{u}_j(t)| + |\mathring{w}_j(t)| < \delta_1$.

Moreover,

(5.20)
$$\int_0^\infty |y_j(t,a)|\,da = \int_0^t |y_j(t,a)|\,da + \int_t^\infty |y_j(t,a)|\,da$$
$$\int_0^\infty |y_j(t,a)|\,da = \int_0^t h_j(a)|z_j(t-a)|\,da + \int_0^\infty \frac{h_j(t+a)}{h_j(a)}|y_j(0,a)|\,da$$
$$\int_0^\infty |y_j(t,a)|\,da < \frac{\varepsilon}{2M_1}\int_0^\infty h_j(a)\,da + \int_0^\infty |y_j(0,a)|\,da$$

in view of (5.8) and $h_j(a+t)/h_j(a) < 1$.

Therefore, if $\int_0^\infty |y_j(0,a)| \le \delta$, then by (5.20), $\int_0^\infty |y_j(t,a)|\,da < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since $y_j(t,a) = x_j(t,a) - x_j^*(a)$, we have proved that the endemic equilibrium is stable.

To show that it is asymptotically stable, let $\varepsilon_1 > 0$ be given. By Lemma 5.2 let δ be such that if $\int_0^\infty |y_j(0,a)|\,da \le \delta$ for all j, then $|z_j(t)| < \varepsilon_1$ and $\lim_{t\to\infty} |z_j(t)| = 0$. Given $\varepsilon > 0$, let $\tau \ge 0$ such that for all j

$$|z_j(t)| < \frac{\varepsilon}{3M_1} \quad \text{for } t \ge \tau \tag{5.21}$$

and

$$\int_0^\infty |h_j(u+T)|\,du < \frac{\varepsilon}{3\varepsilon_1}. \tag{5.22}$$

Choose $\tau_1 \ge 2\tau$ so that

$$\int_t^\infty |y_j(t,a)|\,da < \varepsilon/3 \quad \text{for } t \ge \tau_1. \tag{5.23}$$

Now

$$\begin{aligned}\int_0^\infty |y_j(t,a)|\,da &= \int_0^\tau |y_j(t,a)|\,da + \int_\tau^t |y_j(t,a)|\,da \\ &\quad + \int_t^\infty |y_j(t,a)|\,da.\end{aligned} \tag{5.24}$$

Hence, if $t \ge \tau_1$, then $t \ge 2\tau$ and, by (5.21),

(5.25)
$$\int_0^\tau |y_j(t,a)|\,da = \int_0^\tau h_j(a)|z_j(t-a)|\,da < \frac{\varepsilon}{3M_1}\int_0^\tau h_j(a)\,da \le \frac{\varepsilon}{3}.$$

Also by (5.22),
(5.26)
$$\begin{aligned}\int_\tau^t |y_j(t,a)|\,da &= \int_\tau^t h_j(a)|z_j(t-a)|\,da < \varepsilon_1 \int_\tau^t h_j(a)\,da\\ &= \varepsilon_1 \int_0^{t-\tau} h_j(u+\tau)\,du \le \varepsilon_1 \int_0^\infty h_j(u+\tau) < \frac{\varepsilon}{3}.\end{aligned}$$

Hence, using (5.23)–(5.26), if $t \ge \tau_1$,

$$\int_0^\infty |y_j(t,a)|\,da < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore,

$$\lim_{t\to\infty} \int_0^\infty |y_j(t,a)|\,da = 0. \qquad \square$$

6. Global stability of the disease-free equilibrium.

For the rest of this section we assume that the condition $\rho \le 1$ for the existence of a unique (disease-free) equilibrium holds (Theorem 3.1). In the next theorem we prove the global stability of such equilibrium.

Theorem 6.1. *If $\rho < 1$ then for $j = 1, \ldots, n$, $x_j(t,a) \to 0$ and $I_j(t) \to 0$ as $t \to \infty$. Hence, under the threshold, the disease-free equilibrium is globally stable.*

Proof. Consider the model (2.1)–(2.5). Solving along characteristics we obtain

$$x_j(t,a) = \begin{cases} h_j(a)x_j(t-a,0) & \text{for } t > a\\ \frac{h_j(a)}{h_j(a-t)} f_j(a-t) & \text{for } t < a \end{cases} \tag{6.1}$$

where $f_j(a) \stackrel{\text{def}}{=} x_j(0,a)$ and $h_j(a)$ is given in (2.7). Using definition (2.2) of $J_j(t)$ and relation (6.1), we can write

$$\begin{aligned}J_j(t) &= \sum_{k=1}^n \int_0^t \beta_{jk}(t-s)h_k(t-s)x_k(s,0)\,ds\\ &\quad + \sum_{k=1}^n \int_0^\infty \frac{h_j(t+s)}{h_j(s)} \beta_{jk}(t+s) f_j(s)\,ds.\end{aligned} \tag{6.2}$$

Notice that, by (2.5),

$$0 \le x_j(t,0) \le J_j(t). \tag{6.3}$$

Since the $\beta_{jk}(a)$ are bounded for $a \ge 0$ and $\lim_{t\to\infty} h_j(t) = 0$, then by taking the upper limit in (6.3), we obtain, using Fatou's Lemma,

$$\overline{\lim_{t\to\infty}}\, x_j(t,0) \le \sum_{k=1}^{n} \overline{\lim_{t\to\infty}} \int_0^t \beta_{jk}(t-s)h_k(t-s)x_k(s,0)\,ds. \tag{6.4}$$

Denote

$$\overline{\lim_{t\to\infty}}\, x_j(t,0) = \bar{x}_j$$

hence

$$\bar{x}_j \le \sum_{k=1}^{n} \overline{\lim_{t\to\infty}} \left(\int_0^t \beta_{jk}(t-s)h_k(t-s)\,ds \right) \bar{x}_k.$$

Since, for any measurable function $F(t)$ $\int_0^t F(t-s)\,ds = \int_0^t F(s)\,ds$, we obtain

$$\bar{x}_j \le \sum_{k=1}^{n} \left(\int_0^\infty \beta_{jk}(s)h_k(s)\,ds \right) \bar{x}_k. \tag{6.5}$$

We can write (6.5) in vector form

$$\bar{x} \le A\bar{x} \tag{6.6}$$

where $\bar{x} = [\bar{x}_1, \dots, \bar{x}_n]^T$ and $A = [\alpha_{jk}]$, with $\alpha_{jk} = \int_0^\infty \beta_{jk}(s)h_k(s)\,ds$. Since A is a nonnegative irreducible matrix, Perron-Frobenius theory implies ρ is an eigenvalue with corresponding eigenvector $y > 0$. Thus, for all j,

$$\rho y_j = \sum_{k=1}^{n} \alpha_{jk} y_k, \quad y_j > 0. \tag{6.7}$$

Assume $\bar{x} \ne 0$, then we can choose $\tau > 0$ such that $y_k \ge \tau \bar{x}_k$ for all k and $y_j = \tau x_j$ for some j. Then

$$\rho y_j \ge \sum_{k=1}^{n} \alpha_{jk} \tau \bar{x}_k \ge \tau \bar{x}_j = y_j. \tag{6.8}$$

Since $\rho < 1$ and $y_j > 0$, we obtain a contradiction, hence $\bar{x} = 0$. Therefore,

$$\overline{\lim_{t\to\infty}}\, x_j(t,0) = \bar{x}_j = 0. \tag{6.9}$$

Since $x_j(t,0) \geq 0$, we have $\lim_{t\to\infty} x_j(t,0) = 0$. Therefore, from (6.1),

$$\lim_{t\to\infty} x_j(t,a) = 0. \tag{6.10}$$

Now, using Fatou's Lemma in (2.1),

$$0 \leq \overline{\lim_{t\to\infty}}\, I_j(t) \leq \int_0^\infty \overline{\lim_{t\to\infty}}\, x_j(t,a)\, da.$$

Therefore, for all j,

$$\lim_{t\to\infty} I_j(t) = 0. \qquad \square \tag{6.11}$$

7. Summary and remarks. We have proved the following assertions about class-age-dependent models with subpopulations and without a removed class.

a) Under the threshold ($\rho \leq 1$), the unique equilibrium is the disease-free equilibrium which is globally stable when $\rho < 1$.

b) Above the threshold ($\rho > 1$) there is an endemic equilibrium which, under reasonable conditions (5.19), is locally asymptotically stable.

Previous results for SIS models [**18, 6**] suggest that conditions (5.19) are sufficient but not necessary and that in case b) the endemic equilibrium is also globally stable.

Appendix

We assume the usual order on $\mathbf{R}_+^n$: if $x, y \in \mathbf{R}_+^n$, $x \leq y$ makes sense componentwise. A function $F : \mathbf{R}_+^n \to \mathbf{R}_+^n$ is called monotone nondecreasing if $x \leq y$ implies $F(x) \leq F(y)$.

Definition. A function $F : \mathbf{R}_+^n \to \mathbf{R}_+^n$ is called *strictly sublinear* if for fixed $x \in \operatorname{int} \mathbf{R}_+^n$ and for a fixed $r \in (0,1)$ there exists $\varepsilon > 0$ such that

$$F(rx) \geq (1+\varepsilon) rF(x).$$

For a proof of the following theorem, see [**13, 17**].

Theorem. *Assume*

1) $F : \mathbf{R}_+^n \to \mathbf{R}_+^n$ *is a continuous, monotone nondecreasing, strictly sublinear and bounded function.*

2) $F(0) = 0$ *and* $F'(0)$ *is irreducible.*

Then

a) $F(x)$ *does not have a nontrivial fixed point on the boundary of* $\mathbf{R}_+^n$.

b) $F(x)$ *has a fixed point* $p > 0$ *if and only if* $\rho(F'(0)) > 1$, *where* ρ *is the spectral radius of* $F'(0)$. *If a positive fixed point exists, then it is unique.*

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