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THE NUMERICAL SOLUTION OF THE GENERALIZED AIRFOIL EQUATION

D. BERTHOLD, W. HOPPE AND B. SILBERMANN

1. Introduction. Singular integral equations on an interval and their numerical solution have been studied by many authors [**3, 5, 6, 7, 11, 12, 13, 14, 16, 18, 19, 23, 25, 32**] in the recent two decades. There the case when the integral operator contains an additional term with a weakly singular kernel of the type $\log |y-x|$ possesses special importance for applications in aerodynamics as well as in the diffraction theory and in the two-dimensional elasticity theory [**9, 11, 12, 14, 33**].

In the present paper we will study a quadrature method for the equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{v(y)\,dy}{y-x}-\frac{\nu}{\pi}\int_{-1}^{1}\ln|y-x|v(y)\,dy+\frac{1}{\pi}\int_{-1}^{1}k(x,y)v(y)\,dy=f(x), \tag{1.1}$$

$x\in(-1,1)$. Here f and k are given Hölder-continuous functions, ν is a complex number, and v is the sought function. The paper [**12**] elaborates a collocation method, the collocation points of which are just the zeros of certain orthogonal polynomials (Jacobi polynomials). Furthermore, the approximate solutions are sought in the form of linear combinations of other Jacobi polynomials, multiplied by the corresponding weight. The analytical treatment of [**16, 25**] for the operator

$$cI+dS_o,\quad S_o v(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{v(y)}{y-x}\,dy,\quad x\in(-1,1), \tag{1.2}$$

with Hölder-continuous coefficients c and d, where the action of the operator $cI+dS_o$ is described by an invariance relation between certain orthogonal polynomials, suggests that Golberg and Fromme's method [**12**] is a very natural one and gives cause to expect reliable results. Nevertheless, a disadvantage of this method consists of the fact that

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integrals of the type $\int_{-1}^{1} \ln|y-x|u(y)\,\sigma(y)\,dy$, where u are certain polynomials and $\sigma(y) = (1+y)^{\frac{1}{2}}(1-y)^{-\frac{1}{2}}$, have to be evaluated numerically. At least, no procedure is given on how to do it exactly. But, on using some mapping properties of the weakly singular integral operator

$$W_0 v(x) = -\frac{1}{\pi}\int_{-1}^{1} \ln|y-x|v(y)\,dy, \quad x \in (-1,1), \tag{1.3}$$

(cf. [**8, 11, 27, 31**]), we are able to establish a quadrature method for solving the equation (1.1), which is, contrary to that one of [**12**], fully discretized. Moreover, we are in a position to treat $S_o + \nu W_o$ (instead of merely S_o) as the dominant part of the operator in (1.1) and the discretization of this part is quite simple and can be done without any integration. This approach seems to be very natural and finally leads to iterative two-grid and multiple grid methods for determining an approximation solution of (1.1) with a computational complexity of $O(n^2)$.

2. Action of the Cauchy-type singular integral operator on special orthogonal polynomials. Denote by J the interval $[-1,1]$, by ρ any nonnegative integrable function on J, and by L^2_ρ, as usual, the Hilbert space of complex-valued functions on J for which the norm $||f||_\rho$ is finite, where

$$||f||_\rho := \sqrt{(f,f)_\rho}, \qquad (f,g)_\rho := \frac{1}{\pi}\int_J f(x)\overline{g(x)}\rho(x)\,dx.$$

Now we are looking for appropriate spaces in which equation (1.1) is to be considered. Following [**4, 12, 16**] we choose two weight functions $\sigma(x) = (1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}$ and $\mu(x) = [\sigma(x)]^{-1}$ in accordance with the underlying physical problem. The weight σ can be interpreted as a function, describing the behavior of the solution v of (1.1) near the endpoints of the interval J.

Further, let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be systems of orthogonal polynomials with respect to the scalar products $(.,.)_\sigma$ and $(.,.)_\mu$, respectively. In particular, if we have

$$\begin{aligned} p_n(x) &= \cos[(n+(1/2))s][\cos(s/2)]^{-1},\\ q_n(x) &= \sin[(n+(1/2))s][\sin(s/2)]^{-1}, \end{aligned} \tag{2.1}$$

$n = 0, 1, \ldots,\ s \in (0, \pi)$, $\cos s = x$, then the following useful relations will be valid (cf. [**4, 16, 25**]).

Lemma 2.1. *If $n = 0, 1, \ldots,$ then $S_o\sigma p_n = q_n$ and $S_0\mu q_n = -p_n$.*

Corollary 2.2. *The operator $S_o\sigma I : L^2_\sigma \to L^2_\mu$ is continuously invertible, both $S_0\sigma I$ and its inverse are isometrical isomorphisms, and $(S_o\sigma I)^{-1} = -S_o\mu I$.*

Therefore, following a standard idea, we substitute $v = \sigma u$ in (1.1) and consider in the sequel the equation

$$Au := (S + \nu W + K)u = f \tag{2.2}$$

instead of (1.1), where $S = S_o\sigma I$, $W = W_o\sigma I$, and

$$Ku(x) = \frac{1}{\pi}\int_{-1}^{1} k(x, y)u(y)\sigma(y)\, dy, \quad x \in [-1, 1].$$

Note that A is a Fredholm operator with index zero (cf. Theorem 3.6 below).

3. Analytical properties of the weakly singular operator W_o. The aim of this section is to summarize some mapping properties of the operator W. In this connection, we also refer to [**11**]. Results about the compactness of W and the invertibility of $S + \nu W$ are given.

Let $\tau(x) = (1 - x^2)^{-1/2}$ and $t_n(x) = \cos(n \arccos x)$, $n = 0, 1, \ldots,$ be the Chebyshev weight and the Chebyshev polynomials, respectively.

Lemma 3.1. [**8, 27**] *For all $x, y \in (-1, 1)$ with $x \neq y$ we have*

$$\ln|y - x| = -\ln 2 - \sum_{i=1}^{\infty}(2/i)t_i(x)t_i(y).$$

Theorem 3.2. *The polynomials t_n fulfill the relations*

$$W_0\tau t_0 = t_0 \ln 2 \quad \textit{and} \quad W_0\tau t_n = t_n/n \quad \textit{for } n = 1, 2, \ldots .$$

Proof. Denote by $s_n(x,y)$ the partial sum

$$s_n(x,y) = -\ln 2 - \sum_{i=1}^{n} (2/i) t_i(x) t_i(y).$$

Now we have for any $x \in J$ and $n, m \in \mathbf{N}$ with $n < m$

$$||s_n(x,\cdot) - s_m(x,\cdot)||_\tau^2 = \frac{1}{\pi} \int_J \left\{ \sum_{i=n+1}^{m} (2/i) t_i(x) t_i(y) \right\}^2 \tau(y)\, dy.$$

Since $(t_i, t_j)_\tau = \delta_{ij}/2$, $j \neq 0$, we conclude that

$$\begin{aligned} ||s_n(x,.) - s_m(x,.)||_\tau^2 &= \sum_{i=n+1}^{m} \frac{2}{i^2} t_i^2(x) \\ &\leq \sum_{i=n+1}^{\infty} \frac{2}{i^2} \to 0, \end{aligned}$$

as $n \to \infty$. Hence $\{s_n(x,.)\}$ is a fundamental sequence in L_τ^2 for each $x \in J$. So we conclude by the completeness of this space that

$$\frac{1}{\pi} \int_J [\ln|y-x| - s_n(x,y)]^2 \tau(y)\, dy \underset{n\to\infty}{\to} 0, \quad x \in J,$$

because we know from Lemma 3.1 the limit of $s_n(x,.)$ in the sense of convergence almost everywhere. Now, Cauchy-Schwarz's inequality gives for $k = 0, 1, \dots$

$$\frac{1}{\pi} \left| \int_J [\ln|y-x| - s_n(x,y)] t_k(y) \tau(y)\, dy \right| \underset{n\to\infty}{\to} 0,$$

and consequently

$$W_0 \tau t_k(x) = -\frac{1}{\pi} \lim_{n\to\infty} \int_J s_n(x,y) t_k(y) \tau(y)\, dy, \quad x \in J.$$

In virtue of the orthogonality of the Chebyshev polynomials t_k, the assertion follows. $\square$

Corollary 3.3. *The operator W acts on polynomials according to the rule*: $Wp_o = q_1/2 + (\ln 2 - (1/2))q_o$ *and*

$$Wp_n = \frac{1}{2}\left[\frac{q_{n+1}}{n+1} + \frac{q_n}{n(n+1)} - \frac{q_{n-1}}{n}\right], \quad n = 1, 2, \dots .$$

Proof. Use the relations $p_n(x)(1+x) = t_n(x) + t_{n+1}(x)$, $q_{n+1}(x) - q_n(x) = 2t_{n+1}(x)$, which can be verified by a straightforward computation based upon the trigonometric representations (2.1), and $\sigma(x) = (1+x)\tau(x)$. $\square$

For $t \in J$ and $\delta > 0$ we denote by $J_\delta(t)$ the interval $(t-\delta, t+\delta) \cap J$. Further, let

$$h(t, x, \delta) = \int_{J_\delta(t)} \ln^2\left|\frac{y-x}{y-t}\right| \sigma(y)\, dy.$$

Lemma 3.4. *We have the relation $\chi(\delta) := \sup h(t, x, \delta) \to 0$ as $\delta \to 0$, where the supremum is taken over all points $(x, t) \in J \times J$.*

Proof. By suitable estimates of the integrand, one can verify that the function h depends continuously on each of the variables $t, x \in J$, $\delta \geq 0$. Hence, h is uniformly continuous on $J \times J \times [0, \delta_0]$ for some fixed positive δ_0. Therefore, the assertion follows from the obvious relation $h(t, x, \delta) \to 0$ as $\delta \to 0$, $t, x \in J$. $\square$

In the sequel, $\mathcal{C}$ denotes the Banach space of all continuous functions $f : J \to \mathbf{C}$, endowed with the norm $||f||_\infty = \max\{|f(x)| : x \in J\}$.

Theorem 3.5. *The operator $W : \mathbf{L}^2_\sigma \to \mathcal{C}$ is compact.*

Proof. Let $\varepsilon > 0$ be arbitrary. According to Lemma 3.4 we can choose $\delta > 0$ so that $\chi(\delta) < \varepsilon^2\pi/4$. Now let $u \in \mathbf{L}^2_\sigma$ with $||u||_\sigma \leq 1$. Using

Cauchy-Schwarz's inequality, we obtain

$$|(Wu(x) - Wu(t)| \leq \left\{ \frac{1}{\pi} \int \ln^2 \left| \frac{y-x}{y-t} \right| \sigma(y)\, dy \right\}^{1/2} + \left\{ \frac{1}{\pi} \int \ln^2 \left| \frac{y-x}{y-t} \right| \sigma(y)\, dy \right\}^{1/2},$$

where the first integral is taken over $J_\delta(t)$ and the second one over $J \backslash J_\delta(t)$. Thus, the first term is less than $\varepsilon/2$. To make the second one less than $\varepsilon/2$, too, we take into account the uniform (with respect to $y \in J \backslash J_\delta(t)$) convergence $|(y-x)(y-t)^{-1}| \to 1$ as $x \to t$. Consequently, there exists $\delta' > 0$ such that $|Wu(x) - Wu(t)| < \varepsilon$ if only $|x - t| < \delta'$, i.e., $\{Wu : ||u||_\sigma \leq 1\} \subset \mathcal{C}$ is equicontinuous.

Suppose that $u \in \mathbf{L}^2_\sigma$ and define $u_i := (u, p_i)_\sigma$. Attending to Corollary 3.3 (and its proof) and $|t_n(x)| \leq 1$, $x \in J$, we obtain

$$|Wu(x)| \leq (1 + \ln 2)|u_o| + 2 \sum_{i=1}^{\infty} \frac{|u_i|}{i} \leq C ||u||_\sigma$$

for all $x \in J$, where $C^2 = (1 + \ln 2)^2 + 4 \sum_{i=1}^{\infty} 1/i^2$ is finite. Now it remains to apply the Arzela-Ascoli Theorem. □

In the following, we will always assume that the condition

(K) $\qquad \dim \ker A = 0 \quad \text{and} \quad \dim \ker B = 0 \quad \text{in } \mathbf{L}^2_\sigma$

is fulfilled, where $A := S + \nu W + K$ and $B := S + \nu W$.

Theorem 3.6. *The operators $A, B : \mathbf{L}^2_\sigma \to \mathbf{L}^2_\mu$ are continuously invertible if* (K) *and the condition*

$$\int_J \int_J |k(x,y)|^2 \sigma(y)\, dy \mu(x)\, dx < \infty \tag{3.1}$$

are satisfied.

Proof. Condition (3.1) implies the compactness of K (cf. [**22**]). Further, from Theorem 3.5 we easily derive the compactness of W :

$\mathbf{L}^2_\sigma \to \mathbf{L}^2_\mu$. Since S is invertible, both A and B are Fredholm operators with index zero. This yields, together with (K), the assertion. $\square$

Of course, it is useful to know a sufficient and necessary condition for $\ker B = \{0\}$. Therefore, we introduce the following function

$$m(\nu) := I_o(\nu) + \nu(I_o(\nu) + I_1(\nu)) \ln 2, \quad \nu \in \mathbf{C},$$

where I_k denotes the modified Bessel function of the first kind of order k, and formulate the following criterion.

Theorem 3.7. *The set* $\ker B$ *is trivial in* $\mathbf{L}^2_\sigma$ *if and only if* $m(\nu) \neq 0$.

Proof. Consider the equation

$$Bu = (S + \nu W)u = 0. \tag{3.2}$$

Following [**29**], we shall substitute $u = -S_o \mu w$, $w \in \mathbf{L}^2_\mu$. Then, by Corollary 2.2, we have $w - \nu W S_o \mu w = 0$, i.e.,

$$w(x) + \frac{\nu}{\pi^2} \int_J w(z)\mu(z) \bigg\{ \int_J \frac{\ln|y-x|\sigma(y)}{z-y}\, dy \bigg\}\, dz = 0, \tag{3.3}$$

$x \in J$. By the aid of [**26**, formula 5.4.15.3] or [**29**, formula (9)] we find that

$$\frac{\mu(z)}{\pi} \int_J \frac{\ln|y-x|\sigma(y)}{z-y}\, dy = \arcsin z + \mu(z) \ln 2 \begin{cases} -\frac{1}{2}\pi, & z > x \\ +\frac{1}{2}\pi, & z < x. \end{cases}$$

Hence, (3.3) is equivalent to

$$Mw := w + \nu w_o = 0, \tag{3.4}$$

where

$$\begin{aligned} w_o(x) := \frac{1}{\pi} \int_J & w(y)[\arcsin y + \mu(y) \ln 2 - \pi/2]\, dy \\ & + \int_{-1}^{x} w(y)\, dy, \end{aligned}$$

$x \in J$. Since $w_o'(x) = w(x)$, we can transform equation (3.4) into an equivalent ordinary differential equation

$$w_o' + \nu w_o = 0, \tag{3.5}$$

with the additional condition

$$\int_J w_o(x)\tau(x)\,dx - \ln 2 \int_J w_o'(x)\mu(x)\,dx = 0, \tag{3.6}$$

which is obtained from the definition of w_o. Its general solution is given by $w_o(x) = C_o e^{-\nu x}$; thus $w(x) = w_o'(x) = -\nu C_o e^{-\nu x}$. Inserting this in (3.6), we get

$$C_o \int_J e^{-\nu x}\tau(x)[1 + \nu(1-x)\ln 2]\,dx = 0. \tag{3.7}$$

We conclude from this equality that (3.2) possesses a unique solution u $(\equiv 0)$ if and only if (3.7) has a unique solution C_o $(= 0)$, i.e., if

$$\int_J e^{-\nu x}\tau(x)[1 + \nu(1-x)\ln 2]\,dx \neq 0. \tag{3.8}$$

Formula (49.3.8) from [**30**] yields by a straightforward computation that

$$\frac{1}{\pi}\int_J (-1)^k x^k e^{-\nu x}\tau(x)\,dx = I_k(\nu), \quad k = 0, 1, \quad \nu \in \mathbf{C}. \tag{3.9}$$

Combining (3.8) and (3.9) the assertion follows. $\square$

Corollary 3.8. *If* $\nu \in \mathbf{R}$, *then* $\ker B = \{0\}$.

Proof. If $\nu \geq 0$, then obviously $m(\nu) > 0$. For $\nu < 0$ we shall apply asymptotic formulas for the Bessel functions. We know from [**21**, Section 6] that, for $x \in \mathbf{R}$, $x > 1$,

$$I_0(x) = e^x(1 + 1/(8x) + r_0(x)/x^2)/\sqrt{2\pi x},$$
$$I_1(x) = e^x(1 - 3/(8x) + r_1(x)/x^2)/\sqrt{2\pi x},$$

where r_0, r_1 satisfy the estimates

$$|r_0(x)| \le \sqrt{\pi}(63/128), \qquad |r_1(x)| \le \sqrt{\pi}(165/256), \quad x > 1. \tag{3.10}$$

Since I_0 is even and I_1 is odd, we have for $\nu = -x,\ x > 1$

$$m(\nu) = I_0(x) - x(I_0(x) - I_1(x)) \ln 2 = m_0(x) e^x x^{-2}/\sqrt{2\pi x},$$

where

$$m_0(x) = (1 - \frac{1}{2}\ln 2)x^2 + (\frac{1}{8} - (r_0 - r_1)\ln 2)x + r_0.$$

After some calculation one concludes that both zeros x_1, x_2 of the quadratic polynomial m_0 do not belong to the set $\{x \in \mathbf{R}, x \ge 4\}$, whenever the parameters r_0 and r_1 satisfy (3.10). Hence, $m(\nu) \neq 0$ for all $\nu \le -4$.

Unfortunately, the asymptotic formulas for $I_k(x)$ for small values of x are difficult to be exploited for our purpose. Therefore, in case $-4 < \nu < 0$, we use the following method, which, in turn, fails for $\nu \le -4$. Let $u_i := (u, p_i)_\sigma$. Comparing the Fourier coefficients with respect to the orthonormal basis $\{q_i\}_{i=0}^\infty \subset \mathbf{L}^2_\mu$ in equation (3.2), we get in view of Lemma 2.1 and Corollary 3.3 that $u_1 = (2/\nu - 1 + 2\ln 2)u_0$ and

$$u_{i+1} = (1 + 1/i)u_{i-1} + [(2i+2)/\nu + 1/i]u_i, \quad i = 1, 2, \dots .$$

Assuming that $u_0 \neq 0$ one easily verifies that the signs of u_i alternate and $|u_{i+1}| > |u_{i-1}|$, $i = 1, 2, \dots$, holds. Thus, the sequence $\{u_i\}$ is not square summable, which contradicts Parseval's equation. Consequently, the assumption $u_0 \neq 0$ is false. But $u_0 = 0$ obviously means that $u_i = 0$ for all i, i.e., $u = 0$. $\square$

4. Description of the quadrature method. In accordance with the considerations in Sections 2 and 3, we shall seek an approximate solution u_n of equation (2.2) in the class Π_n of all polynomials of degree less than n with complex coefficients. In the following, let

$$y_{nk} = \cos((2k-1)\pi/(2n+1)), \quad k = 1, \dots, n, \tag{4.1}$$

be the zeros of the n-th orthogonal polynomial p_n and

$$l_{nk}^{\sigma}(y)=\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{y-y_{ni}}{y_{nk}-y_{ni}}=\frac{p_n(y)}{(y-y_{nk})p_n'(y_{nk})} \tag{4.2}$$

the corresponding Lagrangian fundamental polynomials. For an arbitrary function $u:[-1,1]\to\mathbf{C}$ the Lagrangian interpolation operator L_n^{σ} is defined by

$$L_n^{\sigma}u(y)=\sum_{k=1}^{n}l_{nk}^{\sigma}(y)u(y_{nk}), \tag{4.3}$$

and by

$$\sigma_{nk}=\frac{1}{\pi}\int_J l_{nk}^{\sigma}(y)\sigma(y)\,dy=\frac{q_n(y_{nk})}{p_n'(y_{nk})} \tag{4.4}$$

we denote the Christoffel numbers of the polynomial p_n. A straightforward calculation yields that

$$\sigma_{nk}=(2/(2n+1))(1+y_{nk}). \tag{4.5}$$

Let $\mathbf{R}_{\sigma}$ be the linear space of all bounded functions on J being Riemann-integrable with respect to the weight σ. Then for $u\in\mathbf{R}_{\sigma}$ the Gaussian quadrature rule

$$\frac{1}{\pi}\int_J u(y)\sigma(y)\,dy\sim\sum_{k=1}^{n}u(y_{nk})\sigma_{nk} \tag{4.6}$$

holds, which is exact for polynomials of degree less than $2n$. We denote the zeros of the orthogonal polynomials q_n by

$$x_{nj}=\cos(2j\pi/(2n+1)),\quad j=1,\dots,n, \tag{4.7}$$

and we define the Lagrangian fundamental polynomials l_{nj}^{μ}, the Christoffel numbers μ_{nj}, $j=1,\dots,n$, and the interpolation operators L_n^{μ} in the same way as above. Then an analogous Gaussian quadrature rule with the degree of accuracy $2n-1$ is valid.

In order to approximate the solution of equation (2.2) we consider the following equations

$$(S + \nu L_n^\mu W + K_{nn})u_n = L_n^\mu f, \quad u_n \in \Pi_n, \tag{4.8}$$

where the operators K_{mn} are defined on $\mathbf{L}_\sigma^2$ by (cf. [**2**])

$$K_{mn}u(x) = \frac{1}{\pi}\int_J k_{mn}(x,y)u(y)\sigma(y)\,dy,$$
$$k_{mn}(x,y) = L_{mx}^\mu L_{ny}^\sigma k(x,y).$$

(The second subscripts x and y of the interpolation operators mark the variable according to which the interpolation is done.)

Remark 4.1. Usually, in the case of quadrature method, we approximate the operator K by $L_n^\mu \tilde{K}_n$, where

$$\tilde{K}_n u_n(x) = \frac{1}{\pi}\int_J L_{ny}^\sigma[k(x,y)u_n(y)]\sigma(y)\,dy, \quad u_n \in \Pi_n.$$

These operators are defined only on a subset of $\mathbf{L}_\sigma^2$. But using the (exact) Gaussian quadrature rule (4.6), we observe that K_{nn} is the extension of $L_n^\mu \tilde{K}_n$ to all of $\mathbf{L}_\sigma^2$. This fact is very important in further investigations.

For the remaining part of the paper, let the kernel $k(x,y)$ fulfill the following conditions:

(a) $\int_J [k(x,y)]^2\sigma(y)\,dy \in \mathbf{R}_\mu$,

(b) $\int_J k(x,y)\pi(y)\sigma(y)\,dy \in \mathbf{R}_\mu$,

for each polynomial π of arbitrary degree,

(c) $k(x,\cdot) \in \mathbf{R}_\sigma$ uniformly with respect to $x \in J$.

Let $\mathcal{C}^{l,\gamma}$ ($l \geq 0$ integer, $0 \leq \gamma \leq 1$) denote the subspace of $\mathcal{C}$ containing all l-times continuously differentiable functions whose l-th derivative satisfies a Hölder condition with exponent γ. If we introduce the norm

$$||u||_{l,\gamma} := \sum_{i=0}^{l} ||D^i u||_\infty + C_{l,\gamma}(u),$$
$$C_{l,\gamma}(u) := \sup |D^l u(x) - D^l u(y)|\,|x-y|^{-\gamma},$$

where the supremum is taken over all $x, y \in J$, $x \neq y$ and D denotes the operator of differentiation, $\mathcal{C}^{l,\gamma}$ becomes a Banach space.

Lemma 4.2. *Let ρ be one of the weights σ or μ. Then the following assertions are valid*:

(i) $||f - L_n^\rho f||_\rho \underset{n\to\infty}{\to} 0$ *for each* $f \in \mathbf{R}_\rho$.

(ii) *If* $f \in \mathcal{C}^{l,\gamma}$, *then* $||f - L_n^\rho f||_\rho \leq Cn^{-l-\gamma}$, *where C does not depend on n.*

Proof. For (i) see [**10**], for (ii) see [**16, 25**]. ◻

Definition 4.3. The kernel k is said to satisfy the assumption $(*)$, if

$$k(\cdot, y) \in \mathcal{C}^{l,\gamma} \text{ uniformly with respect to } y \in J \text{ and}$$
$$k(x, \cdot) \in \mathcal{C}^{r,\delta} \text{ uniformly with respect to } x \in J.$$

Lemma 4.4. *We have*

$$||K_{mn} - K||_{\sigma\to\mu} =: \varepsilon_{mn} \underset{m,n\to\infty}{\to} 0,$$

where $||\cdot||_{\sigma\to\mu}$ means the norm in the space $\mathcal{L}(\mathbf{L}^2_\sigma, \mathbf{L}^2_\mu)$. Moreover, if assumption $(*)$ is fulfilled, then

$$\varepsilon_{mn} \leq C(m^{-l-\gamma} + n^{-r-\delta}).$$

Proof. Let $u \in \mathbf{L}^2_\sigma$. For each $x \in J$, we have

$$K_{mn}u(x) - Ku(x) = g(x) + h(x),$$

where

$$g(x) := \frac{1}{\pi} L^\mu_{mx} \int_J [L^\sigma_{ny} k(x,y) - k(x,y)] u(y)\sigma(y)\, dy$$

and

$$h(x) := \frac{1}{\pi} \int_J [L^{\mu}_{mx} k(x,y) - k(x,y)] u(y) \sigma(y) \, dy.$$

Now, $||K_{mn}u - Ku||_{\mu} \leq ||g||_{\mu} + ||h||_{\mu}$. The first norm can be estimated as follows:

$$\begin{aligned} ||g||_{\mu} &\leq C \sup_{x \in J} \left| \int_J [L^{\sigma}_{ny} k(x,y) - k(x,y)] u(y) \sigma(y) \, dy \right| \\ &\leq C \sup_{x \in J} \left\{ \int_J [L^{\sigma}_{ny} k(x,y) - k(x,y)]^2 \sigma(y) \, dy \right\}^{\frac{1}{2}} ||u||_{\sigma}. \end{aligned}$$

By Lemma 4.2, the expression in the braces tends to zero as $n \to \infty$. If assumption $(*)$ is satisfied, then the same expression can be estimated by $Cn^{-r-\delta} ||u||_{\sigma}$.

Consider the function $h(x)$. Since

$$|h(x)|^2 \leq \frac{1}{\pi} \int_J [L^{\mu}_{mx} k(x,y) - k(x,y)]^2 \sigma(y) \, dy ||u||^2_{\sigma},$$

we obtain

$$||h||_{\mu} \leq \frac{1}{\pi} \left\{ \int_J \int_J [L^{\mu}_{mx} k(x,y) - k(x,y)]^2 \sigma(y) \, dy \mu(x) \, dx \right\}^{1/2} ||u||_{\sigma}.$$

Now from [**16**] Theorem 3.2, it follows that the integral expression converges to zero as $m \to \infty$. Changing the order of integration, using the relation $||v||_{\sigma} \leq c \, \sup\{|v(y)| : y \in J\}$, and applying Lemma 4.2 again, we get $||h||_{\mu} \leq Cm^{-l-\gamma} ||u||_{\sigma}$, if assumption $(*)$ is valid. Thus, the assertions are proved. □

Remark 4.5. We conclude from this lemma that K_{mn} is uniformly bounded. We even are able to give a concrete bound. It is well-known that

$$||K_{mn}||^2_{\sigma \to \mu} \leq \frac{1}{\pi^2} \int_J \int_J [k_{mn}(x,y)]^2 \sigma(y) \, dy \mu(x) \, dx.$$

Since k^2_{mn} is a polynomial of degree $2m-2$ and $2n-2$ in the variable x and y, respectively, we get, using the (exact) Gaussian quadrature rules,

$$||K_{mn}||^2_{\sigma \to \mu} \leq C_{\sigma} C_{\mu} \sup |k(x,y)|^2, \tag{4.9}$$

where the supremum is taken over all pairs $(x, y) \in J \times J$ and C_ρ is defined by $C_\rho := \pi^{-1} \int_J \rho(y)\, dy$.

Lemma 4.6. *We have*

$$||L_n^\mu W - W||_{\sigma\to\mu} \underset{n\to\infty}{\to} 0.$$

Proof. Recall that W is compact from $\mathbf{L}^2_\sigma$ into $\mathcal{C}$ (Theorem 3.5). Since L_n^μ converges strongly on $\mathcal{C}$ with respect to the $\mathbf{L}^2_\mu$-norm to the identical operator I (Lemma 4.2), the convergence $L_n^\mu W \to W$ is uniform. □

Now we are able to show

Lemma 4.7. *If condition* (K) *is satisfied, then for sufficiently large n the operators $A_n := S + \nu L_n^\mu W + K_{nn}$ and $B_n := S + \nu L_n^\mu W$ are invertible in the pair of spaces $(\mathbf{L}^2_\sigma, \mathbf{L}^2_\mu)$ and*

$$\sup_n ||A_n^{-1}||_{\mu\to\sigma} < \infty, \qquad \sup_n ||B_n^{-1}||_{\mu\to\sigma} < \infty.$$

Proof. From the Lemmata 4.4 and 4.6 we get

$$||B - B_n||_{\sigma\to\mu} \underset{n\to\infty}{\to} 0, \qquad ||A - A_n||_{\sigma\to\mu} \underset{n\to\infty}{\to} 0,$$

where B and A are invertible operators (cf. Theorem 3.6). The assertions follow immediately. □

Thus, we can conclude that for large n equation (4.8) possesses a unique solution $u_n \in \mathbf{L}^2_\sigma$. Moreover, since $u_n = S^{-1}(L_n^\mu f - \nu L_n^\mu W u_n - K_{nn} u_n)$, we get from Lemma 2.1 that the solution is an element of Π_n. If we seek u_n of the form

$$u_n(y) = \sum_{k=1}^{n} (\xi_n)_k l_{nk}^\sigma(y),$$

the vector $\xi_n \in \mathbf{C}^n$ is given by the solution of the system of equations $(S_n + \nu W_n + K_n)\xi_n = f_n$. The matrices S_n, W_n and K_n are obtained from (4.8) by choosing the collocation points x_{nj}, $j = 1, \dots, n$ and using quadrature rules. It is well known that ([**16, 5**])

$$\begin{aligned}
(S_n)_{j,k} &= \sigma_{nk}(y_{nk} - x_{nj})^{-1}, \quad j,k = 1,\dots,n,\\
(K_n)_{j,k} &= \sigma_{nk} k(x_{nj}, y_{nk}), \quad j,k = 1,\dots,n,\\
(f_n)_j &= f(x_{nj}), \quad j = 1,\dots,n.
\end{aligned}$$

Considering $S + \nu W$ and $S + \nu L_n^\mu W$ as dominant parts of the operators A and A_n, respectively, our aim is now to find a simple representation of the entries of the matrix W_n. Moreover, we want to get a representation of the inverse of $S_n + \nu W_n$.

Since $\{l_{nk}^\sigma\}_{k=1}^n$ and $\{l_{nj}^\mu\}_{j=1}^n$ are bases in Π_n, the matrix $S_n + \nu W_n$ is easily seen to be the matrix representation of the operator $S + \nu L_n^\mu W$ in this pair of bases. On the other hand, the matrix representation H_n of this operator in the pair of bases $\{p_i\}_{i=0}^{n-1}$ and $\{q_i\}_{i=0}^{n-1}$ is very simple. Using Lemma 2.1, Corollary 3.3, and $L_n^\mu q_n = 0$, we get $H_n = I_n + \nu G_n$, where I_n denotes the unit matrix of order n and

$$G_n = \begin{pmatrix} -1/2 + \ln 2 & -1/2 & & O \\ 1/2 & 1/4 & \ddots & \\ & \ddots & \ddots & \frac{-1}{2(n-1)} \\ O & & \frac{1}{2(n-1)} & \frac{1}{2n(n-1)} \end{pmatrix}.$$

By P_n^σ and Q_n^μ, we denote the matrices with the entries

$$(P_n^\sigma)_{i,k} = p_{i-1}(y_{nk}), \qquad (Q_n^\mu)_{i,j} = q_{i-1}(x_{nj}), \quad i,j,k = 1,\dots,n.$$

From (4.3) we deduce that P_n^σ is the transformation matrix from the basis $\{p_i\}$ to the basis $\{l_{nk}^\sigma\}$. Analogously, the matrix Q_n^μ transforms from $\{q_i\}$ to $\{l_{nj}^\mu\}$.

Lemma 4.8. *The matrices P_n^σ and Q_n^μ are invertible, its inverses are $(P_n^\sigma)^{-1} = (P_n^\sigma D_n^\sigma)^T$ and $(Q_n^\mu)^{-1} = (Q_n^\mu D_n^\mu)^T$, where D_n^σ and D_n^μ are diagonal matrices with the diagonal entries σ_{nk} and μ_{nk}, $k = 1, \dots, n$, respectively.*

Proof. We carry out the proof for P_n^σ. Consider the matrix product $P_n^\sigma D_n^\sigma (P_n^\sigma)^T$. Its entry in the i-th row and j-th column is

$$\sum_{k=1}^{n} p_{i-1}(t_{nk}) p_{j-1}(t_{nk}) \sigma_{nk}.$$

Using the (accurate) Gaussian quadrature rule (4.6) and the orthonormality of the system $\{p_i\}$, we get $P_n^\sigma D_n^\sigma (P_n^\sigma)^T = I_n$, whence the first assertion follows immediately. The second one can be shown analogously. □

Now, with a little algebra, we derive

$$S_n + \nu W_n = (Q_n^\mu)^T [I_n + \nu G_n] P_n^\sigma D_n^\sigma. \tag{4.10}$$

Consequently, applying the shorted Gaussian algorithm to the tridiagonal matrix $I_n + \nu G_n$, we are able to solve equations of the form

$$(S_n + \nu W_n)\xi_n = g_n, \quad g_n \in \mathbf{C}^n, \tag{4.11}$$

effectively.

5. Convergence theorems. In this section we shall prove two theorems about the convergence of the quadrature method for the equation $Au = f$. In the first one we show that on very weak conditions the approximate solution u_n converges to the exact solution u. But it is often necessary to know the rates of convergence, especially in the solution of the approximate equation by the help of multiple grid methods (see Section 6). Thus, we formulate a second convergence theorem, in which on certain assumptions about the smoothness of the kernel k and the right-hand side f we get estimates of the convergence rate.

Theorem 5.1. *If, in addition to the assumptions made up to now, $f \in \mathbf{R}_\mu$ is satisfied, then the equations* (4.8) *are uniquely solvable for all sufficiently large n, and for the unique solutions u and u_n of* (2.2) *and* (4.8), *resepectively, one has $||u - u_n||_\sigma \to 0$ as $n \to \infty$.*

Proof. The existence of unique solutions u and u_n (for large n) of the equations $Au = f$ and $A_n u_n = L_n^\mu f$, respectively, follows from Theorem 3.6 and Lemma 4.7. Using

$$||u - u_n||_\sigma \leq ||A_n^{-1}||_{\mu\to\sigma}[||A_n - A||_{\sigma\to\mu}||u||_\sigma + ||f - L_n^\mu f||_\mu],$$

Lemma 4.7 (together with its proof) and Lemma 4.2 we get the assertion. □

Now we proceed to estimates of the convergence rates. To do this, we need the following lemma.

Lemma 5.2. *If u is the solution of $Bu = (S + \nu W)u = g$, $g \in \mathcal{C}^{l,\gamma}$, then $Wu \in \mathcal{C}^{l+1,\gamma}$. In case $1/2 < \gamma < 1$, the solution u is an element of $\mathcal{C}^{l,\gamma-\frac{1}{2}}$.*

Proof. Consider the ordinary differential equation $w' + \nu w = g$. A little thought shows that its general solution belongs to $\mathcal{C}^{l+1,\gamma}$. Using Lemma 2.1, Corollary 3.3, and the trigonometric representations of the polynomials p_n and q_n we get $(Wv)' = Sv$ for each polynomial v of arbitrary degree. This relation can be extended by continuity to the whole space $\mathbf{L}^2_\sigma$ in the sense of a generalized derivative, i.e., $(Wv)' = Sv$, $v \in \mathbf{L}^2_\sigma$. Now let $u \in \mathbf{L}^2_\sigma$ be the unique solution of $(S + \nu W)u = g$ (cf. Theorem 3.6). The considerations above show that $w = Wu$ satisfies the ordinary differential equation and hence $Wu \in \mathcal{C}^{l+1,\gamma}$. Therefore, $u = S^{-1}(g - \nu Wu)$ with $g - \nu Wu \in \mathcal{C}^{l,\gamma}$. With the regularity statements for the operator S in [**15, 18**] we obtain $u \in \mathcal{C}^{l,\gamma-1/2}$. □

Theorem 5.3. *If, in addition to the assumptions made up to now, the kernel k satisfies $(*)$ and $f \in \mathcal{C}^{l,\gamma}$, then the equations* (4.8) *are uniquely solvable for all sufficiently large n, and for the unique solutions u and u_n of* (2.2) *and* (4.8), *respectively, we have*

$$(5.1) \qquad ||u - u_n||_\sigma \leq Cn^{-\alpha},$$

where $\alpha = \min\{l + \gamma, r + \delta\}$. If $1/2 < \gamma < 1$, then u is an element of $\mathcal{C}^{l,\gamma-1/2}$. Finally, if $\alpha > 3/2$, then

$$(5.2) \qquad ||u - u_n||_\infty \leq Cn^{-\alpha+3/2}.$$

Proof. Let u be the unique solution of $Au = f$. Then $Bu = f - Ku \in \mathcal{C}^{l,\gamma}$. From Lemma 5.2, we conclude that $Wu \in \mathcal{C}^{l+1,\gamma}$. Applying the Lemmata 4.2 and 4.4 we can estimate $||u - u_n||_\sigma$

$$\leq ||A_n^{-1}||_{\mu\to\sigma}[||\nu(L_n^\mu - I)Wu||_\mu + ||K_{nn} - K||_{\sigma\to\mu}||u||_\sigma + ||f - L_n^\mu f||_\mu]$$
$$\leq C(n^{-l-\gamma} + n^{-r-\delta}).$$

The relation $u \in \mathcal{C}^{l,\gamma-1/2}$ follows from Lemma 5.2. Using (5.1) and [**17**, Lemma 4.11] we get (5.2). $\square$

6. Effective numerical solution of the approximating equations. In generalization of [**17, 2**] we are going to present an iteration process for solving (4.8). This iterative process seems to be a very natural one, since it exactly reflects the subdivision of the operators A and A_n into the dominant parts B and B_n, respectively, and the perturbation. Set $m \ll n$. After choosing $u_{n,0} \in \Pi_n$ we construct a sequence $\{u_{n,i}\}_{i=0}^\infty \subset \Pi_n$ by a so-called two-grid method:

$$B_n u_{n,i+1/2} = L_n^\mu f - K_{nn} u_{n,i}, \tag{6.1}$$
$$A_m e_{m,n,i} = K_{mn}(u_{n,i} - u_{n,i+1/2}), \quad e_{m,n,i} \in \Pi_m, \tag{6.2}$$
$$u_{n,i+1} = u_{n,i+1/2} + e_{m,n,i}, \tag{6.3}$$

where the second and the third equations are motivated by

$$A_n(u_n - u_{n,i+1/2}) = L_n^\mu f - (B_n + K_{nn})u_{n,i+1/2} = K_{nn}(u_{n,i} - u_{n,i+1/2}).$$

With respect to the effective solution of the first equation we refer to the remarks at the end of Section 4. The relations (6.1)–(6.3) can be written as iterative equations for a fixpoint equation in the following form:

$$u_{n,i+1} = T_{mn} L_n^\mu f + (I - T_{mn} A_n) u_{n,i}, \tag{6.4}$$

where $T_{mn} := (I - A_m^{-1} K_{mn}) B_n^{-1}$. It is easy to see that the solution u_n of (4.8) is a fixpoint of (6.4). Whence, for the convergence of $u_{n,i}$ to u_n $(i \to \infty)$ it is sufficient that $||I - T_{mn} A_n||_{\sigma\to\sigma} < 1$ for large m and n. In order to verify this, we have to do some further preparations. The proof of the following lemma is due to D. Jackson [**24**, Chapter VI, Section 2].

Lemma 6.1. *Let $u \in \mathcal{C}^{l,\gamma}$ and $n > l+1$. Then there exists a polynomial $u_n \in \Pi_n$ such that $u_n^* = u - u_n$ satisfies*

$$||u_n^*||_\infty \le Cn^{-l-\gamma}||u||_{l,\gamma},$$

where the constant C depends only on l and γ.

Lemma 6.2. *We have*

$$||W - L_n^\mu W||_{\mathcal{C}^{l,\gamma} \to \mathbf{L}^2_\mu} \le Cn^{-l-\gamma}$$

with a constant C not depending on n.

Proof. Let $u \in \mathcal{C}^{l,\gamma}$. By Lemma 6.1, u has a representation of the form $u = u_{n-1} + u_{n-1}^*$ with $u_{n-1} \in \Pi_{n-1}$ and $||u_{n-1}^*||_\infty \le Cn^{-l-\gamma}||u||_{l,\gamma}$. From Corollary 3.3, we derive that $Wu_{n-1} \in \Pi_n$ and, consequently,

$$||Wu - L_n^\mu Wu||_\mu = ||(W - L_n^\mu W)u_{n-1}^*||_\mu \le ||W - L_n^\mu W||_{\sigma\to\mu}||u_{n-1}^*||_\sigma.$$

The uniform boundedness of the first norm follows from Lemma 4.6. Using $||u_{n-1}^*||_\sigma \le C||u_{n-1}^*||_\infty$ we get the assertion. □

Lemma 6.3. *The following assertions are valid*:

(i) $||B_n^{-1} - B^{-1}||_{\mu\to\sigma} \underset{n\to\infty}{\to} 0$,

(ii) $||B_n^{-1} - B^{-1}||_{\mathcal{C}^{l,\gamma}\to\mathbf{L}^2_\sigma} \le Cn^{-l-\gamma+1/2}$, *if* $1/2 < \gamma < 1$.

Proof. The first assertion follows from Lemma 4.7 (together with its proof) and

$$||B_n^{-1} - B^{-1}||_{\mu\to\sigma} \le ||B_n^{-1}||_{\mu\to\sigma}||B - B_n||_{\sigma\to\mu}||B^{-1}||_{\mu\to\sigma}.$$

Now let $f \in \mathcal{C}^{l,\gamma}$, $1/2 < \gamma < 1$. We derive the existence of a constant C such that $||B_n^{-1}f - B^{-l}f||_\sigma \le Cn^{-l-\gamma+1/2}$. According to Lemma 5.2, the relation $B^{-1}f \in \mathcal{C}^{l,\gamma-1/2}$ is valid. Applying Lemma 6.2, we can estimate

$$\begin{aligned}
&||B_n^{-1}f - B^{-1}f||_\sigma \\
&\qquad \le ||B_n^{-1}||_{\mu\to\sigma}||\nu W - \nu L_n^\mu W||_{\mathcal{C}^{l,\gamma-1/2}\to\mathbf{L}^2_\mu} \cdot ||B^{-1}f||_{l,\gamma-1/2} \\
&\qquad \le Cn^{-l-\gamma+1/2},
\end{aligned}$$

where C does not depend on n. Consider now the operators $C_n \in \mathcal{L}(\mathcal{C}^{l,\gamma}, \mathbf{L}^2_\sigma)$, $C_n := n^{l+\gamma-1/2}(B_n^{-1} - B^{-1})$. From the principle of uniform boundedness [**20**, Chapter VII] we conclude that $||C_n||_{\mathcal{C}^{l,\gamma}\to\mathbf{L}^2_\sigma}$ is uniformly bounded. Therefore, the second assertion is proved. □

Lemma 6.4. *One has*

$$||B_m^{-1}K_{mn} - B^{-1}K||_{\sigma\to\sigma} =: \delta_{mn} \underset{m,n\to\infty}{\to} 0.$$

Moreover, if the kernel k satisfies assumption $(*)$ with $1/2 < \gamma < 1$, then

$$\delta_{mn} \le C(m^{-l-\gamma+1/2} + n^{-r-\delta}).$$

Proof. We write $B_m^{-1}K_{mn} - B^{-1}K = B_m^{-1}(K_{mn} - K) + (B_m^{-1} - B^{-1})K$. Now the first assertion follows from $K \in \mathcal{L}(\mathbf{L}^2_\sigma, \mathbf{L}^2_\mu)$ and the Lemmata 4.4, 4.7, and 6.3. In order to prove the second assertion, we remark that in this case $K \in \mathcal{L}(\mathbf{L}^2_\sigma, \mathcal{C}^{l,\gamma})$. Consequently, with Lemma 6.3, we get

$$||B_m^{-1} - B^{-1}||_{\mathcal{C}^{l,\gamma}\to\mathbf{L}^2_\sigma}||K||_{\mathbf{L}^2_\sigma\to\mathcal{C}^{l,\gamma}} \le Cm^{-l-\gamma+1/2}.$$

The Lemmata 4.4 and 4.7 yield

$$||B_m^{-1}||_{\mu\to\sigma}||K_{mn} - K||_{\sigma\to\mu} \le C(m^{-l-\gamma} + n^{-r-\delta}).$$

Thus, the lemma is proved. □

Now we can show the validity of the following theorem.

Theorem 6.5. *We have*

$$||I - T_{mn}A_n||_{\sigma\to\sigma} =: \zeta_{mn} \underset{m,n\to\infty}{\to} 0.$$

Moreover, if assumption $(*)$ with $1/2 < \gamma < 1$ is satisfied, we can estimate ($\beta = \min\{l + \gamma - 1/2, r + \delta\}$)

$$\zeta_{mn} \le C(n^{-\beta} + m^{-\beta}).$$

Proof. The operator $I - T_{mn}A_n$ admits the representation

$$A_m^{-1}[(K_{mn} - K_{mm})B_n^{-1}K_{nn} + K_{mm}(B_n^{-1}K_{nn} - B_m^{-1}K_{mn})] \\ + (B_m^{-1}K_{mn} - B_n^{-1}K_{nn}).$$

From the Lemmata 4.4, 4.7 and 6.4, we obtain $\zeta_{mn} \to 0$ as $m, n \to \infty$. Furthermore, if $(*)$ with $1/2 < \gamma < 1$ holds true, then the same lemmata yield the sought estimate. $\square$

Now, the iteration error $||u_{ni} - u_n||$ of the two-grid method can be estimated, using (6.4) and the fixpoint property of the solution u_n of (4.8), by

$$||u_{ni} - u_n||_\sigma \le \zeta_{mn}^i ||u_{n0} - u_n||_\sigma. \tag{6.5}$$

Choosing, for example, $u_{n0} = 0$, we get

$$||u_{ni} - u_n||_\sigma \le C\zeta_{mn}^i, \tag{6.6}$$

and if there exists a constant c so that $n \le cm$, we obtain

$$||u_{ni} - u_n||_\sigma \le Cn^{-i\beta}. \tag{6.7}$$

The total error after i iteration steps, that means $||u_{ni} - u||_\sigma$, can be estimated by (use Theorem 5.3)

$$||u_{ni} - u||_\sigma \le Cn^{-\min\{l+\gamma, i(l+\gamma-1/2), r+\delta\}}. \tag{6.8}$$

We see that, for example, if $l + \gamma \ge 1$, then after two iteration steps the order of the total error has reached the order of the approximation error (see Theorem 5.3). Considering the computational complexity of this two-grid method we find out that it is of the order $O(n^3)$ since $n \le cm$ is required. But this disadvantage does not occur if we use multiple grid methods. The idea consists in using again a two-grid method to solve (6.2), etc. We define such a method by the help of an iteration process analogously to (6.4). First, we choose two positive integers $N \ge 2$ and n_0 and define $\{n_p\}_{p=0}^\infty$ by $n_p = Nn_{p-1}$, $p = 1, 2, \dots$. The subscript p is called the level. For simplicity, in the following we will denote by A_p, B_p, $K_{p-1,p}$, L_p^μ, T_p, u_p, ζ_p the operators A_{n_p}, B_{n_p}, K_{n_{p-1},n_p},

$L^{\mu}_{n_p}$, T_{n_{p-1},n_p}, the function u_{n_p}, and the number ζ_{n_{p-1},n_p}, respectively. From Theorem 6.5, we conclude that

$$\zeta_p \le v_p := Cn_p^{-\beta}. \tag{6.9}$$

Following [**28**], we define an operator $M_p \in \mathcal{L}(\mathbf{L}^2_\mu, \mathbf{L}^2_\sigma)$ for each p recurrently as follows

$$\begin{aligned} M_o &= A_0^{-1}, \\ M_p &= (I - Q_{p-1}K_{p-1,p})B_p^{-1}, \quad p = 1, 2, \dots, \\ Q_{p-1} &= (2I - M_{p-1}A_{p-1})M_{p-1}, \quad p = 1, 2, \dots . \end{aligned}$$

Let us consider the iteration process

$$u_{p,i+1} = M_p L^{\mu}_p f + (I - M_p A_p)u_{p,i}. \tag{6.10}$$

It can be shown that (6.10) realizes the idea described above. Define $\xi_p := ||I - M_p A_p||_{\sigma\to\sigma}$. Based on the convergence of the two-grid method on level 1 (i.e., $n := n_1$, $m := n_0$) the convergence of the multiple grid method can be proved in the following way.

Theorem 6.6. *If the kernel k satisfies assumption* $(*)$ *with* $1/2 < \gamma < 1$ *and* v_1 *(from* (6.9)*) is less than*

$$b := \min\{1, (1/2d)[\sqrt{C_K^2 + 4d(\sqrt{d} - d)} - C_K]\},$$

where $d := N^{-\beta}$ *and* $C_K := \sup\{||B_p^{-1}K_{pp}||_{\sigma\to\sigma} : p = 0, 1, \dots\}$, *then*

$$\xi_p \le \sqrt{v_p}, \quad p = 1, 2, \dots .$$

Proof. (cf. also [**28**]). We only give the idea of the proof. The operator $I - M_p A_p$ admits the representation

$$(I - T_p A_p) - [I - M_{p-1}A_{p-1}]^2[(I - T_p A_p) + B_p^{-1}K_{pp}].$$

Thus, we conclude that

$$\xi_p \le \zeta_p + \xi_{p-1}^2(\zeta_p + ||B_p^{-1}K_{pp}||_{\sigma\to\sigma}).$$

Defining recurrently $w_1 := v_1$, $w_p := v_p + w_{p-1}^2(v_p + C_K)$, $p = 2, 3, \ldots$ we get from [**1**, Lemma 17], $w_p^2 < v_p$, $p = 1, 2, \ldots$, if only $v_1 < b$ is fulfilled. It is easy to see that $M_1 = T_1$, consequently, $\xi_1 = \zeta_1$. By induction, we obtain $\xi_p \leq w_p$, $p = 2, 3, \ldots$. This proves the assertion. $\square$

Returning to (6.10), we deduce

$$||u_{pi} - u_p||_\sigma \leq \xi_p^i ||u_{po} - u_p||_\sigma, \tag{6.11}$$

where u_p is the solution of $A_p u_p = L_p^\mu f$. Choosing, for example, $u_{po} = 0$ we get

$$||u_{pi} - u_p||_\sigma \leq C\xi_p^i \leq C n_p^{-ib/2}, \tag{6.12}$$

and for the total error (with α from Theorem 5.3)

$$||u_{pi} - u||_\sigma \leq C n_p^{-\min\{\alpha, i\beta/2\}}. \tag{6.13}$$

A little thought shows us that the computational complexity of this multiple grid method is $O(n_p^2) + O(n_o^3)$, which is of the order $O(n_p^2)$.

Remark 6.7. If we modify the condition on v_1 in Theorem 6.6 to $v_1 < b := (1/2d)[\sqrt{C_K^2 + d^2} - C_K]$, which is a stronger one in case $d < 16/25$, then we can improve the error estimate (6.12) to $||u_{pi} - u_p|| \leq C n_p^{-i\beta}$. For the proof, see [**28**, Chapter 2], where it was remarked, however, that in practice this stronger condition is often not realizable without loss of efficiency.

7. Quadrature method for a more general integral equation. The treatment of the operator $S + \nu W + K$ gives rise to the following more comprehensive generalization of the Cauchy type singular integral operator, namely the equation

$$(S^{(l)} + K)u = f, \tag{7.1}$$

where

$$
\begin{aligned}
&S^{(l)} = \sum_{j=0}^{l} c_j Z_j \sigma I, \quad c_j \in \mathbf{C}, \quad c_0 \neq 0,\\
&Z_j v(x) = \frac{1}{\pi}\int_J z_j(y-x)v(y)\,dy, \quad j = 0,1,\dots\,,\\
&z_0(x) = x^{-1},\\
&z_j(x) = \frac{(-x)^{j-1}}{(j-1)!}\bigg[-\ln|x| + \sum_{i=1}^{j-1}\frac{1}{i}\bigg], \quad j = 1,2,\dots\,.
\end{aligned}
$$

With these notations, we have $Z_0 = S_0$, $Z_1 = W_0$, and $z_j(x) = -z'_{j+1}(x)$, $x \neq 0$, $j = 0,1,\dots$. The investigation of the operators Z_j and the construction of the approximating equations proceeds on the basis of Sections 3 and 4. We shall enlighten this strategy by applying it to the case $j = 2$. First find out the series expansion of $z_j(y-x)$ in terms of the Chebyshev polynomials of the first kind:

$$
\begin{aligned}
z_2(y-x) &= (y-x)(\ln|y-x| - 1)\\
&= (y-x)\bigg[-\ln 2 - \sum_{k=1}^{\infty}\frac{2}{k}t_k(x)t_k(y) - 1\bigg]\\
&= (\ln 2)t_1(x)t_0(y) + [(-\ln 2)t_0(x) + (1/2)t_2(x)]t_1(y)\\
&\quad + \sum_{k=2}^{\infty}\bigg[-\frac{t_{k-1}(x)}{(k-1)k} + \frac{t_{k+1}(x)}{(k+1)k}\bigg]t_k(y).
\end{aligned}
$$

Here we took into consideration Lemma 3.1 and the recurrence formula $t_{k+1}(x) = 2xt_k(x) - t_{k-1}(x)$.

The second step consists of determining $Z_j\sigma p_k$, $k = 0,1,\dots$, which proceeds in a similar manner to the proofs of Theorem 3.2 and Corollary 3.3; for example, we get

$$
\begin{aligned}
Z_2\sigma p_0 &= -q_0\ln 2 + ((\ln 2)/2 - 1/8)q_1 + (1/8)q_2,\\
Z_2\sigma p_1 &= \left(\frac{1}{8} - \frac{\ln 2}{2}\right)q_0 - \frac{1}{4}q_1 + \frac{1}{12}q_2 + \frac{1}{24}q_3,\\
Z_2\sigma p_k &= \frac{1}{4}\bigg\{\frac{q_{k-2}}{(k-1)k} - \frac{2q_{k-1}}{(k-1)k(k+1)} - \frac{2q_k}{k(k+1)} + \frac{2q_{k+1}}{k(k+1)(k+2)}\\
&\quad + \frac{q_{k+2}}{(k+1)(k+2)}\bigg\}, \quad k = 2,3,\dots\,.
\end{aligned}
$$

In the third step, we establish the approximating equations. The quadrature method developed here replaces the operator $S^{(l)}$ by the approximating ones

$$S_n^{(l)} = c_0 S + \sum_{j=1}^{l} c_j L_n^\mu Z_j \sigma I, \quad n = 1, 2, \dots ,$$

acting on Π_n. Consider as in Section 4 the matrix representation $H^{(l)}$ of $S^{(l)}$ with respect to the bases $\{p_i\}_{i=0}^\infty$ and $\{q_i\}_{i=0}^\infty$. It will be a band matrix the band width of which exactly equals $2l-1$. When switching over to the representation $H_n^{(1)}$ of $S_n^{(1)}$, the band structure is disturbed by nonzero entries in the last $l-1$ columns which come from the decomposition of $L_n^\mu q_{n+1}, \dots , L_n^\mu q_{n+l-1}$ with respect to the basis $\{q_0, \dots , q_{n-1}\}$ of Π_n.

Taking into account the basis transformations $\{p_i\}_{i=0}^{n-1} \mapsto \{l_{nk}^\sigma\}_{k=1}^n$ and $\{q_i\}_{i=0}^{n-1} \mapsto \{l_{nj}^\mu\}_{j=1}^n$, we get the approximating system

$$[(Q_n^\mu)^T H_n^{(1)} P_n^\sigma D_n^\sigma + K_n]\xi_n = f_n, \tag{7.2}$$

where $(f_n)_j = f(x_{nj})$, $j = 1, \dots , n$, and $u_n(x) = \sum_{k=1}^n (\xi_n)_k l_{nk}^\sigma(x)$. The proofs of the convergence Theorems 5.1 and 5.3 are applicable to this general case in a slightly adapted way, too. Note that $[Z_j\sigma u]' = Z_{j-1}\sigma u$, $j = 1, 2, \dots$, such that we can conclude the smoothness of $Z_j\sigma u$ from the smoothness of Ku and f. Furthermore, the compactness of $Z_j\sigma u$ ($j \geq 2$) follows from the continuity of z_j.

Finally, we can establish an analogous iteration process and verify its convergence to the exact solution of (7.2) under the same conditions as in Theorems 5.1 and 5.3 when substituting the assumption $\dim\ker B = 0$ by $\dim\ker S^{(l)} = 0$.

Remark 7.1. Although we have not found any treatment in the literature where the equation (7.1) with $l \geq 2$ arises from some physical or technological problem, we believe that the present generalization is not only of purely theoretical interest. Suppose that the functions k and f in equation (2.2) are such that f possesses Hölder continuous derivatives of higher order than k and that one can find appropriate coefficients c_j such that

$$\tilde{k}(x,y) := k(x,y) - \sum_{j=2}^{l} c_j z_j(y-x)$$

is smoother than k. In this situation, it is recommendable to solve the equivalent equation (7.1) (where K is defined by the kernel $\tilde{k}$ instead of k) by the quadrature method described here. The advantage will consist of an increasing convergence rate (cf. Theorem 5.3).

8. Numerical results.

We applied our method to the equation (2.2) with $\nu = 1$, $k(x,y) = |x| + |y|$, and $f(x) = x + |x| + 1 + \ln 2 + 2/\pi$. Then (2.2) possesses the exact solution $u(x) \equiv 1$. In order to solve the approximate equations (4.8) we used the two-grid method (6.1)–(6.3) with the start polynomial $u_{n,0} \equiv 0$ and $m = n/3$.

Notice that in this case $(f, k(.,y), k(x,.) \in \mathcal{C}^{0,1})$ we can expect from the error estimate (6.8) that

$$||u_{n1} - u||_\sigma \le Cn^{-1/2}, \qquad ||u_{ni} - u||_\sigma \le Cn^{-1}, \quad i = 2, 3, \dots .$$

In the following table we present the total error $||u_{ni} - u||_\sigma$ for various values of n and i.

n	1	2	3	4
6	3.21E-1	6.65E-3	1.28E-3	1.12E-3
12	5.79E-2	3.07E-4	2.40E-4	2.40E-4
18	3.17E-2	1.16E-4	9.98E-5	9.98E-5
24	2.04E-2	5.94E-5	5.41E-5	5.41E-5

We remark that the error was computed exactly by using the Gaussian quadrature rule (4.6).

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Technical University of Chemnitz, Department of Mathematics, PSF 964, 0-9010 Chemnitz, Germany.