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NUMERICAL SOLUTIONS FOR WEAKLY SINGULAR HAMMERSTEIN EQUATIONS AND THEIR SUPERCONVERGENCE

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ABSTRACT. In the recent paper [**7**], it was shown that the solutions of weakly singular Hammerstein equations satisfy certain regularity properties. Using this result, the optimal convergence rate of a standard piecewise polynomial collocation method and that of the recently proposed collocation-type method of Kumar and Sloan [**10**] are obtained. Superconvergence of both of these methods are also presented. In the final section, we discuss briefly a standard product-integration method for weakly singular Hammerstein equations and indicate its superconvergence property.

1. Introduction. We consider the Hammerstein equation with weakly singular kernel

$$(1.1)\qquad \varphi(s) - \int_a^b g_\alpha(|s-t|)k(s,t)\psi(t,\varphi(t))\,dt = f(s), \quad a \le s \le b,$$

where

$$g_\alpha(s) = \begin{cases} s^{\alpha-1} & \text{for } 0<\alpha<1 \\ \log s & \text{for } \alpha = 1. \end{cases}$$

Throughout this paper, we assume that

(i) $k \in C([a,b]\times[a,b])$

(ii) $\psi \in C([a,b]\times(-\infty,\infty))$ and satisfies the Lipschitz condition $|\psi(t,y_1)-\psi(t,y_2)| \le A|y_1-y_2|$.

In the recent paper [**7**], it was shown that under assumptions (i), (ii) and

(iii) $AG<1$, where $G \equiv \sup_{a\le s\le b}\int_a^b |g_\alpha(|s-t|)k(s,t)|\,dt$,

there is a unique solution to equation (1.1).

Generalizing the argument of C. Schneider [**14**], regularity properties of the solution φ were also obtained in [**7**]. For our present purposes, these results can be summarized as follows:

Theorem A. *Let $m \in N_0$, where N_0 is the set of nonnegative integers. Let $k \in C^{m+1}([a,b] \times [a,b])$, $f \in C^{(0,\alpha)}[a,b] \cap C^m(a,b)$, and assume that, for $m = 0,1$, $\psi \in C^{(0,1)}([a,b] \times (-\infty,\infty))$ and, for $m \geq 2$, $\psi \in C^{m-1}([a,b] \times (-\infty,\infty))$. Moreover, assume that the functions $y_i(s) \equiv (s-a)^i(b-s)^i f^{(i)}(s)$, for $i = 1,\dots,m$, are α-Hölder continuous on $[a,b]$. Then for $0 < \alpha < 1$ there exists a $\beta \in [\alpha, m]$ or for $\alpha = 1$ there exists $\beta \in [1-\varepsilon, m]$ for any $\varepsilon \in (0,1)$ such that the solution φ of (1.1) belongs to $C^m(a,b) \cap C^{(0,\beta)}[a,b]$. Also, we have $\varphi_i(s) \equiv (s-a)^i(b-s)^i \varphi^{(i)}(s)$ belongs to $C^{(0,\beta)}[a,b]$ for each $i = 1,\dots,m$.*

Here $C^{(0,\alpha)}[a,b]$ denotes the class of α-Hölder continuous functions defined on $[a,b]$. For most cases, $\beta = \alpha$ when $0 < \alpha < 1$ and $\beta = 1 - \varepsilon$ when $\alpha = 1$. Thus we cannot expect the solution to belong to $C^1[a,b]$. In the terminology of J. Rice [**12**], we can also say that φ belongs to the class Type $(\beta, m, \{a,b\})$. The results described in Theorem A parallel the regularity result on the solution of weakly singular Fredholm equations obtained by Schneider in [**14**].

The purpose of this paper is two-fold. First, we present a standard piecewise collocation method and show that the error of its approximation is $\mathcal{O}(n^{-m})$ where n is the number of breakpoints and $m-1$ is the degree of polynomials used. We also show that there is a superconvergence at the collocation points. This is done in Section 2. Second, we review the collocation-type method of Kumar and Sloan [**10**] and prove that their method when applied to equation (1.1) also provides an approximation whose error is of order $\mathcal{O}(n^{-m})$. With a certain condition on the set of partition points, a global superconvergence is attained for the method. This will be presented in Section 3. The results described in Sections 2 and 3 are not unexpected since similar results are available for the Fredholm equations, [**15, 18**]. Moreover, superconvergence results can be obtained by borrowing the arguments from [**15**] and [**18**]. Despite these viewpoints, the results of this paper are new and the authors believe that they will give useful information toward solving equation (1.1) numerically. In addition to the aforementioned papers on weakly singular Fredholm equations, there are many other notable papers on the subject, see, e.g., [**4, 5, 6** and **13**].

In order to make this paper self-contained, several known results will be stated as they are needed. In the final section, Section 4, we make a few comments concerning the product-integration scheme applied to equation (1.1).

2. Piecewise polynomial collocation scheme. Following the idea of Vainikko and Uba [**18**] which is based on the result of Rice [**12**], for $0 < \mu \le m$, $m \in \mathbf{N} \equiv$ the set of all positive integers, let $q = \mu/\alpha$. Also let $n \in \mathbf{N}$. Then the breakpoints of our piecewise polynomial approximation are selected as

$$\begin{aligned} t_i &= a + ((b-a)/2)(2i/n)^q \quad && \text{for } i = 0, 1, \dots, [n/2] \\ t_i &= (b+a) - t_{n-i} && \text{for } i = [n/2]+1, \dots, n. \end{aligned} \tag{2.1}$$

Define a sequence $\{\xi_i\}_{i=1}^m$ of points such that $0 \le \xi_1 < \cdots < \xi_m \le 1$. Also, we let

$$t_{ij} = t_i + \xi_j(t_{i+1} - t_i), \quad i = 0, 1, \dots, n-1; \quad j = 1, \dots, m \tag{2.2}$$

so that

$$t_i \le t_{i1} < t_{i2} < \cdots < t_{im} \le t_{i+1}, \quad i = 0, 1, \dots, n-1.$$

The approach here is to construct the approximate solution φ_n of equation (1.1) as a piecewise polynomial of degree $m-1$ with breakpoints (2.1).

For each $i = 0, 1, \dots, n-1$, let l_{ik} denote the Lagrange fundamental polynomial for the knots $\{t_{ik}\}_{k=1}^m$, so that

$$l_{ik}(\sigma) = \prod_{\substack{l=1 \\ l \ne k}}^{m} \frac{(\sigma - t_{il})}{(t_{ik} - t_{il})}, \quad t_i \le \sigma \le t_{i+1}. \tag{2.3}$$

We require the approximate solution φ_n to satisfy

$$\varphi_n(t_{ij}) - \int_a^b g_\alpha(|t_{ij} - t|) k(t_{ij}, t) \psi(t, \varphi_n(t))\, dt = f(t_{ij}), \tag{2.4}$$

for $i = 0, 1, \dots, n-1$; $j = 1, 2, \dots, m$. If $\xi_1 = 0$ and $\xi_m = 1$, then we assume that (2.4) is satisfied only once at the breakpoints.

For the particular choice of basis functions selected in (2.3), we have $\varphi_n(t) = \sum_{k=1}^m a_{ik} l_{ik}(t)$, for $t_i \le t \le t_{i+1}$, $i = 0, 1, \dots, n-1$; thus (2.4) can be written as

$$\begin{aligned} a_{ij} - \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} & g_\alpha(|t_{ij} - t|) k(t_{ij}, t) \\ & \cdot \psi\bigg(t, \sum_{k=1}^m a_{pk} l_{pk}(t)\bigg)\, dt = f(t_{ij}), \end{aligned} \tag{2.5}$$

for $i = 0, 1, \dots, n-1$ and $j = 1, 2, \dots, m$.

In order to describe equations (1.1) and (2.5) in operator form, we let $\Psi(\varphi)(t) = \psi(t, \varphi(t))$ and

$$(K\Psi)(\varphi)(s) = \int_a^b g_\alpha(|s-t|) k(s,t) \psi(t, \varphi(t))\, dt.$$

Moreover, for any continuous function φ, P_n denotes the interpolation projector defined by

$$(P_n \varphi)(s) = \sum_{k=1}^m \varphi(t_{ik}) l_{ik}(s), \quad t_i \le s \le t_{i+1}, \quad i = 0, 1, \dots, n-1.$$

Then equations (1.1) and (2.5) can be written respectively as

$$\varphi - K\Psi\varphi = f \tag{2.6}$$

and

$$\varphi_n - P_n K \Psi \varphi_n = P_n f. \tag{2.7}$$

Let $||\quad||$ denote the norm in $L_\infty = L_\infty[a,b]$. It is straightforward to verify that $\{||P_n||\}_{n=1}^\infty$ is bounded (considered as operators on $C[a,b]$) and that

$$||P_n \varphi - \varphi|| \to 0 \quad \text{as } n \to \infty, \quad \text{for } \varphi \in C[a,b], \tag{2.8}$$

for our choice of breakpoints, (2.1).

The following propositions of Vainikko and Uba [**18**] concerning the projectors P_n will be used to prove the superconvergence of numerical solutions at the collocation points.

Proposition V-U-1. *If $q = \mu/\alpha \ge 1$ and $\mu \le m$, then $||u - P_n u|| \le cn^{-\mu}$ for each $u \in \text{Type}\,(\alpha, m, \{a,b\})$ where c is a constant.*

Proposition V-U-2. *If $q = \mu/\alpha \ge 1$, $\mu \le m$ and $p = 1/\alpha$, then $||u - P_n u||_p \le c(\mu)\delta_n$ for each $u \in$ Type $(\alpha, m, \{a,b\})$. Here $c(\mu)$ denotes a constant which depends upon μ and*

$$\delta_n = \begin{cases} n^{-m} & \text{for } \mu > m/2 \\ n^{-m}(\ln n)^{\alpha} & \text{for } \mu = m/2 \\ n^{-2\mu} & \text{for } \mu < m/2. \end{cases}$$

Put $T\varphi \equiv K\Psi\varphi + f$ and $T_n\varphi_n \equiv P_n K\Psi\varphi_n + P_n f$ so that (2.6) and (2.7) become

$$\varphi = T\varphi \tag{2.9}$$

and

$$\varphi_n = T_n\varphi_n. \tag{2.10}$$

We are now in a position to state and prove our first theorem. A proof is provided for completeness. The reader who is interested in additional discussion on the solution of nonlinear equations may consult [**3, 8** and **19**].

Theorem 2.1. *Assume that the hypotheses of Theorem* A *are satisfied, and for $0 < \mu \le m$, $m \in \mathbf{N}$, let $q = \mu/\alpha$. Let $\varphi_0 \in C[a,b]$ be an isolated solution of* (1.1). *Assume that the partial derivative Ψ_2 of ψ with respect to the second variable exists and satisfies*

$$|\Psi_2(t, y_1(t)) - \Psi_2(t, y_2(t))| \le C|y_1(t) - y_2(t)| \quad \text{for each } t \in [a,b].$$

Then the piecewise polynomial collocation approximation equation (2.10) *has a unique solution $\varphi_n \in L_\infty[a,b]$ in $||\varphi - \varphi_0|| \le \delta$ for some $\delta > 0$ and* *for sufficiently large n. Moreover, there is a constant Q, $0 < Q < 1$, and α_n such that*

$$\sup_{||\varphi-\varphi_0||\le\delta} ||(I - T_n'(\varphi_0))^{-1}(T_n'(\varphi) - T_n'(\varphi_0))|| \le Q$$

and

$$\alpha_n/(1+Q) \le ||\varphi_n - \varphi_0|| \le \alpha_n/(1-Q).$$

Finally,

$$||\varphi_n - \varphi_0|| = \mathcal{O}(n^{-\mu}).$$

Proof. Since P_n is a bounded linear operator, $(P_nK\Psi)'(\psi_0) = P_n(K\Psi)'(\varphi_0)$, where

$$(K\Psi)'(\varphi_0)(\varphi)(s) = \int_a^b g_\alpha(|s-t|)k(s,t)\Psi_2(t,\varphi_0(t))\varphi(t)\,dt$$

is valid provided that ψ has the first partial derivative Ψ_2 with respect to the second variable. Since $K\Psi$ is a compact operator of $L_\infty[a,b]$ into $C[a,b]$, $(K\Psi)'(\varphi_0)$ is also compact. Now because of (2.8), it is easy to see that $||P_n(K\Psi)'(\varphi_0) - (K\Psi)'(\varphi_0)|| \to 0$ as $n \to \infty$. From this, one can conclude that $(I - P_n(K\Psi)'(\varphi_0))^{-1} = (I - T_n'(\varphi_0))^{-1}$ exists and it is a bounded linear operator for all sufficiently large n, say for all $n \ge N_1$.

Now for $||\varphi - \varphi_0|| \le \delta$ and $n \ge N_1$, we have

$$\begin{aligned}
||T_n'(\varphi) - T_n'(\varphi_0)|| &= ||P_n(K\Psi)'(\varphi) - P_n(K\Psi)'(\varphi_0)|| \\
&\le ||P_n|| \sup_{||\varphi^*||=1} \bigg| \int_a^b g_\alpha(|s-t|)k(s,t)\{\Psi_2(t,\varphi(t)) \\
&\qquad\qquad - \Psi_2(t,\varphi_0(t))\}\varphi^*(t)\,dt \bigg| \\
&\le ||P_n|| GC ||\varphi - \hat{\varphi}_0|| \le M\delta,
\end{aligned}$$

where $M = GC\sup_n ||P_n||$ and G is defined as in (iii) of Section 1. Hence, $\sup_{||\varphi-\varphi_0||\le\delta} ||(I - T_n'(\varphi_0))^{-1}(T_n'(\varphi) - T_n'(\varphi_0))|| \le Q$ with $Q \equiv M\delta||(I - T_n'(\varphi_0))^{-1}||$. Here we take δ so small that $0 < Q < 1$. Now $||T_n(\varphi_0) - T(\varphi_0)|| = ||P_nK\Psi(\varphi_0) - K\Psi(\varphi_0) + P_nf - f||$ and because of (2.8), there exists N_2 so that for $n \geq N_2$, $\alpha_n \equiv ||(I - T_n'(\varphi_0))^{-1}(T_n(\varphi_0) - T(\varphi_0))|| \leq \delta(1-Q)$. Hence, for $n \geq \max\{N_1, N_2\}$, using Theorem 2 of [**17**], one can conclude that (2.10) has a unique solution in $||\varphi - \varphi_0|| \leq \delta$ and the inequalities $\alpha_n/(1+Q) \leq ||\varphi_n - \varphi_0|| \leq \alpha_n/(1-Q)$ hold. To prove the convergence rate, consider

$$\begin{aligned} ||\varphi_n - \varphi_0|| &\leq \frac{\alpha_n}{1-Q} \\ &= \frac{||(I - T_n'(\varphi_0))^{-1}(T_n(\varphi_0) - T(\varphi_0))||}{1-Q} \\ &\leq \frac{||(I - T_n'(\varphi_0))^{-1}||\, ||T_n(\varphi_0) - T(\varphi_0)||}{1-Q} \\ &= \frac{Q}{M\delta(1-Q)} ||P_n K\Psi(\varphi_0) - K\Psi(\varphi_0) + P_n f - f|| \\ &= \frac{Q}{M\delta(1-Q)} ||P_n \varphi_0 - \varphi_0||. \end{aligned}$$

Since P_n is the interpolatory projection defined using the nonuniform breakpoints (2.1), using the regularity result of the solution φ_0 of (1.1) obtained in [**7**], proposition V-U-1 now enables us to conclude that $||\varphi_0 - P_n\varphi_0|| = \mathcal{O}(n^{-\mu})$. $\square$

The next theorem establishes the superconvergence of φ_n to φ_0 at the collocation points. This theorem states in part that in order to obtain the m-th order of accuracy at the collocation points, it is sufficient to take $q = m/2\alpha$. Recall that, to achieve the same order of accuracy in the uniform norm, we must take $q = m/\alpha$ [**18**].

Theorem 2.2. *Let φ_0 and φ_n be defined as in Theorem* 2.1. *If $q = \mu/\alpha > 1$, $\mu \leq m$, then* $\max_{\substack{0\leq i\leq n-1\\ 1\leq j\leq m}} |\varphi_n(t_{ij}) - \varphi_0(t_{ij})| = \mathcal{O}(\varepsilon_n)$ *where, for $0 < \alpha < 1$, $\varepsilon_n = \delta_n(\ln n)^{1-\alpha}$, and for $\alpha = 1$, $\varepsilon_n = \delta_n \ln n$, where δ_n is defined in* V-U-2.

Proof. First we need to show that

(2.11) $||\varphi_n - P_n\varphi_0|| \leq c||K\Psi(P_n\varphi_0) - K\Psi(\varphi_0)||$ for some constant c.

Consider

$$\begin{aligned}\varphi_n - P_n\varphi_0 &= P_n(K\Psi(\varphi_n) - K\Psi(P_n\varphi_0) + K\Psi(P_n\varphi_0) - K\Psi(\varphi_0)) \\ &= P_n(K\Psi)'(P_n\varphi_0 + \theta(\varphi_n - P_n\varphi_0))(\varphi_n - P_n\varphi_0) \\ &\quad + P_n(K\Psi(P_n\varphi_0) - K\Psi(\varphi_0))\end{aligned}$$

for some $0 < \theta < 1$. Hence, 2.11 is obtained. Moreover, using the Lipschitz condition on ψ, we obtain

$$\begin{aligned}&||K\Psi(P_n\varphi_0) - K\Psi(\varphi_0)|| \\ &\qquad \leq A_1||P_n\varphi_0 - \varphi_0|| \sup_{a\leq s\leq b} \int_{\substack{s\in[a,b] \\ |s-t|\leq h}} |g_\alpha(|s-t|)|\, dt \\ &\qquad\quad + A_1||P_n\varphi_0 - \varphi_0||_p \sup_{a\leq s\leq b} \left\{ \int_{\substack{s\in[a,b] \\ |s-t|>h}} |g_\alpha(|s-t|)|^{p'}\, dt \right\}^{1/p'}\end{aligned}$$

where $A_1 = A\sup_{a\leq s,t\leq b}|k(s,t)|$, $0 < h < b-a$, $p = 1/\alpha$ and $1/p + 1/p' = 1$. Using propositions V-U-1 and V-U-2 and noting that $|\varphi_n(t_{ij}) - \varphi_0(t_{ij})| \leq ||\varphi_n - P_n\varphi_0||$, for appropriately chosen values of h (see [**18**, Proposition 3]), we obtain the desired results. □

Example. Consider

$$\varphi(s) - \int_0^1 \frac{\varphi^2(t)}{|s-t|^{1/4}}\, dt = f(s) \quad 0 \leq s \leq 1,$$

where f is selected so that $\varphi(s) = s^{3/4}$ is the solution to be approximated. The collocation points t_{ij} are obtained by mapping the Gaussian points (the zeros of Legendre polynomials) into $[t_i, t_{i+1}]$ for each $i = 0, 1, \ldots, n-1$. We used $m = 2$ and two different values of q for comparison, ($q = m/2\alpha = 4/3$ and $q = m/\alpha = 8/3$ were used in the first and second data respectively). In order to obtain the second order accuracy of the method in the uniform norm, we must take $q = 8/3$ whereas the same order is achieved at the collocation points with $q = 4/3$.

Data 1. ($q = 4/3$)

n	$\max_{t_{ij}} \lvert\varphi(t_{ij}) - \varphi_n(t_{ij})\rvert$	decay exp.	$\lvert\lvert\varphi - \varphi_n\rvert\rvert$	dec. exp.
8	3.16 D-03		1.99 D-02	
16	7.96 D-04	1.99	9.41 D-03	1.08
32	1.95 D-04	2.03	4.57 D-03	1.04
64	4.84 D-05	2.01	2.22 D-03	1.04

Data 2. ($q = 8/3$)

n	$\max_{t_{ij}} \lvert\varphi(t_{ij}) - \varphi_n(t_{ij})\rvert$	decay exp.	$\lvert\lvert\varphi - \varphi_n\rvert\rvert$	dec. exp.
8	2.46 D-03		7.46 D-03	
16	6.47 D-04	1.92	1.74 D-03	2.10
32	1.47 D-04	2.13	4.06 D-04	2.10
64	3.43 D-05	2.10	9.54 D-05	2.09

$||\varphi - \varphi_n||$ was approximated by $\max\{|\varphi(x_i) - \varphi_n(x_i)| : x_i = i/50$ for $i = 0, \dots, 50\}$.

In the remainder of this section, we make some remarks concerning the discrete form of equation (2.5)—i.e., the form obtained when the integrals in (2.5) are replaced by some numerical quadrature. Up to this point, we have paid no attention to the discrete form of equation (2.5). In many of the practical problems, the integrals involved in (2.5) must be computed numerically. In order to accomplish this, it is convenient to utilize some type of product-integration scheme. Consider the integrals in (2.5),

$$\int_{t_i}^{t_{i+1}} g_\alpha(|t_{ij} - t|) k(t_{ij}, t) \psi\bigg(t, \sum_{k=1}^{m} a_{ik} l_{ik}(t)\bigg)\, dt. \tag{2.12}$$

A simple change of variables will transform (2.12) into

$$\frac{t_{i+1} - t_i}{2} \int_{-1}^{1} g_\alpha(|t_{ij} - \xi(x)|) k(t_{ij}, \xi(x)) \psi\bigg(\xi(x), \sum_{k=1}^{m} a_{ik} l_{ik}(\xi(x))\bigg)\, dx$$

where $\xi(x) = [(t_{i+1} - t_i)x + t_i + t_{i+1}]/2$.

Now we outline the results of I. Sloan [**16**] in order to demonstrate a method of approximating the integral in (2.12). Let $\{x_{mk}\}_{k=1}^{m}$ be the

Chebyshev points and $l_{mr}(x_{mk}) = \delta_{rk}$. Then the integral above can be approximated by

$$\frac{t_{i+1}-t_i}{2}\sum_{r=1}^{m} w_{mr}k(t_{ij},\xi(x_{mr}))\psi(\xi(x_{mr}),\sum_{k=1}^{m} a_{ik}l_{ik}(\xi(x_{mr})) \tag{2.13}$$

where $w_{mr} = \int_{-1}^{1} g_\alpha(|t_{ij}-\xi(x)|)l_{mr}(x)\,dx$.

Now, for the m-th degree Chebyshev polynomial of the first kind T_m, we have

$$l_{mr}(x) = \frac{T_m(x)}{(x-x_{mr})T_m'(x_{mr})}. \tag{2.14}$$

By virtue of the Christoffel-Darboux identity (2.14) becomes

$$l_{mr}(x) = \frac{2}{m}\left[\frac{1}{2}+\sum_{k=1}^{m} T_k(x)\cos k\theta_{mr}\right] \quad \text{with } \theta_{mr} = (2r-1)/2m. \tag{2.15}$$

Using (2.13),

$$\begin{aligned} w_{mr} &= \frac{2}{m}\left[\frac{1}{2}a_0 + \sum_{k=1}^{m-1} a_k \cos k\theta_{mr}\right] \\ \text{with } a_k &= \int_{-1}^{1} g_\alpha(|t_{ij}-\xi(x)|)T_k(x)\,dx. \end{aligned} \tag{2.16}$$

For the weakly singular kernel $g_\alpha(s) = s^{\alpha-1}$, $0 < \alpha < 1$, a recurrence relation is available to compute a_k in (2.16) efficiently. See [**11** and **16**] for more detailed discussion on the subject. If $F(x) \equiv k(t_{ij},\xi(x))\psi(\xi(x),\sum_{k=1}^{m} a_{ik}l_{ik}(\xi(x)))$ is sufficiently smooth, e.g., $F \in C^m[-1,1]$, then it is proved [**16**, Theorem 3] that (2.13) approximates (2.12) in the order that is consistent with the order of approximation of φ_0 by φ_n obtained in Theorem 2.1.

3. Method of Kumar and Sloan. In this section, as an alternative approach to the standard collocation method described in the previous section, we present the new collocation-type method of S. Kumar and

I. Sloan [**10**] and improve their results by giving the rate of convergence of numerical solutions. Following the argument given by Schneider in [**15**], we are able to provide a global superconvergence result provided that the partition points $\{\xi_j\}_{j=1}^m$ in (2.2) are chosen properly. In [**10**], the following interesting observation was made. If the system of nonlinear equations (2.5) were to be solved by an iterative scheme, then the integrals in (2.11) must be computed at each step of the iteration. To circumvent this difficulty, we let

$$z(s) \equiv \psi(s, \varphi(s)). \tag{3.1}$$

Upon substituting (3.1) into (1.1),

$$\varphi(s) - \int_a^b g_\alpha(|s-t|)k(s,t)z(t)\,dt = f(s) \quad a \le s \le b. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$z(s) = \psi(s, f(s) + \int_a^b g_\alpha(|s-t|)k(s,t)z(t)\,dt). \tag{3.3}$$

z is approximated by $z_n(t) = \sum_{k=1}^m a_{pk} l_{pk}(t)$ for $t_p \le t \le t_{p+1}$, $p = 0, \dots, n-1$. Using the collocation scheme,

$$z_n(t_{ij}) = \psi(t_{ij}, f(t_{ij}) + \int_a^b g_\alpha(|t_{ij}-t|)k(t_{ij},t)z_n(t)\,dt) \tag{3.4}$$
$$\text{for } i = 0, \dots, n-1 \quad \text{and} \quad j = 1, \dots, m,$$

or equivalently

(3.5)
$$a_{ij} = \psi(t_{ij}, f(t_{ij}) + \sum_{p=0}^{n-1}\sum_{k=1}^{m} a_{pk} \int_{t_p}^{t_{p+1}} g_\alpha(|t_{ij}-t|)k(t_{ij},t)l_{pk}(t)\,dt)$$
$$\text{for } i = 0, \dots, n-1 \quad \text{and} \quad j = 1, \dots, m.$$

In this new system of nonlinear equations, the $a'_{ij}s$ are ingeniously extracted out of the integrals, making the computations of the integrals necessary only once throughout the iteration process. Upon computing a_{ij}, the approximation $\bar{\varphi}_n$ to φ is obtained by

$$(3.6)\qquad \bar{\varphi}_n(s) = f(s) + \sum_{p=0}^{n-1}\sum_{k=1}^{m} a_{pk}\int_{t_p}^{t_{p+1}} g_\alpha(|s-t|)k(s,t)l_{pk}(t)\,dt.$$

The convergence of $\bar{\varphi}_n$ to φ is guaranteed by [**10**, Theorem 2]. The convergence rate is given by Kumar and Sloan through the inequality $||z-z_n|| \leq$ const. $||z-p_nz||$. Hence, the actual rate of convergence depends heavily upon the smoothness of z which, for weakly singular equation (1.1), is not normally in $C^1[a,b]$. The regularity result of Theorem A now allows us to establish the optimal convergence rate.

Theorem 3.1. *Let $m \in N_0$, $k \in C^{m+1}([a,b]\times[a,b])$, $f \in C^m(a,b)$ and the functions $y_i(s) \equiv (s-a)^i(b-s)^i f^{(i)}(s)$, for $i = 0,\dots,m$, be α-Hölder continuous on $[a,b]$. Assume that, for $m = 0,1, \psi \in C^{(0,1)}([a,b]\times(-\infty,\infty))$ and for $m \geq 2$, $\psi \in C^{m-1}([a,b]\times(-\infty,\infty))$. Let $\varphi_0, \bar{\varphi}_n, z$ and z_n be the solutions of* (1.1), (3.6), (3.3) *and* (3.4), *respectively. Here we assume φ_0 to be an isolated solution. If* 1 *is not an eigenvalue of the compact linear operator $(Tw)(s) \equiv \Psi_2(s, f(s) + (Kz)(s))(Kw)(s)$ defined on $C[a,b]$ with*

$$(Kw)(s) \equiv \int_a^b g_\alpha(|s-t|)k(s,t)w(t)\,dt, \text{ then } ||\varphi_0 - \bar{\varphi}_n|| = \mathcal{O}(n^{-m}).$$

Proof. From [**9**, Theorem 2], we have

$$(3.7)\qquad ||\varphi_0 - \bar{\varphi}_n|| \leq c^*||K(z-P_nz)|| \leq c^*||K||\,||z-P_nz||$$

where c^* is a constant independent of n and $||K|| = \sup_{a\leq s\leq b}\int_a^b |g_\alpha(|s-t|)||k(s,t)|\,dt$. In [**7**], it was shown that $\varphi \in \text{Type}\,(\alpha, m, \{a,b\})$ and that $z \in \text{Type}\,(\alpha, m, \{a,b\})$. Since $\{t_i\}_{i=0}^n$ are selected according to (2.1), the result of Rice [**12**] and (3.7) would yield $||\varphi_0 - \bar{\varphi}_n|| = \mathcal{O}(n^{-m})$ upon choosing $q = m/\alpha$ or $q = m/1-\varepsilon$ for any $\varepsilon \in (0,1)$ in the logarithmic case. □

Our next theorem discusses a global superconvergence of $\bar{\varphi}_n$ to φ_0. The inequality (3.7) again serves as a critical factor in obtaining this result.

Theorem 3.2. *Assume that the conditions in Theorem* 3.1 *are satisfied. Also with* $m \geq 1$ *assume that* $M_1 \equiv \int_0^1 \prod_{j=1}^m (\xi_j - s)\, ds = 0$ *where* $\{\xi_j\}_{j=1}^m$ *are the points used in* (2.2). *Let* φ_0 *be an isolated solution of* (1.1) *and, for* $\alpha \leq \beta \leq m+1$, *assume* $\varphi_0 \in \mathit{Type}\,(\beta, m+1, \{a,b\})$ *or* $\varphi_0 \in \mathit{Type}\,(\beta-\varepsilon, m+1, \{a,b\})$ *for any* $\varepsilon \in (0,\beta)$ *in the logarithmic case. Then with* $q = (\alpha+m+1)/(\alpha+\beta)$ *and* $q = (\alpha+m+1)/(\alpha+\beta-\varepsilon)$ *in the logarithmic case used as the graded exponent in* (2.1), *we have*

$$||\varphi_0 - \bar{\varphi}_n|| = \begin{cases} \mathcal{O}(n^{-m-\alpha}) & 0 < \alpha < 1 \\ \mathcal{O}(n^{-m-1}\ln n) & \alpha = 1. \end{cases} \tag{3.8}$$

Proof. Arguing as in the previous theorem, $z \in \mathrm{Type}\,(\beta, m+1, \{a,b\})$ or $z \in \mathrm{Type}\,(\beta-\varepsilon, m+1, \{a,b\})$. For such z, following the proof of Theorem 3 of Schneider [**15**], we can establish the following upper bounds,

$$\begin{aligned} |E(g_\alpha, z)| &\equiv \left| \int_a^b g_\alpha(|s-t|) k(s,t) \{z(t) - (P_n z)(t)\}\, dt \right| \\ &\leq \begin{cases} cn^{-m-\alpha} & 0 < \alpha < 1 \\ cn^{-m-1}\ln n & \alpha = 1, \end{cases} \end{aligned}$$

where c is a constant. Combining this result with the first inequality in (3.7), we obtain (3.8). $\square$

This type of superconvergence arises from the fact that $\varphi_n(= f + Kz_n)$, being the iterate of z_n, has the possibility that it converges faster than z_n. Kumar [**9**] demonstrates that this superconvergence does occur under the assumption that z belongs to some Sobolev space. Such a strong condition on the smoothness of z is not necessary in the present paper. Now we present examples.

Example 1. Consider the previous example of Section 2,

$$\varphi(s) - \int_0^1 \frac{\varphi^2(t)}{|s-t|^{1/4}}\, dt = f(s), \quad 0 \leq s \leq 1,$$

where f is selected so that $\varphi(s) = s^{3/4}$ is a solution. Let $e_n = ||\varphi - \bar{\varphi}_n||$. In the first data, $q = (\alpha+m+1)/(\alpha+\beta)$ with $\alpha = \beta = 3/4$. In the second data, $q = 1$ (uniformly spaced breakpoints) is tested for comparison. In both of these two cases, t_{ij} are obtained by mapping the Gaussian points into $[t_i, t_{i+1}]$ for each $i = 0, 1, \ldots, n-1$. Hence, $M_1 = 0$. The e_n's are estimated as before.

$m = 2$

n	e_n data-1	decay exp.	e_n data-2	decay exp.
8	4.72 D-03		1.62 D-02	
16	9.06 D-04	2.38	6.83 D-03	1.24
32	1.52 D-04	2.57	1.84 D-03	1.92
64	2.43 D-05	2.65	4.96 D-04	1.89

$m = 3$

8	8.24 D-04		5.27 D-03	
16	7.70 D-05	3.42	1.06 D-03	2.31
32	6.57 D-06	3.55	1.64 D-04	2.69
64	5.12 D-07	3.68	2.43 D-05	2.72

Example 2. Consider

$$\varphi(s) - \int_0^1 \frac{\sin \varphi(t)}{\sqrt{|s-t|}}\, dt = f(s), \quad 0 \le s \le 1,$$

where f is selected so that $\varphi(s) = \sqrt{s}$ is a solution. In the first data, $q = (\alpha + m + 1)/(\alpha + \beta)$ with $\alpha = \beta = 1/2$ and $M_1 = 0$ (the Gaussian points are used as in Example 1). In the second data, we used the same q but $M_1 \neq 0$. Namely, for $m = 2$, $t_{i1} = t_i$ and $t_{i2} = t_{i+1}$, and for $m = 3$, $t_{i1} = t_i$, $t_{i2} = (2t_i + t_{i+1})/3$ and $t_{i3} = t_{i+1}$. In the third data, the case for $q = 1$ and $M_1 = 0$ was tested for comparison.

$m = 2$

n	e_n data-1	dec. exp.	e_n data-2	dec. exp.	e_n data-3	dec. exp
8	5.75 D-03		3.26 D-02		1.43 D-02	
16	1.22 D-03	2.23	8.15 D-03	2.00	7.12 D-03	1.00
32	2.52 D-04	2.28	2.01 D-03	2.02	3.15 D-03	1.17
64	4.94 D-05	2.35	5.03 D-04	2.00	1.36 D-03	1.21

$m = 3$

n	e_n data-1	dec. exp.	e_n data-2	dec. exp.	e_n data-3	dec. exp
8	9.31 D-04		6.77 D-03		4.04 D-03	
16	9.39 D-05	3.31	9.01 D-04	2.91	9.36 D-04	2.11
32	8.77 D-06	3.42	1.13 D-04	2.99	2.32 D-04	2.01
64	8.08 D-07	3.44	1.43 D-05	2.98	5.05 D-05	2.20

4. Production-integration method. In this final section, we discuss the product-integration method for equation (1.1). Since the stated results which follow can be proved with minor modifications to our earlier proofs, we omit the proofs.

Let P_n be the interpolatory projection which was defined in Section 2 with the breakpoints (2.1) and the interpolation points (2.2). The product-integration method was investigated by Atkinson [**1,2**] to obtain a numerical solution of the weakly singular second kind Fredholm integral equation. His results were extended by Schneider in [**15**] who, using the graded breakpoints (2.1) of Rice, gave the optimal rate of convergence of the numerical solution. It is straightforward to see that the product-integration method can be applied to our Hammerstein equation (1.1). Furthermore, it is evident, from the discussion of Section 3, that a global superconvergence is possible.

In the product-integration method, we seek a function φ_n, taken usually as a piecewise polynomial, which satisfies the following equation,

(4.1)
$$\varphi_n(s) - \int_a^b g_\alpha(|s-t|)P_n[k(s,\cdot)\psi(\cdot,\varphi_n(\cdot))](t)\,dt = f(s), \quad a \le s \le b.$$

This can be written as

$$\varphi_n - K_\alpha P_n \hat{\Psi}\varphi_n = f \tag{4.2}$$

where $K_\alpha(\varphi)(s) = \int_a^b g_\alpha(|s-t|)\varphi(t)\,dt$ and $\hat{\Psi}(\varphi)(t) = k(s,t)\psi(t,\varphi(t))$. Using an argument similar to the one used to prove Theorem 2.1 or using the results from [**3** or **19**], the following theorem can be established.

Theorem 4.1. *Assume that the hypotheses of Theorem* A *are satisfied. Let* φ_0 *be an isolated solution of* (1.1) *and for* $\beta \in [\alpha, m]$, φ_0

is of Type $\{\beta, m, \{a, b\})$ *or* φ_0 *is of* Type $(\varphi - \varepsilon, m, \{a, b\})$ *for any* $\varepsilon \in (0, \beta)$ *in the logarithmic case. Let* Ψ_2 *satisfy the Lipschitz condition as in Theorem* 2.1. *Then, with* $q = m/\alpha$ *or* $q = m/\beta - \varepsilon$ *in the logarithmic case, the solution* φ_n *of* (4.2) *exists and* $||\varphi_0 - \varphi_n|| = \mathcal{O}(n^{-m})$.

For a result of superconvergence, we obtain

Theorem 4.2. *Let the hypotheses of Theorem* 3.2 *be satisfied. Then for the solution* φ_n *of* (4.2) *and an isolated solution* φ_0 *of* (1.1), *we have*

$$||\varphi_0 - \varphi_n|| = \begin{cases} \mathcal{O}(n^{-m-\alpha}) & 0 < \alpha < 1 \\ \mathcal{O}(n^{-m-1} \ln n) & \alpha = 1. \end{cases}$$

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REFERENCES

1. K. Atkinson, *The numerical solution of Fredholm integral equations of the second kind*, SIAM J. Num. Anal. **4** (1967), 337–349.

2. ———, *The numerical solution of Fredholm integral equations of the second kind with singular kernels*, Num. Math. **19** (1972), 248–259.

3. ———, *The numerical evaluation of fixed points of completely continuous operators*, SIAM J. Num. Anal. **10** (1973), 799–807.

4. G.A. Chandler, *Galerkin method for boundary integral equations on polygonal domains*, J. Austral. Math. Soc. Ser. B **26** (1984), 1–13.

5. I. Graham, *Singular expansions for the solution of second kind Fredholm integral equations with weakly singular convolution kernels*, J. Integral Equations Appl. **4** (1982), 1–30.

6. I. Graham and C. Schneider, *Product integration for weakly singular integral equations*, in *Constructive methods for the practical treatment of integral equations* (G. Hammerlin and K. Hoffmann, eds.), ISNM 73, Birkhauser, 1985, 156–164.

7. H. Kaneko, R. Noren and Y. Xu, *Regularity of the solution of Hammerstein equations with weakly singular kernel*, Integral Equations Operator Theory, **13** (1990), 660–670.

8. M. Krasnoselskii, *Topological methods for nonlinear integral equations*, Pergamon Press, New York, 1964.

9. S. Kumar, *Superconvergence of a collocation-type method for Hammerstein equations*, IMA J. Num. Anal. **7** (1987), 313–325.

10. S. Kumar and I. Sloan, *A new collocation-type method for Hammerstein integral equations*, Math. Comp. **48** (1987), 585–593.

11. R. Piessens and M. Branders, *Numerical solution of integral equations of mathematical physics using Chebyshev polynomials*, J. Comp. Phys. **21** (1976), 178–196.

12. J. Rice, *On the degree of convergence of nonlinear spline approximations* in *Approximation with special emphasis on spline functions* (I.J. Schoenberg, ed.), Academic Press, New York, 1969, 349–365.

13. G. Richter, *On weakly singular Fredholm integral equations with displacement kernels,* J. Math. Anal. Appl. **55** (1976), 32–42.

14. C. Schneider, *Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind,* Integral Equations Operator Theory **2** (1979), 62–68.

15. ———, *Product integration for weakly singular integral equations*, Math. Comp. **36** (1981), 207–213.

16. I. Sloan, *On the numerical evaluation of singular integrals*, BIT **18** (1978), 91–102.

17. G. Vainikko, *Perturbed Galerkin method and general theory of approximate methods for nonlinear equations*, Zh. Vychisl. Mat. i Mat. Fiz. **7** (1967), 723–751. Engl. Translation, U.S.S.R. Comp. Math. and Math. Phys. **7**, No. 4 (1967), 1–41.

18. G. Vainikko and P. Uba, *A piecewise polynomial approximation to the solution of an integral equation with weakly singular kernel*, J. Austral. Math. Soc. Series B, **22** (1981), 431–438.

19. R. Weiss, *On the approximation of fixed points of nonlinear compact operators*, SIAM J. Num. Anal. **11** (1974), 550–553.

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