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STABILITY OF COLLOCATION METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. We investigate the stability properties of exact and discretized collocation methods, with respect to Volterra integro-differential equations, with degenerate kernel and the basic test equation.

1. Introduction. This paper concerns the stability analysis of the collocation methods for the Volterra integro-differential equation (hereafter referred to as VIDE):

$$(1.1)\qquad y'(t) = f\left(t, y(t), \int_{t_0}^{t} K(t,s,y(s)\,ds\right), \quad t \in [t_0, T], \qquad y(t_0) = y_0$$

where the given functions f and K are assumed to be continuous respectively for $t \in [t_0, T]$ and $(t,s) \in S^* = \{(t,s) : t_0 \leq s \leq t \leq T\}$. For the sake of completeness, the collocation methods and their relationship with the Runge-Kutta methods are described in Section 2.

At present, there are few general analyses on the stability properties of numerical methods for VIDE, and in particular no results are known to the authors about the collocation methods. Until now, the stability analysis has been carried out on the basic test equation (see for example [**3, 5, 6, 11, 15, 16**]) and on positive-definite kernels ([**13, 14**]). In this paper we analyze the stability of the collocation methods, both exact and discretized, with respect to the linear VIDE with degenerate kernel of rank n [**2, 5**]:

$$(1.2)\qquad y'(t) = g(t) + q(t)y(t) + \int_{t_0}^{t} \sum_{l=1}^{n} a_l(t) b_l(s) y(s)\,ds$$

where g, q, a_l, b_l; $l = 1, \dots, n$ are assumed to be continuous.

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We observe that, from the Stone-Weierstrass theorem, it follows that the class of degenerate kernels:

$$K(t,s) = \sum_{l=1}^{n} a_l(t) b_l(s) \tag{1.3}$$

is dense in the class of all continuous kernels and therefore (1.2) can be considered as a significant test equation for stability analysis. Conditions ensuring the stability of the analytical solution of the general VIDE (1.1) can be found in [**7**], while some hypotheses for the stability of the degenerate kernel VIDE (1.2) are given in [**1, 5,** p. 427, **8**].

In order to carry out the stability analysis we construct, in Section 3, the stability matrix of the collocation methods and we prove some local stability conditions. These are independent of the kernel decomposition and require the localization of the roots of a polynomial, whose degree is one plus the minimum of the rank of the kernel and the number of the collocation parameters. In the particular case of a degenerate convolution kernel and $q(t)$ constant, the above conditions become *a priori* stability conditions. Moreover, under the hypothesis of dissipativity of the "associated ODE system" (see Section 3), we prove that the numerical solutions, obtained with a particular class of discretized collocation methods, are contractive.

In Section 4, the particular case of the basic test equation

$$y'(t) = \lambda y(t) + \mu \int_{t_0}^{t} y(s)\, ds, \quad \lambda < 0,\ \mu < 0$$

is considered and we characterize in the $\{h\lambda, h^2\mu\}$ plane the stability regions of the exact collocation methods and of a subclass of the fully implicit discretized collocation methods. We prove that, if the collocation parameters are symmetric, the above methods cannot be A_0-*stable*.[1] The stability properties of some collocation methods coincide with those of a suitable associated Runge-Kutta Nyström method applied to the test equation $y'' - \lambda y' - \mu y = 0$. However, there are no general results for such methods.

[1] The method is said to be A_0-*stable*, if its stability region contains the halfplane $\{(h\lambda, h^2\mu) : \lambda < 0, \mu < 0\}$ (see [**5**, 7.4.1, p. 470]).

More precisely, we show that the stability regions are infinite along the direction of the horizontal axis $h\lambda$, whereas they are bounded along the direction of the vertical axis $h^2\mu$. Moreover, if the collocation parameters are the zeros of the ultraspherical polynomial $P_m^\alpha(t)$, we give a lower bound, depending on α and m, to the size of the boundary of the stability regions on the vertical axis. In the case of the one-point exact collocation methods, we prove that there is only one A_0-*stable* method and it is the Backward Euler-Trapezoidal method. Last, for the implicit discretized collocation method we find a particular class of methods, related to the A-stable Runge-Kutta methods, which are A_0-*stable*.

2. The collocation methods. Let us consider the initial value problem (1.1) and let $t_i = t_0 + ih$, $i = 0, \dots, N$, $t_N = T$, be a partition of the interval $[t_0, T]$ whose subintervals are $\sigma_i = [t_i, t_{i+1}]$, $i = 0, \dots, N-1$. The collocation method approximates the exact solution $y(t)$ of (1.1) by an element $u(t)$ of the polynomial spline space:

$$S_m^{(d)} = \{u : u \in C^d[t_0, T], u_i := u|_{\sigma_i} \in \Pi_m,\ i = 0, \dots, N-1\}.$$

Here Π_m is the space of real polynomial of degree not exceeding m. Let $0 \le c_1 < c_2 \cdots < c_m \le 1$ be given collocation parameters and define the collocation points by $t_{ij} = t_i + c_j h$, $j = 1, \dots, m$. Then the collocation solution is the element $u(t)$ of $S_m^{(0)}$ satisfying (1.1) at the discrete set of collocation points t_{ij}. This leads to the collocation equations:

$$u'(t_{ij}) = f\left(t_{ij}, u(t_{ij}), \int_{t_0}^{t_i} K(t_{ij}, s, u(s))\, ds + \int_{t_i}^{t_{ij}} K(t_{ij}, s, u(s))\, ds\right) \tag{2.2}$$
$$j = 1, \dots, m,\ i = 0, \dots, N-1$$

which, together with the continuity conditions: $u_i(t_i) = u_i(t_{i-1})$ uniquely determine $u \in S_m^{(0)}$ as:

$$u(t_i + sh) = u_i(t_i) + h \sum_{k=1}^{m} \int_0^s L_k(\theta)\, d\theta u'(t_{ik}) \tag{2.3}$$

where

$$L_k(\theta) = \prod_{\substack{j=1 \\ j\neq k}}^{m} (\theta - c_j)/(c_k - c_j).$$

If the integrals occurring in the collocation equations (2.2) are calculated analytically the collocation method is said to be *exact*, but in most applications they have to be evaluated by means of suitable quadrature rules and the method is referred to as *discretized*. Let u'_{ij} and u_{i+1} be an approximation of $u'(t_{ij})$ and $u(t_{i+1})$, respectively, and let us denote with Ψ^i_j and Φ^i_j a quadrature formula approximating respectively the first and the second integral of (2.2). With this notation and remembering (2.3), the discretized version of the collocation method is:

$$\begin{aligned} u_{ij} &= f(t_{ij}, u_i + h\sum_{k=1}^{m} w_{jk}u'_{ik}, \Psi^i_j + \Phi^i_j) \\ u_{i+1} &= u_i + h\sum_{k=1}^{m} w_k u'_{ik} \end{aligned} \tag{2.4}$$

where

$$w_{jk} = \int_0^{c_j} L_k(\theta)\,d\theta, \quad j,k = 1,\dots,m \tag{2.5}$$

$$w_k = \int_0^{1} L_k(\theta)\,d\theta, \quad k = 1,\dots,m. \tag{2.6}$$

Following [**5**, Section 5.4.3] two of the most common choices for Ψ^i_j and Φ^i_j are as follows

A) *Fully implicit discretization*:

$$\begin{aligned} \Psi^i_j &= h\sum_{\nu=0}^{i-1}\sum_{k=0}^{m} w_k K(t_{ij}, t_{\nu k}, u_\nu + h\sum_{l=1}^{m} w_{lk}u'_{\nu l}) \\ \Phi^i_j &= h\sum_{k=1}^{m} c_j w_k K(t_{ij}, t_i + c_j c_k h, u_i + h\sum_{l=1}^{m} a^l_{jk}u'_l) \end{aligned} \tag{2.7}$$

where

$$a^l_{j,k} = \int_0^{c_j c_k} L_l(s)\, ds, \quad j,k,l = 1,\dots,m$$

and w_k are given by (2.6). This method requires the evaluation of K only in its domain of definition S^* and it coincides with a de Hoog and Weiss implicit Runge-Kutta method [**4, 11**].

B) *Implicit discretization*: Ψ^i_j given by (2.7)

$$\Phi^i_j = h \sum_{k=1}^{m} w_{jk} K(t_{ij}, t_{ik}, u_i + h \sum_{l=1}^{m} w_{kl} u'_{il})$$

where w_{jk} are defined in (2.5). This method requires the evaluation of K also outside S^* and therefore it may cause some trouble if the extension of K is not smooth enough out of S^* and it coincides with an extended m-stage Pouzet Runge-Kutta method (compare also [**5**, p. 291]).

3. Stability results. In this section some stability theorems are derived for the method applied to VIDE with degenerate kernel (1.2). They hold both for the exact and discretized collocation methods and are independent of the choice of the discretization formula. The first step is the construction of the stability matrix. We need the following definitions:

$$\alpha^i_{jl} = a_l(t_{ij}), \quad j = 1,\dots,m;\ l = 1,\dots,n$$
$$\gamma_k(s) = \int_0^s L_k(\theta)\, d\theta, \quad k = 1,\dots,m$$

and those in Table 1.

TABLE 1. Definitions.

	Exact	Discretized A	Discretized B
z_l^i $l=1,\dots,m$	$\int_{t_0}^{t_i} b_l(s)u(s)\,ds$	$\sum_{\nu=0}^{i-1}\sum_{k=1}^{m} b_l(t_{\nu k})[u_\nu + h\sum_{r=1}^{m} w_{rk}u'_{\nu r}]$	$\sum_{\nu=0}^{i-1}\sum_{k=1}^{m} b_l(t_{\nu k})[u_\nu + h\sum_{r=1}^{m} w_{rk}u'_{\nu r}]$
s_{jk}^i $j,k=1,\dots,m$	$hw_{jk}q(t_{ij}) + h^2\int_0^{c_j} K(t_{ij}, t_i + sh)\gamma_k(s)\,ds$	$hw_{jk}q(t_{ij}) + h^2\sum_{r=1}^{m} c_j w_r a_{jr}^k K(t_{ij}, t_i + c_j c_r h)$	$hw_{jk}q(t_{ij}) + h^2\sum_{r=1}^{m} w_{jr}w_{rk}K(t_{ij}, t_{ir})$
β_{lk}^i $l=1,\dots,n$ $k=1,\dots,m$	$h\int_0^1 b_l(t_i + sh)\gamma_k(s)\,ds$	$h^2\sum_{r=1}^{m} w_r w_{rk} b_l(t_{ir})$	$h^2\sum_{r=1}^{m} w_r w_{rk} b_l(t_{ir})$
p_j^i $j=1,\dots,m$	$q(t_{ij}) + h\int_0^{c_j} K(t_{ij}, t_i + sh)\,ds$	$q(t_{ij}) + h\sum_{k=1}^{m} c_j w_k K(t_{ij}, t_i + c_j c_k h)$	$q(t_{ij}) + h\sum_{k=1}^{m} w_{jk} K(t_{ij}, t_{ik})$
d_l^i $l=1,\dots,n$	$h\int_0^1 b_l(t_i + sh)\,ds$	$h\sum_{k=1}^{m} w_k b_l(t_{ik})$	$h\sum_{k=1}^{m} w_k b_l(t_{ik})$

Let I be the identity matrix and

$$S_i = (s_{jk}^i); \qquad A_i = (\alpha_{ij}^i); \qquad B_i = (\beta_{jk}^i);$$
$$P_i = (p_j^i); \qquad D_i = (d_j^i); \qquad w = (w_j);$$

$$U' = [u'_{i1}, \dots, u'_{im}]^T, \qquad Z_{i+1} = [z_1^{i+1}, \dots, z_n^{i+1}]^T,$$
$$G_i = [g(t_{i1}), \dots, g(t_{im})]^T; \qquad Y_i = [u_i, Z_i]^T,$$

$$M_i = \begin{pmatrix} 1 + hw(I - S_i)^{-1}P_i & hw(I - S_i)^{-1}A_i \\ D_i + B_i(I - S_i)^{-1}P_i & I + B_i(I - S_i)^{-1}A_i \end{pmatrix}, \tag{3.1}$$
$$T_i = [hw(I - S_i)^{-1}G_i, B_i(I - S_i)^{-1}G_i]^T.$$

Then the following theorem holds:

Theorem 3.1. *The application of the collocation method on* VIDE *with degenerate kernel leads to the finite recurrence relation*

$$Y_{i+1} = M_i Y_i + T_i. \tag{3.2}$$

Proof. Let us consider the fully implicit discretized method. Using the above notation we have

$$\Psi_j^i = h \sum_{l=1}^{n} a_l(t_{ij}) z_l^i$$

and z_l^i satisfies

$$z_l^{i+1} = z_l^i + u_i d_l^i + \sum_{r=1}^{m} \beta_{lr}^i u'_{ir}.$$

Therefore, (2.4) written in vectorial form becomes

$$u_{i+1} = u_i + hwU'_{i+1} \tag{3.3}$$

$$(I - S_i)U'_{i+1} = G_i + u_i P_i + A_i Z_i \tag{3.4}$$

$$Z_{i+1} = Z_i + u_i D_i + B_i U'_{i+1}. \tag{3.5}$$

Assuming that $(I - S_i)^{-1}$ exists, computing U'_{i+1} from (3.4) and replacing it in (3.3) and (3.5), we get that the method applied to (1.2) leads to recursive relation (3.2). An analogous proof applies both to the exact and to the implicit discretized methods. For the former such a proof can also be found in [**8**]. □

We observe that the elements of M_i depend on the step number i, therefore we first derive some local stability results.

Referring to [**5**, p. 432] we give the following

Definition 3.1. The one step collocation method is called

a) locally stable in the strong sense if all the eigenvalues of M_i lie within the unit circle;

b) locally stable if all the eigenvalues of M_i are within or on the unit circle and those on the unit circle are weakly stable, i.e., they correspond to Jordan blocks of order one in the Jordan normal form of the matrix M_i.

Moreover, let us define the $m+1$ dimensional matrix:

$$E(x) = \begin{pmatrix} x-1 & -hw \\ -(x-1)P_i - A_iD_i & (x-1)(I-S_i) - A_iB_i \end{pmatrix}$$

and the polynomial

$$C(x) = \det\,(E(x)).$$

Hereafter we will proceed analogously to [**10**]. Therefore, we will report only the fundamental steps of the proof of the following results. The details can be found in [**10**].

Theorem 3.2. *The exact discretized collocation method is locally stable in t_i in the strong sense if all the zeros of*

$$(x-1)^{n-m}C(x) = 0$$

are within the unit circle, it is locally stable if they are within the unit circle and those on the unit circle correspond to weakly stable eigenvalues of M_i.

Proof. We give the proof in the case $m = n$. It can be easily verified that in this case

$$\det E(x) = \det \left(\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ -P_i & (I-S_i) & 0 & A_i \\ \hdashline 0 & hw & & \\ & & & (x-1)I \\ D_i & B_i & & \end{array}\right) \tag{3.6}$$

since the matrix blocks have the same dimension $m+1$, by using a known property of determinants, we can write

$$\begin{aligned} C(x) &= \det \begin{pmatrix} 1 & 0 \\ -P_i & (I-S_i) \end{pmatrix} \\ &\quad \cdot \det \left\{ (x-1)I - \begin{pmatrix} 0 & hw \\ D_i & B_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -P_i & (I-S_i) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & A_i \end{pmatrix} \right\} \\ &= \det \begin{pmatrix} 1 & 0 \\ -P_i & (I-S_i) \end{pmatrix} \\ &\quad \cdot \det \left\{ (x-1)I - \begin{pmatrix} hw(I-S_i)^{-1}P_i, & hw(I-S_i)^{-1}A_i \\ D_i + B_i(I-S_i)^{-1}P_i, & B_i(I-S_i)^{-1}A_i \end{pmatrix} \right\} \end{aligned}$$

which proves that the zeros of $C(x)$ are equal to the eigenvalues of $M_i(x)$. The proof can be easily extended to the case $m \neq n$ by adding a suitable number of rows and columns, which do not change the value of the determinant and make again the blocks appearing in (3.6) of the same order. $\square$

Remark 3.1. The polynomial $C(x)$ does not depend on the kernel decomposition. In fact, the elements of the vector A_iD_i and of the matrix A_iB_i have the following expression:

$$(A_iD_i)_j = \begin{cases} h\int_0^1 K(t_{ij}, t_i + sh)\,ds & \text{for exact collocation methods} \\ h\sum_{k=1}^m w_k K(t_{ij}, t_{ik}) & \text{for discretization A) and B)} \end{cases}$$

$$(A_iB_i)_{jk} = \begin{cases} \int_0^1 K(t_{ij}, t_i + sh)\gamma_k(s) & \text{for exact coll. methods} \\ h^2\sum_{r=1}^m w_r w_{rk} K(t_{ij} t_{ir}) & \text{for discretization A) and B).} \end{cases}$$

Remark 3.2. The calculation of the $n+1$ eigenvalues of M_i is reduced, in virtue of Theorem 3.2, to the determination of the roots of a polynomial whose degree is $\min\{m, n\} + 1$.

Remark 3.3. In the case $n > m$, the $n - m$ eigenvalues equal to 1 are weakly stable since their algebraic and geometrical multiplicities coincide. In fact, the matrix $M_i - I$ can be written as

$$M_i - I = \begin{bmatrix} hw & 0 \\ B_i & I \end{bmatrix} \begin{bmatrix} (I-S_i)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_i & A_i \\ D_i & 0 \end{bmatrix}$$

and then

$$\operatorname{rank}(M_i - I) \leq \operatorname{rank} \begin{bmatrix} P_i & A_i \\ D_i & 0 \end{bmatrix} \leq m+1.$$

Theorem 3.3. *If $q(t)$ is constant and $K(t,s)$ is convolution degenerate kernel $K(t-s)$ then the eigenvalues of the matrix M_i are constant with respect to i.*

Proof. The elements of the matrix E are independent of i, as can be easily seen by the variable change $s = t_i + \theta h$. □

Now let us consider the implicit discretization (Table 1B) and give the following

Definition 3.2. An implicit discretized collocation method whose parameters are $c = (c_j)$, $W = (w_{jk})$, $w = (w_j)$ and a Runge-Kutta ODE method characterized by the same Butcher array $\begin{array}{c|c} c & W \\ \hline & w \end{array}$ are said to be *associated*.

Before establishing the subsequent result we recall that the following ODE system can be defined as "the ODE system associated to the degenerate VIDE (1.2)" (compare [**1, 5**]):

$$\begin{cases} \phi'(t) = F(t, \phi(t)) \\ \phi(t_0) = \phi_0^* \end{cases} \tag{3.7}$$

with

$$\phi = [\phi_0, \dots, \phi_n]^T, \qquad F = [F_0, \dots, F_n]^T \qquad \phi_0^* = [y_0, 0, \dots, 0]^T$$

$$F_0(t, \phi(t)) = g(t) + q(t)\phi_0(t) + \sum_{l=1}^{n} a_l(t)\phi_l(t)$$

$$F_l(t, \phi(t)) = b_l(t)\phi_0(t), \quad l = 1, \dots, n.$$

The following theorem relates the stability properties of the collocation methods to those of the associated Runge-Kutta method. We refer to [**12**] for the definitions of *dissipative* ODE system (p. 17), *B-stability* and *contractivity* (p. 97).

Theorem 3.4. *If the* ODE *associated system is dissipative, and the associated* ODE *Runge-Kutta method is* B*-stable, then the numerical solutions obtained with the implicit discretized collocation method applied to* (1.2) *are contractive.*

Proof. We prove that the numerical solution obtained by applying the collocation method to (1.2) is equal to the numerical solution obtained by applying the associated Runge-Kutta method to the associated ODE system. Therefore, let us put

$$v_k = [v_k^0, \dots, v_k^n]^T, \quad \text{where } v_k^0 = u'_{ik},$$
$$v_k^l = b_l(t_{ik})\left[u_i + h\sum_{r=1}^{m} w_{kr}u'_{ir}\right], \quad l = 1, \dots, n$$

where the subscript i in v has been omitted for simplicity. With these positions and recalling that $Y_i = [u_i, z_1^i, \dots, z_m^i]$, we can rewrite (3.3) and (3.5) in a unique vectorial form:

$$Y_{i+1} = Y_i + h\sum_{k=1}^{m} w_k v_k. \tag{3.8}$$

But using (3.4), v_j^0 can be written as:

$$v_j^0 = g(t_{ij}) + q(t_{ij})\left[u_i + h\sum_{k=1}^{m} w_{jk}v_k^0\right] + \sum_{l=1}^{n} a_l(t_{ij})\left[z_l^i + h\sum_{k=1}^{m} w_{jk}v_k^l\right], \quad j = 1, \dots, m$$

which implies

$$v_j^0 = F_0\left(t_{ij}, Y_i + h\sum_{k=1}^{m} w_{jk}v_k\right), \tag{3.9}$$

and, in the same way, it turns out that

$$v_j^l = F_l\left(t_{ij}, Y_{i+1} + h\sum_{k=1}^{m} w_{jk}v_k\right). \tag{3.10}$$

The proof is completed by observing that (3.9) together with (3.10) and (3.8) represents the associated Runge-Kutta ODE method applied to the system (3.7). □

4. The basic test equation. In this section we will consider the particular case of the basic test equation

$$(4.1)\qquad \begin{gathered} y'(t) = \lambda y(t) + \mu \int_{t_0}^{t} y(s)\,ds, \quad t \in (t_0, T] \\ y(t_0) = y_0, \quad \lambda < 0, \quad \mu < 0. \end{gathered}$$

Referring to [**6**] we give the following

Definition 4.1. A region $\mathcal{R}$ of the $\{h\lambda, h^2\mu\}$ plane is said to be the region of absolute stability of a method if for all $(h\lambda, h^2\mu) \in \mathcal{R}$ the numerical solutions tend to zero.

4a. The exact collocation methods. In order to characterize, in the $\{h\lambda, h^2\mu\}$ plane, the stability regions of these methods we will proceed analogously to [**9** and **11**]. Theorem 2.2 in [**11**] shows that some results obtained there also hold for the exact collocation method. To make the paper self-contained, let us report the statements of the theorems (proofs can be found in [**11**]).

Let η_1 and η_2 be the solution of the equation:

$$\eta^2 - h\lambda\eta - h^2\mu = 0,$$

and let

$$V(t) = \prod_{j=1}^{m} (t - c_j),$$

$$\Theta(t, x) = \sum_{k=0}^{m} V^k(t) x^k,$$

$$\Gamma(t, x) = \sum_{k=0}^{m+1} [kV^{k-1}(t) + V^k(t)] x^k,$$

$$\Delta = \Gamma(0, 1/\eta_1)\Theta(0, 1/\eta_2) - \Gamma(0, 1/\eta_2)\Theta(0, 1/\eta_1),$$

$$
\begin{aligned}
m_{11} = {} & 1/(\Delta(\eta_2-\eta_1))\{[\eta_1\Gamma(0,1/\eta_2)-\eta_2\Gamma(0,1/\eta_1)] \\
& [\Theta(0,1/\eta_1)-\Theta(0,1/\eta_2)]+[\eta_2\Theta(0,1/\eta_1)-\eta_1\Theta(0,1/\eta_2)] \\
& [\Gamma(1,1/\eta_1)-\Gamma(1,1/\eta_2)]\}, \\
m_{12} = {} & h/(\Delta(\eta_2-\eta_1))\{[\Gamma(0,1/\eta_2)-\Gamma(0,1/\eta_1)] \\
& [\Theta(1,1/\eta_1)-\Theta(1,1/\eta_2)]+[\Theta(0,1/\eta_1)-\Theta(0,1/\eta_2)] \\
& [\Gamma(1,1/\eta_1)-\Gamma(1,1/\eta_2)]\}, \\
m_{21} = {} & -(\eta_1\eta_2)/(h\Delta(\eta_2-\eta_1))\{[\eta_1\Gamma(0,1/\eta_2)-\eta_2\Gamma(0,1/\eta_1)] \\
& [\Theta(1,1/\eta_1)/\eta_1-\Theta(1,1/\eta_2)/\eta_2]+[\eta_2\Theta(0,1/\eta_1)-\eta_1\Theta(0,1/\eta_2)] \\
& [\Gamma(1,1/\eta_1)/\eta_1-\Gamma(1,1/\eta_2)/\eta_2]\}, \\
m_{22} = {} & -(\eta_1\eta_2)/(\Delta(\eta_2-\eta_1))\{[\Gamma(0,1/\eta_2)-\Gamma(0,1/\eta_1)] \\
& [\Theta(1,1/\eta_1)/\eta_1-\Theta(1,1/\eta_2)/\eta_2]+[\Theta(0,1/\eta_1)-\Theta(0,1/\eta_2)] \\
& [\Gamma(1,1/\eta_1)/\eta_1-\Gamma(1,1/\eta_2)/\eta_2]\}.
\end{aligned}
$$

Then the following theorem holds

Theorem 4.1. *The stability region of the exact collocation method for* VIDE *is the set of values* $\{h\lambda, h^2\mu\}$ *such that the eigenvalues of the matrix* $M^* = (m_{ij})$, $i, j = 1, 2$, *are in modulus less than* 1.

Proof. See [**11**, Theorem 2.7]. □

Theorem 4.2. *If the parameters* c_j *are symmetric in* $[0, 1]$ *the exact collocation method cannot be* A_0-stable *and its stability region is unbounded along the direction of the horizontal axis* $\mu = 0$.

Proof. See [**11**, Theorem 2.9]. □

Now set

$$d_{11}(z) = \sum_{k=0}^{[m/2]} V^{(2k+1)}(1) z^{[(m+1)/2]-k}$$

$$d_{12}(z) = \sum_{k=0}^{[m/2]} [(2k+1)V^{(2k)}(1) + V^{(2k+1)}(1)] z^{[(m+1)/2]-k}$$

$$d_{21}(z) = \sum_{k=0}^{[m/2]} V^{(2k)}(1) z^{[(m+1)/2]-k}$$

$$d_{22}(z) = \sum_{k=0}^{[m+1/2]} [(2k)V^{(2k-1)}(1) + V^{(2k)}(1)] z^{[(m+1)/2]-k}$$

$$R(z) = d_{11}(z) d_{21}(z) [2d_{12}(z) - d_{11}(z)][2d_{22}(z) - d_{21}(z)]$$

Theorem 4.3. *Let r be the largest negative zero of odd multiplicity of the polynomial $R(z)$. The boundary of the stability regions of the exact collocation method contains the range*

$$\begin{cases} h\lambda = 0 \\ r \le h^2\mu \le 0. \end{cases}$$

Proof. See [**11**, Theorem 2.10]. □

Finally, putting

$$g(\alpha, m) = 4[2m^2 + m(2\alpha + 1) + 2\alpha - 1]/[m^2 + m\alpha + \alpha]$$

there results:

Theorem 4.4. *If the collocation parameters are the zeros of an ultraspherical polynomial $P_m^{(\alpha)}(t)$ the boundary of the stability regions of the exact collocation method contains at least*

$$\begin{cases} h\lambda = 0 \\ -g(\alpha, m) \le h^2\mu \le 0. \end{cases}$$

Proof. See [**11**, Theorem 2.12]. □

Corollary 4.1. *If the collocation parameters are the zeros of an ultraspherical polynomial, then the boundary of the stability region contains at least*

$$\begin{cases} h\lambda = 0 \\ -8 \le h^2\mu \le 0. \end{cases}$$

Proof. For every α, $g(\alpha, m)$ is decreasing with respect to m and its lower bound is 8. □

In the particular case of $m = 1$, Theorem 4.1 leads to:

Corollary 4.2. *If $m = 1$, there exists only one A_0-*stable *exact collocation method and it is the Backward-Euler-Trapezoidal method.*

Proof. In this case the characteristic polynomial of the stability matrix is:

$$x^2 - \left[(4 - h\lambda(4c_1 - 2)) + h^2\mu\frac{(-2c_1^2 + 2c_1 + 1)}{(2 - 2h\lambda c_1 - h^2\mu c_1^2)}\right]x + \frac{h\lambda(-2c_1 + 2) + h^2\mu(12 - c_1)^2 + 2)}{(2 - 2h\lambda c_1 - h^2)} = 0 \tag{4.3}$$

and applying the Routh-Hurwitz conditions it can easily be seen that $c_1 = 1$ is the only value such that the roots of (4.3) are in modulus less than one for every $\lambda < 0$ and $\mu < 0$. □

In consequence of (4.3), it can be easily seen that:

Corollary 4.3. *The stability region of the exact collocation method with $m = 1$ and $c_1 = 1/2$ is the following strip of the $\{h\lambda, h^2\mu\}$ plane*

$$\begin{cases} -\infty < h\lambda \leq 0 \\ -8 \leq h^2\mu \leq 0. \end{cases}$$

Remark 4.1. As is known, in each interval $[t_i, t_{i+1}]$ the solution $u_i(t)$ furnished by the exact collocation method is equal to the solution y_i of the second order differential equation

$$\begin{cases} y_i''(t) - \lambda y_i'(t) - \mu y_i(t) = [\tau_0^i + \tau_1^i(t - t_i)] \prod_{j=1}^m (t - t_{ij}) \\ y_i(t_i) = y_{i-1}(t_i) \\ y_i'(t_i) = y_{i-1}'(t_i) \end{cases}$$

and, therefore, M^* is also the stability matrix for the ODE collocation method based on the collocation parameters c_j, $j = 1, \dots, m$, and applied to the test equation $y'' - \lambda y' - \mu y = 0$. Therefore, all of the above theorems also hold for these methods.

4b. Fully implicit discretized methods. As we remarked in Section 2, these methods coincide with de Hoog and Weiss implicit Runge-Kutta methods, whose stability analysis has been performed in [**11**].

4c. Implicit discretized collocation methods. First, we recall that these methods are extended Pouzet Runge-Kutta methods. It can be easily proved that the stability matrix M_i, given in (3.1) coincides, in the case of the basic test equation, with the stability matrix of the Runge-Kutta methods derived in [**5**, p. 490] and [**15**, Theorem 5.2]. Then, recalling Definition 3.1, we get

Theorem 4.5. *If the associated* ODE *Runge-Kutta method is A-stable, the implicit discretized collocation method is* A_0-stable.

Proof. The thesis follows from the equivalence of the implicit discretized collocation Pouzet Runge-Kutta methods, taking account of a result of Baker [**3**, Theorem 4.1, **5**, Theorem 7.6]. □

Finally, in the particular case of only one collocation point, we obtain:

Corollary 4.4. *Every one-point implicit discretized collocation method with* $c_1 \geq 1/2$ *is* A_0-stable.

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