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UNCONVENTIONAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS

EZIO VENTURINO

ABSTRACT. A simple method for the solution of second-kind singular integral equations with negative index is investigated. It makes use of Gaussian quadrature that is not of the type suggested by the theory. The major advantage is its simplicity. The error analysis shows that under reasonable assumptions on the smoothness of the solution, the proposed method is convergent. Numerical experiments reveal a higher convergence rate than the one obtained theoretically.

1. Introduction. In the recent literature, the numerical solution of singular integral equations (SIE's) has received considerable attention. It is possible to reduce SIE's to Fredholm integral equations, but in practice direct methods are preferred. The unknown function is replaced by the product of a smooth function times the fundamental function of the problem, with the latter taken as the weight of a quadrature rule. For variable coefficient SIE's, the weight function is nonclassical and the nodes and weights of the quadrature rule must be constructed from scratch. For constant coefficient SIE's, this reduces to Jacobi quadrature.

In this paper we want to analyze the replacement of the possibly nonclassical weights and nodes by the weights and zeros of the Chebyshev polynomials. This is a simpler approach than the standard one. It also has the basic advantage that in doubling the size of the system, the values of the kernel evaluated at the earlier run can be reused. It may also lead to a fast method for second kind SIE's. Recently a fast method has been proposed for first kind equations [**11**], but for second kind equations, one of the difficulties seems to be related to the asymmetry of the Jacobi nodes. In [**21**] an algorithm which uses arbitrary nodes is proposed, but no error analysis is provided. Here we consider

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nodes which are one of the best choices from the approximation theoretic point of view and we want to develop the error analysis for the proposed method. Another perspective of this study is to investigate the effect on its solution of perturbing the coefficients of the equation, while still retaining the original quadrature rule. In contrast with some recent local methods based on spline approximation of the solution [**16** and the literature cited therein, **17**], this study concerns only global polynomial approximations.

The SIE considered here has constant coefficients so that we can compare the method with Gauss-Jacobi quadrature. The index is negative, which implies the fundamental function is bounded at the endpoints. The error analysis can be carried out under two restrictive assumptions. First, we require the solution to possess a Hölder continuous derivative, a similar requirement being necessary in the convergence proof for first kind equations based on an interpolation scheme [**18**]. Second, a bound on the size of the coefficients is imposed. Since we are unable to find a closed form inverse for the matrix of the discretized system, we perform the error analysis by treating the singular operator as a perturbation of the identity. In the experiments no problems were encountered when this last condition was lifted.

The paper is organized as follows. In the next section the problem is stated, and Section 3 presents the numerical scheme. Section 4 deals with the error analysis, and Section 5 considers the complete equation.

2. Mathematical preliminaries. The dominant part of the second kind singular integral equation with real constant coefficients can be written as

$$a\phi(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{\phi(t)\,dt}{t-x} = \tilde{f}(x), \quad -1 < x < 1 \tag{2.1}$$

where the singular integral is interpreted in the Cauchy principal value sense. It is not restrictive to assume the coefficients to satisfy $a > 0$, $a^2 + b^2 = 1$. The fundamental function $\rho(x)$ of the problem is defined by $\rho(x) = (1-x)^\alpha (1+x)^\beta$ with

$$\alpha = \frac{1}{2\pi i} \ln \frac{a - ib}{a + ib} + M, \qquad \beta = -\frac{1}{2\pi i} \ln \frac{a - ib}{a + ib} + N,$$

M and N being integers chosen so that $|\alpha|, |\beta| < 1$. The index of the equation is $\chi = -(\alpha+\beta) = -(M+N)$. We assume here $\chi = -1$, which implies $0 \le \alpha, \beta \le 1$. The solution to (2.1) exists provided the following orthogonality condition is satisfied

$$\int_{-1}^{1} f(x)[\rho(x)]^{-1}\,dx = 0. \tag{2.2}$$

Even though this condition has been used [**13**] to investigate the overdetermined system arising in the discretization of the SIE, as proposed in [**9**, p. 596], we will not make use of it. From a well-known formula [**12**, p. 290] and the assumption on the index, we obtain

$$\begin{aligned}\int_{-1}^{1} \rho(t)(t-x)^{-1}\,dt &= \pi\rho(x)\cot(\pi\alpha)\\ &\qquad - 2B(\alpha,\beta+1)_2F_1\biggl(-1,1;1-\alpha;\frac{1-x}{2}\biggr)\\ &= -\pi[a/b\rho(x) - 2\csc(\pi\alpha)P_1^{(-\alpha,-\beta)}(x)]\end{aligned} \tag{2.3}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of degree n relative to the weight function $\rho(x)$, and $P_n^{(-\alpha,-\beta)}(x)$ the one relative to $[\rho(x)]^{-1}$.

Usually the unkown is rewritten by explicitly expressing the singular behavior at the endpoints via the function ρ and introducing a new unknown function g: $\phi(x) = \rho(x)g(x)$. On using (2.3), (2.1) becomes:

$$\frac{b}{\pi}\int_{-1}^{1} \rho(t)\frac{g(t)-g(x)}{t-x}\,dt - 2b\csc(\pi\alpha)P_1^{(-\alpha,-\beta)}(x)g(x) = \tilde{f}(x). \tag{2.4}$$

In [**15**, p. 114] the solution of the SIE is sought among the functions satisfying a Hölder condition, and the right hand side is assumed to satisfy the same hypothesis. Also, by reducing the SIE to a Fredholm integral equation, [**15**, p. 134–139], it is shown that if $\tilde{f}$ is assumed to be Hölder continuous, the solution ϕ is also. In practice, the function g is assumed to be smooth. The integral in (2.4) exists and can be discretized by an ordinary Gaussian quadrature. If q_k, $k = 1, \dots, n$ denote the zeros of $P_n^{(\alpha,\beta)}(x)$ and c_j, $j = 0, \dots, n$ those of $P_{n+1}^{(-\alpha,-\beta)}(x)$, the following square system is obtained

$$
\begin{aligned}
&-\left[2\csc(\pi\alpha)P_1^{(-\alpha,-\beta)}(c_j)+\sum_{k=1}^{n}\frac{w_k}{q_k-c_j}\right]g(c_j)\\
&\qquad+\sum_{k=1}^{n}\frac{w_k}{q_k-c_j}g(q_k)=f(c_j),\quad j=0,\dots,n
\end{aligned}
\tag{2.5}
$$

$$
g(q_k)=\sum_{j=0}^{n}l_j(q_k)g(c_j)\quad k=1,\dots,n
$$

where $f(x)=\tilde{f}(x)/b$, the w_k's denote the weights of the Gauss-Jacobi quadrature, and $l_j(x)$ are the Lagrange fundamental polynomials constructed on the nodes c_j, $j=0,\dots,n$, [**20**, p. 328]. Observe that $c_j\neq q_k$, $j=0,\dots,n$, $k=1,\dots,n$, since $b\neq 0$, as discussed in [**8**, p. 62]. Note that the system is of order $2n+1$.

Here we want to replace Gauss-Jacobi quadrature with Lobatto-Chebyshev quadrature; see, e.g., [**2**, p. 104 or **19**]. Let

$$
\phi(x)=y^*(x)/\sqrt{1-x^2}
$$

so that

$$
y^*(x)=\tilde{\rho}(x)g(x),\qquad \tilde{\rho}(x)\equiv\sqrt{1-x^2}\rho(x).
\tag{2.6}
$$

Now, proceeding as before, (2.4) is replaced by

$$
-2\csc(\pi\alpha)P_1^{(-\alpha,-\beta)}(x)\frac{y^*(x)}{\tilde{\rho}(x)}+\frac{1}{\pi}\int_{-1}^{1}\frac{\tilde{\rho}(t)}{\sqrt{1-t^2}}\frac{g(t)-g(x)}{t-x}\,dt=f(x).
\tag{2.7}
$$

Let $H_\lambda[-1,1]$ denote the class of Hölder continuous functions of order λ on $[-1,1]$; then

(1) Clearly, $\tilde{\rho}\in H_\lambda[-1,1]$, with $\lambda=\min(\alpha+1/2,\beta+1/2)>1/2$.

(2) If $g(x)$ is required to possess a Hölder continuous first derivative, say $g'\in H_\sigma[-1,1]$, $\sigma>1/2$, we can then define

$$
h(x,t)=\begin{cases}\tilde{\rho}(t)(g(t)-g(x))/(t-x) & -1<x\neq t<1\\ \tilde{\rho}(t)g'(t) & -1<t=x<1\end{cases}.
$$

It follows that $h \in H_\mu[-1,1]$, $\mu = \min(\lambda, \sigma) > 1/2$, as function of the variable t.

Throughout the paper $\{\underline{e}_i\}$ denotes the standard basis in $\mathbf{R}^n$, and $\underline{0}_n$ denotes the null vector in the same space. $\{t_k : k = 1, \dots, n\}$ denotes the zeros of the Chebyshev polynomials of the first kind $T_n(x)$, and $\{s_j : j = 1, \dots, n-1\}$ those of second kind polynomial $U_{n-1}(x)$, with $s_0 = 1$, $s_n = -1$.

3. Analysis of the numerical scheme. On applying Lobatto-Chebyshev quadrature to (2.7) followed by collocation at t_k, and using (2.6) to eliminate the unknown g, we are led to

$$\begin{aligned}\frac{-y^*(t_k)}{\tilde{\rho}(t_k)}&\left[2\csc(\pi\alpha)P_1^{(-\alpha,-\beta)}(t_k) + \frac{1}{n}\sum_{j=0}^{n}{}'' \frac{\tilde{\rho}(s_j)}{s_j - t_k}\right] \\ &+ \frac{1}{n}\sum_{j=0}^{n}{}'' \frac{y^*(s_j)}{s_j - t_k} + \varepsilon_Q(t_k) = f(t_k), \quad k = 1, \dots, n\end{aligned} \tag{3.1}$$

where $\varepsilon_Q(x)$ is the quadrature error. This system is of size n by $(2n+1)$. To obtain a square system we may use Lagrange interpolation:

$$\begin{aligned}y^*(s_j) = \sum_{k=1}^{n} y^*(t_k)&\frac{T_n(s_j)}{(s_j - t_k)T_n'(t_k)} + \varepsilon_I(s_j), \\ &j = 1, \dots, n-1\end{aligned} \tag{3.2}$$

where $\varepsilon_I(x)$ represents the interpolation error. We also need the known information on the behavior of the solution at the endpoints:

$$y^*(s_0) = y^*(s_n) = 0. \tag{3.3}$$

Dropping the error terms, we obtain a square system for the unknown vector

$$\underline{y} = (y(s_0)/\sqrt{2}, y(s_1), \dots, y(s_{n-1}), y(s_n)/\sqrt{2}, y(t_1), \dots, y(t_n))^T$$

approximating the exact solution y^*. The system can be cast in the partitioned form

$$M\underline{y} \equiv \begin{pmatrix} A & B \\ \tilde{I} & \tilde{L} \end{pmatrix} \underline{y} = \underline{f}. \tag{3.4}$$

Here

$$
\begin{aligned}
\underline{\mathrm{f}} &= (f(t_1),\dots,f(t_n),0\dots 0)^T = (\underline{\mathrm{b}}^T,\underline{0}^T_{n+1})^T\\
B &= \operatorname{diag}(\beta_k), \quad k=1,\dots,n\\
\beta_k &= -\frac{1}{\tilde\rho(t_k)}\left[2\csc(\pi\alpha)P_1^{(-\alpha,-\beta)}(t_k)+\frac{1}{n}{\sum_{j=0}^{n}}''\frac{\tilde\rho(s_j)}{s_j-t_k}\right]
\end{aligned}
$$

$$
A_{i,j+1} = \begin{cases} \frac{1}{n}\frac{1}{s_j-t_i} & \text{for } j=1,\dots,n-1 \quad i=1,\dots,n\\ \frac{1}{\sqrt{2}n}\frac{1}{s_j-t_i} & \text{if } j=0,n \quad i=1,\dots,n \end{cases}
$$

$$
\tilde I = \operatorname{diag}(\sqrt{2},1,\dots,1,\sqrt{2})
$$

$$
\tilde L_{j+1,k} = \begin{cases} -\frac{T_n(s_j)}{(s_j-t_k)T_n'(t_k)} = \frac{(-1)^{j+k}\sqrt{1-t_k^2}}{n(s_j-t_k)} & j=1,\dots,n-1\\ & k=1,\dots,n\\ 0 & \text{if } j=0,n \quad k=1,\dots,n. \end{cases}
$$

I and L represent $\tilde I$ and $\tilde L$ once their first and last rows are removed. Note that the system is of order $(2n+1)$, the same size as the system arising from ordinary Gaussian quadrature. But conditions (3.3) are not implemented.

Alternatively, in addition to Lobatto-Chebyshev quadrature we can apply Gauss-Chebyshev quadrature to (2.7) to obtain

$$
\tilde M\underline{\tilde{\mathrm{y}}} \equiv \begin{pmatrix} A & B\\ P & Q\end{pmatrix}\underline{\tilde{\mathrm{y}}} = \underline{\tilde{\mathrm{f}}} \tag{3.5}
$$

with

$$
\begin{aligned}
\underline{\mathrm{f}} &= (f(t_1),\dots,f(t_n),f(s_0),\dots,f(s_n))^T\\
P &= \operatorname{diag}(\sqrt{2},\gamma_1,\dots,\gamma_{n-1},\sqrt{2})\\
\gamma_j &= -\frac{1}{\tilde\rho(s_j)}\left[2\csc(\pi\alpha)P_1^{(-\alpha,-\beta)}(s_j)+\frac{1}{n}{\sum_{j=0}^{n}}''\frac{\tilde\rho(t_k)}{t_k-s_j}\right]
\end{aligned}
$$

$$
Q_{j,k+1} = \begin{cases} \frac{1}{n}\frac{1}{t_k-s_j} & \text{for } j=1,\dots,n-1 \quad k=1,\dots,n\\ 0 & \text{if } j=0,n \quad k=1,\dots,n \end{cases}
$$

Observe that, on using (A.6) and (2.4) we have for $k = 1, \dots, n$,

$$-2\csc(\pi\alpha)P_1^{(-\alpha,-\beta)}(t_k) - \frac{1}{n}\sum_{j=0}^{n}{}'' \frac{\tilde{\rho}(s_j)}{s_j - t_k}$$
$$= \frac{a}{b}\rho(t_k) + \frac{1}{\pi}\int_{-1}^{1}\rho(t)\frac{dt}{t-t_k} - \frac{1}{n}\sum_{j=0}^{n}{}'' \frac{\tilde{\rho}(s_j) - \tilde{\rho}(t_k)}{s_j - t_k}$$
$$\doteq \frac{a}{b}\rho(t_k) + \frac{1}{\pi}\tilde{\rho}(t_k)\int_{-1}^{1}\frac{dt}{(t-t_k)\sqrt{1-t^2}} = \frac{a}{b}\rho(t_k),$$

the equality being approximated because we have suppressed the Lobatto-Chebyshev quadrature error. Proceeding similarly for γ_j's we can then rewrite the coefficients:

$$\beta_k = \frac{a}{b\sqrt{1-t_k^2}}, \quad k = 1, \dots, n, \qquad \gamma_j = \frac{a}{b\sqrt{1-s_j^2}}, \quad j = 1, \dots, n-1. \tag{3.6}$$

For the error analysis, we need two preliminary results on the singular value decompositions of the submatrices

$$C = [A, B] \qquad G = [\tilde{I}, \tilde{L}] \qquad H = [P, Q]. \tag{3.7}$$

The first result is a straightforward generalization of [**10**].

Proposition 1. *The matrix C has full rank. Its singular values are*:

$$\delta_i = [|b|(1-t_i^2)^{1/2}]^{-1} \quad i = 1, \dots, n \tag{3.8}$$

and the corresponding two sets of singular vectors are:

$$\underline{\mathrm{v}}_i^* = \underline{\mathrm{e}}_i \quad i = 1, \dots, n \tag{3.9}$$

$$\underline{\omega}_i = (n\delta_i)^{-1}\Big[\frac{1}{\sqrt{2}}(1-t_i)^{-1}, (s_1 - t_i)^{-1}, \dots, (s_{n-1} - t_i)^{-1}, \frac{-1}{\sqrt{2}}(1+t_i)^{-1}, n\beta_i\underline{\mathrm{e}}_i^T\Big]^T, \tag{3.10}$$

for $i = 1, \ldots, n$. *Furthermore,* $\ker(C)$ *is spanned by the linearly independent set*:

(3.11)
$$\begin{aligned} u_1^* &= (1, \underline{0}_n^T, -(\sqrt{2}n\beta_1)^{-1}(1-t_1)^{-1}, \ldots, -(\sqrt{2}n\beta_n)^{-1}(1-t_n)^{-1})^T \\ \underline{u}_{j+1}^* &= (0, \underline{e}_j^T, -(n\beta_1)^{-1}(s_j-t_1)^{-1}, \ldots, -(n\beta_n)^{-1}(s_j-t_n)^{-1})^T \\ &\qquad j = 1, \ldots, n-1. \end{aligned}$$

$$u_{n+1}^* = (0, \underline{e}_n^T, (\sqrt{2}n\beta_1)^{-1}(1+t_1)^{-1}, \ldots, (\sqrt{2}n\beta_n)^{-1}(1+t_n)^{-1})^T.$$

Also, H *has full rank, and its singular values are*

$$\tilde{\delta}_j = [|b|(1-s_j^2)^{1/2}]^{-1} \quad j = 1, \ldots, n-1, \quad \tilde{\delta}_j = 1, \quad j = 0, n.$$

The associated singular vectors are

$$\begin{aligned} &\underline{v}_i^{**} = \underline{e}_i \quad i = 1, \ldots, n+1, \\ &\underline{\omega}_j^* = (n\tilde{\delta}_j)^{-1}[\underline{0}_j^T, n\gamma_j, \underline{0}_{n-j}^T, (t_1-s_j)^{-1}, \ldots, (t_n-s_j)^{-1}]^T \\ &\qquad j = 1, \ldots, n-1, \\ &\underline{\omega}_0^* = (1, \underline{0}_{2n}^T) \qquad \underline{\omega}_n^* = (\underline{0}_n^T, 1, \underline{0}_n^T). \end{aligned}$$

Finally, $\ker(H)$ *is spanned by the linearly independent set*

$$\underline{u}_k^{**} = [0, (n\gamma_1(t_k-s_1))^{-1}, \ldots, (n\gamma_{n-1}(t_k-s_{n-1}))^{-1}, 0, \underline{e}_k^T], \\ k = 1, \ldots, n.$$

Proposition 2. *The matrix* G *of* (3.7) *has the singular values*

(3.12)
$$\mu_1 = \sqrt{1 + \frac{1}{n}}, \qquad \mu_2 = \cdots = \mu_{n+1} = \sqrt{2}$$

The associated orthonormalized singular vectors are

$$(\underline{v}_1)_k = \begin{cases} (n-1)^{-1/2}(-1)^k & k = 2, \ldots, n \\ 0 & k = 1, n+1 \end{cases}$$

for $j = 2, \ldots, n-1$,

(3.13)
$$(v_j)_k = \begin{cases} (-1)^{j+k}(j(j-1))^{-1/2} & 2 \le k \le j \\ \sqrt{1-\frac{1}{j}} & k = j+1 \\ 0 & k = 1 \ \mathit{or}\ j+2 < k < n+1 \end{cases}$$

$$\underline{v}_n = \underline{e}_1 \quad \underline{v}_{n+1} = \underline{e}_{n+1}$$

and

$$(\underline{u}_1)_k = \begin{cases} (-1)^k[n/(n^2-1)]^{1/2} & 2 \le k \le n \\ (-1)^l[n(n^2-1)]^{-1/2}t_l(1-t_l^2)^{-1/2} & n+1 < k \le 2n+1 \\ & l = k-n-1 \\ 0 & k = 1, n+1 \end{cases}$$

(3.14)
$$(\underline{u}_j)_k = \begin{cases} 0 & k = 1, n+1 \quad j{+}2 \le k \le n{+}1 \\ (-1)^{j+k}(2j(j-1))^{-1/2} & 2 \le k \le j \\ [(j-1)/2j]^{1/2} & k = j+1 \\ [2j(j-1)]^{-1/2}\frac{(-1)}{n}^{j-1+l} & \sqrt{1-t_l^2}\left[\sum_{i=1}^{j-1}\frac{1}{s_i-t_l} - \frac{j-1}{s_j-t_l}\right] \\ & \qquad n{+}2 \le k \le 2n{+}1 \\ & \qquad l = k-n-1 \end{cases}$$

$$j = 2, \dots, n-1 \quad \underline{u}_n = (1, \underline{0}_{2n}^T)^T \quad \underline{u}_{n+1} = (\underline{0}_n^T, 1, \underline{0}_n^T)^T$$

Finally, $\ker(G)$ *is spanned by the following set of linearly independent vectors*:
(3.15)
$$(\underline{\mathrm{r}}_i^*)_k = \begin{cases} \frac{(-1)^{i+k}}{n}\frac{1}{s_{k-1}-t_i} & 2 \le k \le n \\ 0 & k = 1 \quad n{+}1 \le k \le 2n{+}1, \quad k \ne i{+}n{+}1 \\ (1-t_i^2)^{-1/2} & k = n+1+i \end{cases}$$

$$i = 1, \dots, n.$$

Proof. Observe that the matrix GG^T has the block diagonal structure: $GG^T = \operatorname{diag}(2, E, 2)$ with $E \equiv I + LL^T$. Since in view of (A.3) and

(A.4) we have

$$
\begin{aligned}
(LL^T)_{ij} &= \frac{(-1)^{i+j}}{n^2}\sum_{k=1}^{n}\frac{1-t_k^2}{s_j-s_i}\left(\frac{1}{s_i-t_k}-\frac{1}{s_j-t_k}\right)\\
&= \frac{(-1)^{i+j}}{n^2(s_j-s_i)}\Bigg\{(1-s_i^2)\sum_{k=1}^{n}\frac{1}{s_i-t_k}-(1-s_j^2)\sum_{k=1}^{n}\frac{1}{s_j-t_k}\\
&\qquad +\sum_{k=1}^{n}(s_i+t_k-s_j-t_k)\Bigg\} = \frac{(-1)^{i+j+1}}{n}\\
&\qquad i,j=1,\dots,n-1,\quad i\neq j
\end{aligned}
$$

and

$$
\begin{aligned}
(LL)^T_{ii} &= \frac{1}{n^2}\sum_{k=1}^{n}\frac{1-t_k^2}{(s_i-t_k)^2} = \frac{1}{n^2}\Bigg[(1-s_i^2)\sum_{k=1}^{n}\frac{1}{(s_i-t_k)^2}\\
&\qquad +\sum_{k=1}^{n}\left(-1+\frac{2s_i}{s_i+t_k}\right)\Bigg] = 1-\frac{1}{n}\quad i=1,\dots,n-1,
\end{aligned}
$$

E has the form:

$$
E_{ij} = \begin{cases} \frac{(-1)}{n}^{i+j-1} & i\neq j,\quad i,j=1,\dots,n-1\\ 2-\frac{1}{n} & i=j=1,\dots,n-1. \end{cases}
$$

This is a Toeplitz matrix, of which the eigenvalues are needed. Define the elementary matrix R by

$$
R_{ij} = \begin{cases} 1 & \text{if } i=j \text{ or } i=j+1,\quad i,j=1,\dots,n-1\\ 0 & \text{otherwise.} \end{cases}
$$

Let $E(\lambda)=R[E-\lambda I]$. Easily, $\det E(\lambda)=\det[E-\lambda I]$ so that the eigenvalues of $E(\lambda)$ and of E are the same. Now:

$$
\begin{aligned}
E(\lambda)_{1j} &= \begin{cases} 2-\frac{1}{n}-\lambda & j=1\\ \frac{(-1)^j}{n} & j=2,\dots,n-1 \end{cases}\\
E(\lambda)_{ii} &= E(\lambda)_{i,i-1} = 2-\lambda\quad i=2,\dots,n-1\\
E(\lambda)_{ij} &= 0\quad \text{otherwise.}
\end{aligned}
$$

Expanding the determinant of $E(\lambda)$ along the first row the characteristic polynomial can easily be expressed as:

$$p_{n-1}(\lambda) = (2-\lambda-(1/n))(2-\lambda)^{n-2} - (1/n)(n-2)(2-\lambda)^{n-2}.$$

From this the eigenvalues of E are obtained: $\lambda_1 = 1 + 1/n$ and $\lambda_2 = \cdots = \lambda_{n-1} = 2$; thus the singular values (3.14) follow immediately. The corresponding eigenvectors satisfy the equations:

$$(\underline{\mathrm{x}}_1)_{i-1} = -(\underline{\mathrm{x}}_1)_i \quad i = 2, \dots, n-1$$

and

$$\frac{1}{n}\sum_{i=1}^{n-1}(-1)^i(\underline{\mathrm{x}}_j)_i = 0 \quad j = 2, \dots, n-1.$$

It follows that the first eigenvector is

$$(\underline{\mathrm{x}}_1)_k = (-1)^{k-1} \quad k = 1, \dots, n-1$$

while the remaining ones are

$$\begin{aligned}&(\underline{\mathrm{x}}_j)_{j-1} = (\underline{\mathrm{x}}_j)_j = 1 \quad j = 2, \dots, n-1\\ &(\underline{\mathrm{x}}_j)_k = 0 \quad \text{otherwise.}\end{aligned}$$

However, these vectors are not orthogonal. Let

$$\underline{\mathrm{v}}_1 = ||\underline{\mathrm{x}}_1||_2^{-1}\underline{\mathrm{x}}_1 = (n-1)^{-1/2}\underline{\mathrm{x}}_1.$$

Note that $\underline{\mathrm{v}}_1^T\underline{\mathrm{x}}_j = 0$, $j = 2, \dots, n-1$.

Application of the Gram-Schmidt procedure yields $\underline{\mathrm{v}}_2 = (1/\sqrt{2})\underline{\mathrm{x}}_2$.

Now let us assume the general shape (3.13) for $\underline{\mathrm{v}}_j$ and show that the same formula holds for $\underline{\mathrm{v}}_{j+1}$. By induction (3.13) will then be proved. Observe that $\underline{\mathrm{x}}_{j+1}^T\underline{\mathrm{v}}_k = 0$, $k = 1, \dots, j-1$.

In the orthogonalization procedure it is then enough to consider

$$\tilde{\underline{\mathrm{x}}}_{j+1} = \underline{\mathrm{x}}_{j+1} - (\underline{\mathrm{x}}_{j+1}^T\underline{\mathrm{v}}_j)\underline{\mathrm{v}}_j = \underline{\mathrm{x}}_{j+1} - (1-(1/j))^{1/2}\underline{\mathrm{v}}_j.$$

Componentwise:

$$
\begin{aligned}
(\underline{\tilde{x}}_{j+1})_k &= \frac{(-1)^{j+k-1}}{j-1}\frac{j-1}{j} = \frac{(-1)^{j+1+k}}{(j+1)-1} \quad k=1,\dots,j-1 \\
(\underline{\tilde{x}}_{j+1})_j &= 1-\frac{j-1}{j} = \frac{1}{(j+1)-1} \\
(\underline{\tilde{x}}_{j+1})_{j+1} &= 1 \\
(\underline{\tilde{x}}_{j+1})_k &= 0 \quad k=j+2,\dots,n-1.
\end{aligned}
$$

Moreover,

$$
||\underline{\tilde{x}}_{j+1}||^2 = \sum_{k=1}^{j}\frac{1}{j^2}+1 = \frac{j+1}{j}.
$$

From these considerations, the representation (3.13) of the singular vectors of G is obtained by embedding these results in the block structure of GG^T. The other related set of singular vectors is constructed from

$$
\mu_i \underline{u}_i = G^T \underline{v}_i \quad i=1,\dots,n+1.
$$

Since $\underline{v}_i$'s form an orthonormal set of vectors, so will the $\underline{u}_i$'s. Observe that from (A.1),

$$
(L^T \underline{v}_1)_l = \frac{(-1)^{l+1}}{n\sqrt{n-1}}\sqrt{1-t_l^2}\sum_{k=1}^{n-1}\frac{1}{s_k-t_l} = \frac{(-1)^l}{n\sqrt{n-1}}\frac{t_l}{\sqrt{1-t_l^2}}
$$
$$
l=1,\dots,n.
$$

Also, for $j=1,\dots,n-2$,

$$
(L^T \underline{v}_{j+1})_l = \frac{(-1)^{j+l}}{n}\left(\frac{1-t_l^2}{j(j+1)}\right)^{1/2}\left(\sum_{k=1}^{j}\frac{1}{s_k-t_l}-\frac{j}{s_{j+1}-t_l}\right)
$$
$$
l=1,\dots,n.
$$

From these computations (3.14) is immediate. To find the spanning set for ker (G), let $\underline{r}^* = (\underline{\tilde{r}}_1^T, \underline{\tilde{r}}_2^T)^T$ be a solution of the homogeneous system $G\underline{r}^* = \underline{0}$ with $\underline{\tilde{r}}_1 \in \mathbf{R}^{n+1}$, $\underline{\tilde{r}}_2 \in \mathbf{R}^n$. The system can be cast in the form: $\tilde{I}\underline{\tilde{r}}_1 = -\tilde{L}\underline{\tilde{r}}_2$ so that, on taking subsequently $\underline{\tilde{r}}_2 = (1-t_i^2)^{-1/2}\underline{e}_i$, $i=1,\dots,n$, we obtain the linearly independent vectors (3.15). □

For the remainder of this section and the following one, we make the following restrictive assumption on the coefficients of the equation:

$$|b| < a. \tag{3.16}$$

We can now show

Proposition 3. *Under assumption* (3.16), *systems* (3.4) *and* (3.5) *are nonsingular.*

Proof. The diagonal matrix B is square and nonsingular in view of the representation (3.6) of its elements. Let $\underline{c} = (\underline{c}_1^T, \underline{c}_2^T)^T$ be a nontrivial solution of the homogeneous system $M\underline{c} = \underline{0}$. From the second set of equations, $\underline{c}_1 = -\tilde{L}\underline{c}_2$ so that $B(I - B^{-1}A\tilde{L})\underline{c}_2 = \underline{0}$. Observe now that $||\tilde{L}||_2^2 = \max\sigma(L)$ where $\sigma(L)$ denotes the singular values of L. These are obtained from those of E, in Proposition 2, by a shift of -1. Note also that LL^T and L^TL have the same nonzero eigenvalues. It follows that $||L||_2^2 = \max(1, (1/n)) = 1$. Also, from Proposition 1,

$$B^{-1}AA^TB^{-1} = \frac{b^2}{a^2}\mathrm{diag}\sqrt{1-t_k^2}\,\mathrm{diag}\,\frac{1}{1-t_k^2}\,\mathrm{diag}\sqrt{1-t_k^2}$$

yielding in view of the assumption $||B^{-1}A||_2 = |b|/a < 1$. The matrix $I - B^{-1}A\tilde{L}$ is thus invertible and thus $\underline{c}_2 = \underline{0}$, implying $\underline{c}_1 = \underline{0}$. Contradiction. (3.5) is reduced to $B(I - B^{-1}AP^{-1}Q)\underline{c}_2 = \underline{0}$; then observe that

$$||B^{-1}AP^{-1}Q||_2 \leq ||B^{-1}A||_2||P^{-1}Q||_2 \leq b^2a^{-2} < 1. \qquad \square$$

4. Error analysis. In the discretization procedure there are two types of error vectors. Let $\underline{\varepsilon}_L$ be the Lobatto-Chebyshev quadrature error, $\underline{\varepsilon}_G$ the Gauss-Chebyshev quadrature error, and $\underline{\varepsilon}_I$ the interpolation error. The exact solution $\underline{y}^*$ of (3.4) satisfies the system:

$$M\underline{y}^* = \underline{b}^* \equiv (\underline{b}^T - \underline{\varepsilon}_L^T, \underline{\varepsilon}_I^T)^T. \tag{4.1}$$

Define the error vector $\underline{e} = \underline{y}^* - \underline{y} = (\underline{e}_L^T, \underline{e}_I^T)^T$. Subtraction of (3.4) from (4.1) leads to the error equation

$$M\underline{e} = (-\underline{\varepsilon}_L^T, \underline{\varepsilon}_I^T)^T. \tag{4.2}$$

Solving the second set of equations and substituting into the first one yields:

$$(B - AL)\underline{e}_I = -\underline{\varepsilon}_L - A\underline{\varepsilon}_I$$

Taking norms and recalling the proof of Proposition 3,

$$\begin{aligned}||\underline{e}_I||_2 &\le ||(I - B^{-1}AL)^{-1}||_2(||B^{-1}\underline{\varepsilon}_L||_2 + ||B^{-1}A\underline{\varepsilon}_I||_2)\\ &\le (1 - ||B^{-1}A||_2||L||_2)^{-1}\frac{|b|}{a}(||\underline{\varepsilon}_L||_2 + ||\underline{\varepsilon}_I||_2).\end{aligned}$$

Then from the above estimate,

$$||e||_\infty \le 2|b|(a - |b|)^{-1}n^{1/2}\max(||\underline{\varepsilon}_I||_\infty, ||\underline{\varepsilon}_L||_\infty).$$

For (3.5), minor modifications yield the same estimate, with $\underline{\varepsilon}_I$ replaced by $\underline{\varepsilon}_G$. Both the interpolation and the quadrature errors can be given in terms of the best approximation error $E_n(h)$ using the Lebesgue constant Λ_U for the set of nodes $U = \{s_0, \dots, s_n\}$.

The use of Jackson's theorem and the fact that $h \in H_\mu[-1,1]$, together with known bounds on the Lebesgue constant yield

$$\begin{gathered}||\underline{\varepsilon}_L||_\infty, ||\underline{\varepsilon}_G||_\infty \le (1 + \Lambda_U)E_n(h) \le C_1 \ln n\, \omega(h; 1/n) \le C_2 n^{-\mu+\varepsilon},\\ ||\underline{\varepsilon}_I|| \le C_3 n^{-\lambda+\varepsilon},\end{gathered}$$

with $\varepsilon > 0$ arbitrarily small, and where $C_1, \dots, C_5$ represent constants. Summarizing,

Proposition 4. *Under assumption* (3.16) *for the discretization error, the following estimate holds*

$$||\underline{e}||_\infty \le C_4 n^{-\mu+1/2+\varepsilon}. \tag{4.3}$$

We finally address the question of the convergence of the solution, reconstructed from the solution vector $\underline{y}$, to the analytical solution $y^*(x)$. The approximate solution can be represented by the Lagrange interpolatory polynomial $p_{2n-2}(x)$ on the nodes $\{t_1, s_1, \dots, s_{n-1}, t_n\}$.

Since these are the zeros $x_k = \cos(k\pi/2n)$ of $U_{2n-1}(x)$, $k = 1, \dots, 2n-1$,

$$p_{2n-2}(x) = \sum_{k=1}^{2n-1} y(x_k) \frac{U_{2n-1}(x)}{(x-x_k)U'_{2n-1}(x_k)}.$$

Let $p^*_{2n-2}(x)$ be the polynomial of degree $2n-2$ interpolating on the exact values of $y^*(x_k)$. Thus, if Λ_U represents the Lebesgue constant for the nodes x_k,

$$\begin{aligned} &|p^*_{2n-2}(x) - p_{2n-2}(x)| \\ &\qquad \le \sum_{k=1}^{2n-1} |y^*(x_k) - y(x_k)| \left| \frac{U_{2n-1}(x)}{(x-x_k)U'_{2n-1}(x_k)} \right| \le ||\underline{\mathrm{e}}||_\infty \Lambda_U. \end{aligned}$$

Taking the maximum and using this result in the triangular inequality:

$$\begin{aligned} &||y^*(x) - p_{2n-2}(x)||_\infty \le ||y^*(x) - p^*_{2n-2}(x)||_\infty \\ &\qquad + ||p^*_{2n-2}(x) - p_{2n-2}(x)||_\infty \le C_3\omega(y^*; 1/n) + C_4||\underline{\mathrm{e}}||_\infty \ln n. \end{aligned}$$

Combining these estimates with (4.5), since $y^* \in H_\lambda[-1,1]$,

$$(4.4) \qquad ||y^*(x) - p_{2n-2}(x)||_\infty \le C_5 n^{-\eta} \quad \eta = \min(\lambda, \mu - 1/2 - \varepsilon).$$

Summarizing:

Proposition 5. *Under assumption* (3.16) *the convergence of the proposed methods is ensured by the above estimate, which also gives the rate of convergence.*

Remark. A similar result holds for the polynomial $p_{2n}(x)$ constructed on the same nodes as above and on the endpoints $-1, 1$.

5. The complete equation. We consider now the complete equation

$$(5.1) \qquad a\phi(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{\phi(t)\,dt}{t-x} + \int_{-1}^{1} k(x,t)\phi(t)\,dt = \tilde{f}(x).$$

We assume that $k(x,t) \in C^\circ([-1,1]\times[-1,1])$. Introducing the new unknown $y^*(x)$ as done in Section 2 and proceeding with Lobatto-Chebyshev quadrature, we obtain

$$\begin{pmatrix} A+K & B \\ \tilde{I} & \tilde{L} \end{pmatrix} \underline{\mathrm{y}} = \underline{\mathrm{f}} \tag{5.2}$$

with

$$K_{i,j+1} = \begin{cases} \frac{\pi}{nb} k(t_i, s_j) & 1 \le j \le n-1 \\ \frac{\pi}{\sqrt{2}nb} k(t_i, s_j) & j = 0, n \end{cases} \qquad i = 1, \dots, n.$$

Alternatively, we get

$$\begin{pmatrix} A+K & B \\ P & Q+\hat{K} \end{pmatrix} \underline{\mathrm{y}} = \underline{\mathrm{f}} \tag{5.3}$$

$$\hat{K}_{i,j+1} = \begin{cases} \frac{\pi}{nb} k(s_j, t_i) & 1 \le j \le n-1 \\ 0 & j = 0, n \end{cases} \qquad i = 1, \dots, n.$$

The two-norm of the matrix K can be estimated, by means of its Frobenius norm

$$||K||_2 \le ||k||_\infty / |b| \tag{5.4}$$

where

$$||k||_\infty = \max_{-1 \le x,t \le 1} |k(x,t)|.$$

To proceed with the error analysis, we need a further very strong restriction of (3.16)

$$|b| + ||k||_\infty \le a. \tag{5.5}$$

The procedure of Section 4 yields

$$[B - (A+K)L]\underline{\mathrm{e}}_2 = -\underline{\varepsilon}_L - (A+K)\underline{\varepsilon}_I,$$

from which, on taking norms

$$||\underline{\mathrm{e}}_2||_\infty \le \frac{|b|}{a - |b| - ||k||_\infty} [||\underline{\varepsilon}_L||_2 + (1 + ||K||_2)||\underline{\varepsilon}_I||_2].$$

A similar estimate holds for (5.3). From these, we have

Proposition 6. *Under assumption* (5.5) *the methods are convergent with rate* $\mu - 1/2 - \varepsilon$. *If the solution is reconstructed on the whole interval by means of an interpolatory formula, it converges to* $y^*(x)$ *with rate* $\min(\lambda, \mu - 1/2 - \varepsilon)$.

Remark. The previous results show convergence, and the theoretical rate seems to be low and dependent essentially on the endpoint singularities α and β of the fundamental function. Such estimates are commonly accepted in the literature on singular integrals, see e.g., [**1,14**].

6. Conclusions. It is interesting to note that for $\alpha = \beta \cong .5$, i.e., for $a = 0$, the condition number of the system increases. This can be theoretically explained, since from (3.6), $\beta_k = 0$, $k = 1, \ldots, n$. Thus $B \equiv O$; and also, from (A.3), if we sum the rows of A we obtain the zero vector. Thus, the matrix C becomes rank deficient, and the whole system is then singular. For $a = 0$, (2.1) reduces to

$$\tilde{f}(x) = \pi^{-1} \int_{-1}^{1} \frac{\phi(t)\,dt}{t - x} \equiv \pi^{-1} \int_{-1}^{1} y^*(t)(1 - t^2)^{-1/2}(t - x)^{-1}\,dt$$

and after Lobatto-Chebyshev discretization, we get

$$\frac{1}{n} \sum_{j=0}^{n}{}'' \frac{y^*(s_j)}{s_j - t_k} = f(t_k), \quad k = 1, \ldots, n.$$

This is an overdetermined system similar to the one discussed in [**9,13**]. If the discretized orthogonality condition is satisfied by $f(x)$, then in view of (A.3), one of the equations is redundant and can be omitted. No extra unknowns arise and there is no need for the interpolatory formulae. The system has size n.

This method is not a general method for solving SIE's. In engineering applications where systems of moderate or large size are very likely to arise, a simple algorithm in place of an expensive method, if low accuracy is sufficient, may be a viable alternative. This algorithm could be useful also for equations whose solutions possess a singularity in some of the first derivatives, at an unknown location within $(-1,1)$, other than the endpoints. To support this claim we show in the table that this method and Gauss-Jacobi provide similar answers if applied to such a situation. The performance of both algorithms is degraded by the presence of the singularity in the third derivative of the solution. While this dramatically affects the classical rule, it seems to influence much less the algorithm proposed here. The computations have been performed in double precision on a 80386-based machine.

Appendix

In the paper the following identities have been used [**18,10**], with $T_n(t_i)=0$, $i=1,\dots,n$, $U_{n-1}(s_j)=0$, $j=1,\dots,n-1$.

$$\sum_{j=1}^{n-1}\frac{1}{t_k-s_j}=\frac{t_k}{1-t_k^2}\quad k=1,2,\dots,n \tag{A.1}$$

$$\sum_{j=1}^{n-1}\frac{1}{(t_k-s_j)^2}=\frac{n^2}{1-t_k^2}-\frac{1+t_k^2}{(1-t_k^2)^2}\quad k=1,\dots,n \tag{A.2}$$

$$\sum_{k=1}^{n}\frac{1}{s_j-t_k}=0\quad j=1,\dots,n-1 \tag{A.3}$$

$$\sum_{k=1}^{n}\frac{1}{(s_j-t_k)^2}=\frac{n^2}{1-s_j^2}\quad j=1,\dots,n-1 \tag{A.4}$$

$$\sum_{k=1}^{n}\frac{1}{1-t_k^2}=n^2 \tag{A.5}$$

$$\sum_{j=0}^{n}{}''\frac{1}{s_j-t_k}=0\quad k=1,\dots,n \tag{A.6}$$

TABLE. Solution: $(.1+x)^{2.3}$ if $x > -.1$ and $-(.1+x)^{2.3}$ if $x < -.1$.

$a = 3.00\text{E-}1$ $b = 9.54\text{E-}1$ $\alpha = 5.97\text{E-}1$ $\beta = 4.03\text{E-}1$

CONVENTIONAL ALGORITHM g_n denotes the approximate solution of system (2.5)

size of system	condition number	error norm $\|g - g_n\|_\infty$	time used in seconds
3	9.93	.26D+00	.06
7	13.28	.30D−01	.11
15	27.20	.17D−01	.38
31	54.91	.13D−01	1.54
63	118.66	.15D−02	6.98
127	300.07	.36D−03	37.35

total time used in seconds 46.42

$a = 3.00\text{E-}1$ $b = 9.54\text{E-}1$ $\alpha = 5.97\text{E-}1$ $\beta = 4.03\text{E-}1$

LOBATTO CHEBYSHEV ALGORITHM y_n denotes the solution of system (3.4)

size of system	condition number	error norm $\|g - y_n/\tilde{\rho}\|_\infty$	time used in seconds
3	14.92	.67D+00	.06
7	13.02	.99D−01	.16
15	22.29	.11D−01	.33
31	45.14	.17D−02	1.43
63	77.27	.35D−03	6.70
127	222.85	.97D−04	36.47

total time used in seconds 45.15

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Mathematics Department, University of Iowa, Iowa City, IA 52242